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PARTIAL RESULTS ON THE CONVEXITY OF THE PARISI FUNCTIONAL WITH PDE APPROACH

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Abstract. We investigate the convexity problem for the Parisi functional defined on the space of the so-called functional ordered parameters in the Sherrington-Kirkpatrick model. In a recent work of Panchenko, it was proved that this functional is convex along one-sided directions with a probabilistic method. In this paper, we will study this problem with a PDE approach that simplifies the original proof and presents more general results.

1. Introduction and main results

The Sherrington-Kirkpatrick (SK) model was introduced in [5] with the aim of explaining the strange magnetic behaviors of certain alloys. In the past decades, it has been intensively studied in physics; see [2]. One of the most beautiful discoveries was the famous Parisi formula, which states that the thermodynamic limit of the free energy in the SK model with inverse temperature $\beta > 0$ and external field $h \in \mathbb{R}$ can be computed through a variational problem over the space of functional ordered parameters $\mathcal{M}$,

$$\inf_{a\in\mathcal{M}} \left(\log 2 + F_a(\beta h, 1) - \frac{\beta^2}{2}\int_0^1 ta(1-t)dt\right), \tag{1.1}$$

where $\mathcal{M}$ is defined as the collection of all nonincreasing and left-continuous functions from $[0,1]$ to $[0,1]$, and F_a is the solution to the nonlinear parabolic PDE associated with $a \in \mathcal{M}$,

$$\partial_t F_a(x,t) = \frac{1}{2}\left(\partial_{xx}F_a(x,t) + a(t)(\partial_x F_a(x,t))^2\right),\ (x,t) \in \mathbb{R}\times[0,1], \tag{1.2}$$

$$F_a(x,0) = \log\cosh(x),\ x \in \mathbb{R}. \tag{1.3}$$

Note that if a is a step function, F_a can be precisely solved by performing the Hopf-Cole transformation, while for general a, the existence of F_a is assured via an approximation argument using step functions and the uniform L^1-Lipschitz property of $a \mapsto F_a(x,t)$ over arbitrary (x,t); see [1]. The first mathematically rigorous proof for the Parisi formula was presented in the seminal work of Talagrand [6]. Later, its validity in the mixed p-spin model was established by Panchenko [4].

It has been conjectured that the Parisi formula (1.1) has a unique minimizer. In the physicists' picture, such a minimizer is encoded with all information that is needed to completely describe the model. As the third term in the bracket of (1.1)

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is linear in the functional ordered parameter, the uniqueness will follow if one could show that $a \mapsto F_a(\beta h, 1)$ defines a convex functional on $\mathcal{M}$. More generally, let ϕ be a twice differentiable function on $\mathbb{R}$ with

$$\|\phi'\|_\infty \text{ and } \|\phi''\|_\infty < \infty. \tag{1.4}$$

These assumptions guarantee the existence of the solution $F_{\phi,a}$ corresponding to the PDE (1.2) and step-like $a \in \mathcal{M}$ with a new initial condition ϕ in (1.3). They also allow us to adapt a similar argument as Proposition 3.1 in [7] to obtain the uniform L^1-Lipschitz property of $a \mapsto F_{\phi,a}(x,t)$ on all step-like $a \in \mathcal{M}$ over (x,t). So the existence of $F_{\phi,a}$ for arbitrary $a \in \mathcal{M}$ is ensured. Now for any x, define the Parisi functional $P_{\phi,x}$ on $\mathcal{M}$ as $P_{\phi,x}(a) := F_a(x,1)$ for $a \in \mathcal{M}$. Note that $\log\cosh x$ is an even convex function and satisfies (1.4). Numerical simulation suggests:

Conjecture. *The Parisi functional $P_{\phi,x}$ is convex on $\mathcal{M}$ if ϕ is even convex.*

The first related result about this problem was presented in Panchenko [3], where using a probabilistic argument, he showed the convexity along one-sided directions, that is,

$$P_{\phi,x}(\alpha a_1 + (1-\alpha)a_2) \le \alpha P_{\phi,x}(a_1) + (1-\alpha)P_{\phi,x}(a_2),\ \forall \alpha \in [0,1], \tag{1.5}$$

for all $a_1, a_2 \in \mathcal{M}$ with $a_1 \le a_2$. In this paper, we will study the above conjecture via a maximum principle for the nonlinear parabolic PDE (1.2). Although at this point we still have not been able to figure out how to use the present method to give a complete answer, it provides a new way of looking at the convexity problem that simplifies Panchenko's original argument and leads to more general results. Incidentally, it is of independent interest that the numerical evidence seems to support that the conclusion of the Conjecture also holds true when ϕ is nondecreasing. As one shall see, our method can be as well applied to this case and deduce similar convexity along one-sided directions.

We now state the main results. Let $\mathcal{C}$ be the collections of all twice differentiable ϕ on $\mathbb{R}$ satisfying (1.4). Set function spaces

$$\begin{aligned}
\mathcal{F}_1 &= \{(\phi_1,\phi_2) | \phi_1,\phi_2 \in \mathcal{C} \text{ even convex with } \phi_1 \le \phi_2 \text{ and } \phi_1' \le \phi_2'\},\\
\mathcal{F}_2 &= \{(\phi_1,\phi_2) | \phi_1,\phi_2 \in \mathcal{C} \text{ nondecreasing convex with } \phi_1 \le \phi_2\\
&\qquad\qquad \text{and } \phi_1'(x) \le \phi_2'(x)\ \forall x \ge 0\},\\
\mathcal{F} &= \mathcal{F}_1 \cup \mathcal{F}_2.
\end{aligned}$$

Theorem 1.1. *Suppose that $(\phi_1,\phi_2) \in \mathcal{F}$. If $a_1, a_2 \in \mathcal{M}$ with $a_1 \le a_2$, then*

$$P_{\phi,x}(a) \le \alpha P_{\phi_1,x}(a_1) + (1-\alpha)P_{\phi_2,x}(a_2) \tag{1.6}$$

for all $\alpha \in [0,1]$ and $x \in \mathbb{R}$, where $\phi := \alpha\phi_1 + (1-\alpha)\phi_2$ and $a := \alpha a_1 + (1-\alpha)a_2$.

Letting $(\phi_1,\phi_2) \in \mathcal{F}$ with $\phi_1 = \phi_2$, we obtain the convexity along the one-sided directions.

Corollary 1.1. *If $\phi \in \mathcal{C}$ is either nondecreasing convex or even convex, then* (1.5) *holds.*

Example 1.1. Let ϕ be either nondecreasing convex or even convex. For any given $m \in [0,1]$, if $a(t) = m$ for all $0 \le t \le 1$, then $P_{\phi,x}(a) = m^{-1}\log\mathbb{E}\exp m\phi(x+z)$, where z is a standard Gaussian random variable. Thus, Corollary 1.1 implies that $m \mapsto m^{-1}\log\mathbb{E}\exp m\phi(x+z)$ defines a convex function on $[0,1]$. Let us emphasize that this is different from the usual log-convexity of the L^m-norm in $1/m$.

2. Proofs

Throughout this paper, we denote by z the standard Gaussian random variable and set $z_{x,t} = x + \sqrt{t}z$ for $(x,t) \in \mathbb{R} \times [0,1]$. For a given $\phi \in \mathcal{C}$ and a number $0 \le m \le 1$, we will simply use $F_{\phi,m}$ to denote $F_{\phi,a}$ if $a \in \mathcal{M}$ is identically equal to m. In this case, $F_{\phi,m}$ can be explicitly written as

$$F_{\phi,m}(x,t) = \frac{1}{m}\log \mathbb{E}\exp m\phi(z_{x,t}) = \frac{1}{m}\log \mathbb{E}\exp m\phi(x+\sqrt{t}z). \tag{2.1}$$

The central rhythm of the proof for Theorem 1.1 is played by the following proposition.

Proposition 2.1. *Suppose that* $(\phi_1,\phi_2) \in \mathcal{F}$. *Let* $0 < m_1 \le m_2$ *and* $\alpha \in [0,1]$. *Set*

$$n = \alpha m_1 + (1-\alpha)m_2,$$
$$\phi = \alpha\phi_1 + (1-\alpha)\phi_2.$$

Then we have

$$\forall t \in [0,1],\ (F_{\phi_1,m_1}(\cdot,t), F_{\phi_2,m_2}(\cdot,t)) \in \mathcal{F}_i \text{ if } (\phi_1,\phi_2) \in \mathcal{F}_i \tag{2.2}$$

for $i = 1,2$ *and*

$$F_{\phi,n}(x,t) \le \alpha F_{\phi_1,m_1}(x,t) + (1-\alpha)F_{\phi_2,m_2}(x,t) \tag{2.3}$$

for all $(x,t) \in \mathbb{R}\times[0,1]$.

Proof of Theorem 1.1. By virtue of the uniform L^1-Lipschitz property of $P_{\phi,x}$ on $\mathcal{M}$ over x, it suffices to assume that a_1, a_2 are step functions. Furthermore, we may assume without loss of generality that they jump simultaneously at $\{t_j\}_{j=0}^k$ for some $k \ge 1$, where $0 < t_j < t_{j+1} < 1$ for all $1 \le j \le k$. Let $t_0 = 0$ and $t_{k+1} = 1$. For $0 \le j \le k$, let $m_{l,j} = a_l(t_j)$ for $l = 1,2$ and $n_j = a(t_j) = \alpha m_{1,j} + (1-\alpha)m_{2,j}$. Using (2.1) and an induction argument on j, F_{ϕ_l,a_l} and $F_{\phi,a}$ can be solved explicitly as

$$\begin{aligned} F_{\phi_l,a_l}(x,t) &= \frac{1}{m_{l,j}}\log\mathbb{E}\exp m_{l,j}F_{\phi_l,a_l}(x+\sqrt{t-t_j}z, t_j),\\ F_{\phi,a}(x,t) &= \frac{1}{n_j}\log\mathbb{E}\exp n_j F_{\phi,a}(x+\sqrt{t-t_j}z, t_j), \end{aligned} \tag{2.4}$$

whenever $(x,t) \in \mathbb{R}\times(t_j,t_{j+1}]$ for $0 \le j \le k$. Suppose for the moment that there exists some $0 \le j \le k$ such that

$$(F_{\phi_1,a_1}(\cdot,t_j), F_{\phi_2,a_2}(\cdot,t_j)) \in \mathcal{F}_i, \tag{2.5}$$

$$F_{\phi,a}(\cdot,t_j) \le \alpha F_{\phi_1,a_1}(\cdot,t_j) + (1-\alpha)F_{\phi_2,a_2}(\cdot,t_j). \tag{2.6}$$

Note that $m_{1,j} \le m_{2,j}$ for all $0 \le j \le k+1$. Keeping the iteration equations (2.4) in mind and using (2.5), we apply Proposition 2.1 with $F_{\phi_1,a_1}(\cdot,t_j)$, $F_{\phi_2,a_2}(\cdot,t_j)$, $m_{1,j}$, $m_{2,j}$ and $\Delta t_j := t_{j+1}-t_j$ to get by (2.2),

$$(F_{\phi_1,a_1}(\cdot,t_{j+1}), F_{\phi_2,a_2}(\cdot,t_{j+1})) \in \mathcal{F}_i$$

and by (2.6) and then (2.3),

$$\begin{aligned}F_{\phi,a}(x,t_{j+1}) &= \frac{1}{n_j}\log\mathbb{E}\exp n_j F_{\phi,a}(x+\sqrt{\Delta t_j}z,t_j)\\ &\leq \frac{1}{n_j}\log\mathbb{E}\exp n_j\Big(\alpha F_{\phi_1,a_1}(x+\sqrt{\Delta t_j}z,t_j)\\ &\qquad +(1-\alpha)F_{\phi_2,a_2}(x+\sqrt{\Delta t_j}z,t_j)\Big)\\ &\leq \frac{\alpha}{m_{1,j}}\log\mathbb{E}\exp m_j F_{\phi_1,a_1}(x+\sqrt{\Delta t_j}z,t_j)\\ &\qquad +\frac{1-\alpha}{m_{2,j}}\log\mathbb{E}\exp m_2 F_{\phi_1,a_2}(x+\sqrt{\Delta t_j}z,t_j)\\ &= \alpha F_{\phi_1,a_1}(x,t_{j+1})+(1-\alpha)F_{\phi_2,a_2}(x,t_{j+1}).\end{aligned}$$

From this and Proposition 2.1, an induction argument leads to

$$F_{\phi,a}(\cdot,t_{k+1})\leq \alpha F_{\phi_1,a_1}(\cdot,t_{k+1})+(1-\alpha)F_{\phi_2,a_2}(\cdot,t_{k+1}),$$

which gives (1.6) and completes our proof. □

The rest of the paper is devoted to proving Proposition 2.1 that will be divided into two parts. First, we prove (2.2). We begin with a lemma below that gathers a few properties about the expectations for functions of Gaussian random variables as well as two covariance inequalities; the first is a special case of the FKG inequality and the second is taken from [3].

Lemma 2.1. *Suppose that f, f_1, f_2 are real-valued functions on $\mathbb{R}$ and g is a centered Gaussian random variable with $\mathbb{E}g^2=\sigma^2$.*

(*i*) *If f is even, then $\mathbb{E}f(x+g)$ is even in x.*
(*ii*) *If f_1, f_2 are odd with $f_1\leq f_2$ on $[0,\infty)$, then $\mathbb{E}f_1(x+g)\leq\mathbb{E}f_2(x+g)$ for all $x\geq 0$.*
(*iii*) *Let W be a nonnegative function on $\mathbb{R}^2$ with $\mathbb{E}W(x,x+g)=1$ for any x. If f_1, f_2 are nondecreasing, then for any x,*

$$\begin{aligned}&\mathbb{E}f_1(x+g)f_2(x+g)W(x,x+g)\\ &\geq \mathbb{E}f_1(x+g)W(x,x+g)\mathbb{E}f_2(x+g)W(x,x+g).\end{aligned}\tag{2.7}$$

If

$$\begin{cases} f_1 \text{ is even with } f_1'(x)\geq 0 \text{ for } x\geq 0,\\ f_2 \text{ is odd with } f_2'(x)\geq 0 \text{ for } x\geq 0,\\ W(x,y) \text{ is even in } y,\end{cases}\tag{2.8}$$

then (2.7) holds for any $x\geq 0$.

Proof. Since g and $-g$ have the same distribution, (*i*) follows by $\mathbb{E}f(x+g)=\mathbb{E}f(-x-g)=\mathbb{E}f(-x+g)$. As for (*ii*), note that

$$\mathbb{E}f_l(x+g)=\int_{-\infty}^{\infty}f_l(u)\rho(u)du=\int_{-\infty}^{\infty}f_l(u)\rho(u,x)\exp\big(\frac{ux}{\sigma^2}\big)du,$$

where $\rho(u,x)=(2\pi\sigma^2)^{-1/2}\exp(-(u^2+x^2)/2\sigma^2)$ and ρ is the probability density of g. If we first split this integral into two parts $[0,\infty)$ and $(-\infty,0]$ and then use the change of variables $v = -u$ and the assumption that f_l is odd for the integral on $(-\infty, 0]$, it follows that

$$\begin{aligned}\mathbb{E}f_l(x+g) &= \int_0^\infty f_l(u)\rho(u,x)\exp\big(\frac{ux}{\sigma^2}\big)du - \int_0^\infty f_l(v)\rho(v,x)\exp\big(-\frac{vx}{\sigma^2}\big)dv \\ &= 2\int_0^\infty f_l(u)\rho(u,x)\sinh\big(\frac{ux}{\sigma^2}\big)du.\end{aligned}$$

Since $\sinh(ux) \geq 0$ for $x, u \geq 0$ and $f_1 \leq f_2$, this equation gives (ii).

Next we prove (iii). Let g' be an independent copy of g. Denote $g_x = x + g$ and $g'_x = x + g'$. Using $\mathbb{E}W(x, g_x) = \mathbb{E}W(x, g'_x) = 1$, we write

$$\begin{aligned}&\mathbb{E}f_1(g_x)f_2(g_x)W(x,g_x) - \mathbb{E}f_1(g_x)W(x,g_x)\mathbb{E}f_2(g_x)W(x,g_x) \\ &= \mathbb{E}W(x,g_x)W(x,g'_x)(f_1(g_x) - f_1(g'_x))(f_2(g_x) - f_2(g'_x))I(g \geq g').\end{aligned}$$

Applying the change of variables $(s,t) = (g_x, g'_x)$, this integral equals

$$\int_{\{s\geq t\}} K(x,s,t)\exp\Big(-\frac{1}{2\sigma^2}((s-x)^2 + (t-x)^2)\Big)dsdt, \tag{2.9}$$

where

$$K(x,s,t) := \frac{1}{2\pi\sigma^2}W(x,s)W(x,t)(f_1(s) - f_1(t))(f_2(s) - f_2(t)).$$

If f_1, f_2 are nondecreasing, this implies that K is nonnegative for any x and the first assertion follows immediately. Assume that f_1, f_2 satisfy (2.8). Let us split the integral region of (2.9) into two parts $\Omega_1 = \{(s,t) : s \geq t, |s| \leq |t|\}$ and $\Omega_2 = \{(s,t) : s \geq t, |s| \geq |t|\}$. Using the change of variables $(u,v) = (-t,-s)$ and the assumptions that f_1 is even, f_2 is odd and $W(x,y)$ is even in y, we obtain

$$\begin{aligned}&\int_{\Omega_2} K(x,s,t)\exp\Big(-\frac{1}{2\sigma^2}((s-x)^2 + (t-x)^2)\Big)dsdt \\ &= -\int_{\Omega_1} K(x,u,v)\exp\Big(-\frac{1}{2\sigma^2}((u+x)^2 + (v+x)^2)\Big)dudv\end{aligned}$$

and thus (2.9) becomes $\int_{\Omega_1} K(x,s,t)L(x,s,t)dsdt$, where

$$\begin{aligned}L(x,s,t) := \exp&\Big(-\frac{1}{2\sigma^2}((s-x)^2 + (t-x)^2)\Big) \\ &- \exp\Big(-\frac{1}{2\sigma^2}((s+x)^2 + (t+x)^2)\Big).\end{aligned}$$

Note that since $f'_1, f'_2 \geq 0$ for $x \geq 0$, this implies $K \geq 0$ on Ω_1. Also since $x \geq 0$ and $s + t \geq 0$ on Ω_1, this gives $(s+x)^2 + (t+x)^2 - (s-x)^2 - (t-x)^2 = 4x(s+t) \geq 0$. Together these ensure that $L \geq 0$ on Ω_1 and so (2.7) holds for $x \geq 0$. $\square$

Proof of (2.2) *in Proposition* 2.1. For notational convenience, we will denote F_{ϕ_l, m_l} by F_l for $l = 1, 2$ and $F_{\phi,n}$ by F_0. Note that these functions are clearly twice differentiable. Computing directly from (2.1) yields that

$$\partial_x F_l(x,t) = \mathbb{E}\phi'_l(z_{x,t})W_l(x,t), \tag{2.10}$$

$$\begin{aligned}\partial_{xx}F_l(x,t) &= \mathbb{E}\phi''_l(z_{x,t})W_l(x,t) \\ &\quad + m_l\big(\mathbb{E}\phi'_l(z_{x,t})^2 W_l(x,t) - (\mathbb{E}\phi'_l(z_{x,t})W_l(x,t))^2\big),\end{aligned} \tag{2.11}$$

where $W_l(x,t) := \exp m_l\phi_l(z_{x,t})/\mathbb{E}\exp m_l\phi_l(z_{x,t})$. Since ϕ_l satisfies (1.4), the above two equations imply $F_l(\cdot,t)$ satisfies (1.4) too. So $F_l(\cdot,t)\in\mathcal{C}$. From the right-hand side of (2.11), the first term is nonnegative since $\phi_l''\geq 0$, while the second term is nonnegative as well by noting $\mathbb{E}W_l(x,t)=1$ and using Jensen's inequality. Thus, $\partial_{xx}F_l\geq 0$ and F_l is convex in x. Note that if $\phi_1\leq\phi_2$, then the assumption $0\leq m_1\leq m_2$ combined with Jensen's inequality gives

$$\begin{aligned}F_1(x,t)&=\frac{1}{m_1}\log\mathbb{E}\exp m_1\phi_1(z_{x,t})\\&\leq\frac{1}{m_1}\log\mathbb{E}\exp m_1\phi_2(z_{x,t})\\&\leq\frac{1}{m_2}\log\mathbb{E}\exp m_2\phi_2(z_{x,t})=F_2(x,t).\end{aligned}$$

Now, on the one hand, if $(\phi_1,\phi_2)\in\mathcal{F}_1$, then $\phi_1',\phi_2'\geq 0$ and (2.10) yields that $F_1(\cdot,t),F_2(\cdot,t)$ are nondecreasing. On the other hand, if $(\phi_1,\phi_2)\in\mathcal{F}_2$, then $\exp m_1\phi_1,\exp m_2\phi_2$ are even and Lemma 2.1(i) shows that $F_1(\cdot,t),F_2(\cdot,t)$ are also even.

To finish the proof of (2.2), it remains to show that $\partial_xF_1\leq\partial_xF_2$ if $(\phi_1,\phi_2)\in\mathcal{F}_1$ and $\partial_xF_1\leq\partial_xF_2$ for $x\geq 0$ if $(\phi_1,\phi_2)\in\mathcal{F}_2$. Set

$$G(s)=\mathbb{E}\phi_1'(z_{x,t})W(s,x,z_{x,t}),$$

where

$$W(s,x,y):=\frac{\exp((1-s)m_1\phi_1(y)+sm_2\phi_2(y))}{\mathbb{E}\exp((1-s)m_1\phi_1(z_{x,t})+sm_2\phi_2(z_{x,t}))}.$$

Then

$$\begin{aligned}G'(s)=\;&\mathbb{E}f_1(z_{x,t})f_2(z_{x,t})W(s,x,z_{x,t})\\&-\mathbb{E}f_1(z_{x,t})W(s,x,z_{x,t})\mathbb{E}f_2(z_{x,t})W(s,x,z_{x,t}),\end{aligned}$$

where $f_1:=\phi_1'$ and $f_2:=m_2\phi_2-m_1\phi_1$. Suppose that $(\phi_1,\phi_2)\in\mathcal{F}_1$. Since ϕ_1 is convex, $m_2\geq m_1>0$ and $\phi_2'\geq\phi_1'$, it follows that f_1 and f_2 are nondecreasing and the first assertion of Lemma 2.1(iii) yields $G'(s)\geq 0$. Consequently, using $\phi_1'\leq\phi_2'$ and (2.10) gives $\partial_xF_1(x,t)=G(0)\leq G(1)\leq\partial_xF_2(x,t)$. Now, if $(\phi_1,\phi_2)\in\mathcal{F}_2$, then f_1 is odd, f_2 is even and both of them have nonnegative derivatives on $[0,\infty)$. The second assertion of Lemma 2.1(iii) implies $G'(s)\geq 0$ and thus $\partial_xF_1(x,t)=G(0)\leq G(1)$. Note that $\phi_l'(\cdot)\exp m_2\phi_2(\cdot)/\mathbb{E}\exp m_2\phi_2(z_{x,t})$ is odd for $l=1,2$. Using $\phi_1'\leq\phi_2'$ on $[0,\infty)$ and Lemma 2.1(ii) give $G(1)=\mathbb{E}\phi_1'(z_{x,t})W_2(x,t)\leq\mathbb{E}\phi_2'(z_{x,t})W_2(x,t)=\partial_xF_2(x,t)$ for all $x\geq 0$. Thus, we conclude $\partial_xF_1(x,t)\leq\partial_xF_2(x,t)$ for all $x\geq 0$. This finishes the proof of (2.2). □

Next we turn to the proof of (2.3) in Proposition 2.1 that relies on the following:

Lemma 2.2 (Maximum principle)**.** *Let F be a twice differentiable function defined on $\mathbb{R}\times[0,1]$ and satisfy the statement:*

$$\begin{aligned}&\text{whenever there is some } (x,t) \text{ satisfying } \partial_{xx}F(x,t)\leq 0,\\&\partial_xF(x,t)=0 \text{ and } F(x,t)\geq 0, \text{ then } \partial_tF(x,t)\leq 0.\end{aligned}\tag{2.12}$$

If

$$\limsup_{|x|\to\infty}\sup_{0\leq t\leq 1}F(x,t)\leq 0,\tag{2.13}$$

$$F(\cdot,0)\leq 0,\tag{2.14}$$

then $F(\cdot,t)\leq 0$ *for all* $0\leq t\leq 1$.

Proof. For an arbitrary $\varepsilon>0$, set $F_\varepsilon(x,t)=F(x,t)-\varepsilon t$ for $(x,t)\in\mathbb{R}\times[0,1]$. We claim that $F_\varepsilon\leq 0$ on $\mathbb{R}\times[0,1]$. Assume on the contrary that $F_\varepsilon>0$ at (x_0,t_0). From (2.13), there is some $M>0$ such that $F_\varepsilon(x,t)<F_\varepsilon(x_0,t_0)$ for all $(x,t)\in[-M,M]^c\times[0,1]$. So there exists some (x_1,t_1) that realizes the maximum of F_ε over $\mathbb{R}\times[0,1]$. Note that from (2.14), $t_1>0$. At (x_1,t_1), one sees

$$\begin{aligned} F-\varepsilon t_1&>0,\\ \partial_{xx}F&=\partial_{xx}F_\varepsilon\leq 0,\\ \partial_x F&=\partial_x F_\varepsilon=0,\\ \partial_t F-\varepsilon&=\partial_t F_\varepsilon\geq 0.\end{aligned}$$

The first three lines and the statement (2.12) give $\partial_t F(x_1,t_1)\leq 0$. However, the last line reads $\partial_t F(x_1,t_1)\geq\varepsilon$, a contradiction. This completes the proof of our claim and consequently $F(x,t)\leq\varepsilon t$ for all $(x,t)\in\mathbb{R}\times[0,1]$ and $\varepsilon>0$. Letting ε tend to zero finishes our proof. □

Recall F_0,F_1,F_2 from the first part of the proof of Proposition 2.1. As one shall see, we will define $F=F_0-\alpha F_1-(1-\alpha)F_2$ and use Lemma 2.2 to show $F\leq 0$. For technical purposes, we will need two lemmas to simplify our argument. Since their proofs are seemingly independent of our main goal, we will postpone them until the appendix.

Lemma 2.3. *Suppose that* $(\phi_1,\phi_2)\in\mathcal{F}_i$. *Then there exists* $\{(\phi_{1,r},\phi_{2,r})\}_{r\geq 1}\subseteq\mathcal{F}_i$ *such that for* $l=1,2$, $\phi_{l,r}$ *is linear on* $(-\infty,-M_r]\cup[M_r,\infty)$ *for some* $M_r>0$, $\phi_{l,r}\leq\phi_l$ *and* $\phi_{l,r}\to\phi_l$ *pointwise.*

Lemma 2.4. *Let* $m,M\geq 0$. *Suppose that* ϕ *is a continuous function on* $\mathbb{R}$ *with* $\phi(x)=Ax+B$ *for* $x\geq M$ *and* $\phi(x)=A'x+B'$ *for* $x\leq -M$, *where* A,A',B,B' *are constants. Then we have*

$$(\mathbb{E}\exp m\phi(x+\sqrt{t}z))^{\frac{1}{m}}=\begin{cases} O(x,t)\exp\big(Ax+B+\frac{A^2mt}{2}\big),\\[2ex] O'(x,t)\exp\big(A'x+B'+\frac{(A')^2mt}{2}\big),\end{cases}\tag{2.15}$$

where $\lim_{x\to\infty}O(x,t)=1$ *and* $\lim_{x\to-\infty}O'(x,t)=1$ *uniformly over* $0\leq t\leq 1$.

Proof of (2.3) *in Proposition* 2.1. From Lemma 2.3, there exists $(\phi_{1,r},\phi_{2,r})\in\mathcal{F}_i$ such that $\phi_{l,r}$ is linear on $(-\infty,-M_r]\cup[M_r,\infty)$, $\phi_{l,r}\leq\phi_l$ and $\phi_{l,r}\to\phi_l$ pointwise. If we could show that (2.3) holds for all $(\phi_{1,r},\phi_{2,r})$, the dominated convergence theorem implies that (2.3) is also valid for (ϕ_1,ϕ_2). Thus, we may assume without loss of generality that ϕ_1,ϕ_2 are linear on $(-\infty,-M]\cup[M,\infty)$ for some $M>0$.

Define $F=F_0-\alpha F_1-(1-\alpha)F_2$. Our goal is to show that $F\leq 0$ via Lemma 2.2. In order to do so, we now check that the conditions (2.13) and (2.14) are satisfied and the statement (2.12) holds true as follows. First, (2.14) follows immediately from the definitions of F_0,F_1,F_2. Next, we proceed to check (2.13). We show that $\limsup_{x\to\infty}\sup_{0\leq t\leq 1}F(x,t)\leq 0$ first. From the linearity of ϕ_l on $[M,\infty)$, write $\phi_l(x)=A_lx+B_l$ for all $x\geq M$ and some $A_l,B_l\in\mathbb{R}$. Note that no matter $(\phi_1,\phi_2)\in\mathcal{F}_1$ or $\mathcal{F}_2$, we always have $\phi_2'\geq\phi_1'\geq 0$ on $[0,\infty)$. Thus, $A_2\geq A_1\geq 0$.

From Lemma 2.4,

$$\exp F_0(x,t) = O(x,t)\exp\bigg(\alpha(A_1x+B_1)+(1-\alpha)(A_2x+B_1) + \frac{(\alpha A_1+(1-\alpha)A_2)^2nt}{2}\bigg),$$

$$\exp F_l(x,t) = O_l(x,t)\exp\bigg(A_lx+B_l+\frac{A_l^2m_lt}{2}\bigg),$$

where O, O_1, O_2 converge to 1 uniformly over $t\in[0,1]$ as x tends to infinity. So

$$\exp F(x,t) = \frac{O(x,t)}{O_1(x,t)^{\alpha}O_2(x,t)^{1-\alpha}} \cdot \exp\frac{1}{2}(n(\alpha A_1+(1-\alpha)A_2)^2-\alpha m_1A_1^2-(1-\alpha)m_2A_2^2). \tag{2.16}$$

Here the exponent on the right-hand side can be factorized as

$$\frac{1}{2}(A_1-A_2)(c_1A_1-c_2A_2)=\frac{1}{2}(A_1-A_2)((c_1-c_2)A_1+c_2(A_1-A_2)), \tag{2.17}$$

where $c_1:=\alpha(n\alpha-m_1)$ and $c_2:=(1-\alpha)(n(1-\alpha)-m_2)$. Observe that $c_1-c_2=2\alpha(1-\alpha)(m_2-m_1)$ and that $c_2\geq 0$ if and only if $m_1/m_2\geq(2-\alpha)/(1-\alpha)$. Since $m_2\geq m_1$ and $0\leq\alpha\leq 1$, one sees that $c_1\geq c_2$ and $c_2<0$. From the right-hand side of (2.17), these combined with $0\leq A_1\leq A_2$ give that (2.17) ≤ 0 and so $\limsup_{x\to\infty}\sup_{0\leq t\leq 1}F(x,t)\leq 0$. Next, we check $\limsup_{x\to-\infty}\sup_{0\leq t\leq 1}F(x,t)\leq 0$. If $(\phi_1,\phi_2)\in\mathcal{F}_2$, this follows immediately by the symmetry of F in x. If $(\phi_1,\phi_2)\in\mathcal{F}_1$, one may write $\phi_l(x)=A_l'x+B_l'$ for $x\leq -M$ and argue exactly in the same way as above to obtain (2.16) with new parameters A_l' and B_l'. In such case, note that again we have $0\leq A_1'\leq A_2'$ since $0\leq\phi_1'\leq\phi_2'$ on $(-\infty,-M]$. As a result, (2.17) ≤ 0 and $\limsup_{x\to-\infty}\sup_{0\leq t\leq 1}F(x,t)\leq 0$. Thus, (2.13) holds.

Finally, we claim that the statement (2.12) holds. Assume $\partial_{xx}F\leq 0$, $\partial_xF=0$ and $F\geq 0$ at some (x_0,t_0). Using (2.2) and $\partial_xF(x_0,t_0)=0$, we have that at (x_0,t_0), $\partial_tF=\Delta_1/2+\Delta_2/2$, where $\Delta_1:=\partial_{xx}F$ and

$$\Delta_2:=n\left(\alpha\partial_xF_1+(1-\alpha)\partial_xF_2\right)^2-\alpha m_1(\partial_xF_1)^2-(1-\alpha)m_2(\partial_xF_2)^2.$$

Note that by assumption, $\Delta_1\leq 0$. As for Δ_2, it can be factorized as

$$\begin{aligned}\Delta_2&=\left(\partial_xF_1-\partial_xF_2\right)\left(c_1\partial_xF_1-c_2\partial_xF_2\right)\\&=\left(\partial_xF_1-\partial_xF_2\right)\left((c_1-c_2)\partial_xF_1+c_2(\partial_xF_1-\partial_xF_2)\right),\end{aligned} \tag{2.18}$$

where c_1,c_2 are defined in (2.17). Note that c_1,c_2 satisfy $c_2\leq c_1$ and $c_2<0$. Also from (2.2), we have that $0\leq\partial_xF_1\leq\partial_xF_2$ if $(\phi_1,\phi_2)\in\mathcal{F}_1$ and that $0\leq\partial_xF_1\leq\partial_xF_2$ for $x\geq 0$ and $\partial_xF_2\leq\partial_xF_1\leq 0$ for $x\leq 0$ if $(\phi_1,\phi_2)\in\mathcal{F}_2$. Thus, from the right-hand side of (2.18), combining these together yields $\Delta_2\leq 0$ and then $\partial_tF(x_0,t_0)\leq 0$, which means that the statement (2.12) is satisfied. This completes our proof. □

Appendix

Proof of Lemma 2.3. For every $r\geq 1$, let T_r be the smallest integer such that

$$\max\{\max\{|\phi_1'(x)|:x\in[-r,r]\},\max\{|\phi_2'(x)|:x\in[-r,r]\}\}\leq T_r. \tag{A.1}$$

Set $q_{r,p} = p(rT_r)^{-1}$ for $-r^2T_r \le p \le r^2T_r$. In other words, $q_{r,p}$'s form a regular partition of $[-r,r]$ with $2r^2T_r+1$ points. Let $s_{l,r}$ be a continuous piecewise linear function on $\mathbb{R}$ defined as

(A.2)
$$s_{l,r}(x) = \begin{cases} \phi_l(r) - \frac{2}{r} + \phi_l'(r)(x-r), & \text{if } x \ge r, \\ \phi_l(q_{r,p}) - \frac{2}{r} + \frac{\phi_l(q_{r,p+1})-\phi_l(q_{r,p})}{q_{r,p+1}-q_p}(x-q_{r,p}), & \text{if } q_{r,p} < x \le q_{r,p+1}, \\ \phi_l(-r) - \frac{2}{r} + \phi_l'(-r)(x+r), & \text{if } x \le -r. \end{cases}$$

It is easy to check that $s_{l,r}$ is convex and $s_{1,r} \le s_{2,r}$. In addition, if $(\phi_1,\phi_2) \in \mathcal{F}_1$, then $s_{1,r}, s_{2,r}$ are nondecreasing and $s_{1,r}' \le s_{2,r}'$ on $\mathbb{R}$ except at $q_{r,p}$'s; if $(\phi_1,\phi_2) \in \mathcal{F}_2$, then $s_{1,r}, s_{2,r}$ are even and $s_{1,r}' \le s_{2,r}'$ on $[0,\infty)$ except at $q_{r,p}$'s. Note that since the first and third equations on the right-hand side of (A.2) are the supporting lines of $\phi_l - 2/r$ at r and $-r$, respectively, it follows that $\phi_l - 2/r \ge s_{l,r}$ for $|x| \ge r$. On the other hand, if $x \in (q_{r,p}, q_{r,p+1}]$ for some $-r^2T_r \le p \le r^2T_r - 1$, then by the convexity of ϕ_l,

(A.3)
$$\begin{aligned} \phi_l(x) - s_{l,r}(x) &= \phi_l(x) - \left(\phi_l(q_{r,p}) - \frac{2}{r} + \frac{\phi_l(q_{r,p+1}) - \phi_l(q_{r,p})}{q_{r,p+1} - q_{r,p}}(x - q_{r,p})\right) \\ &= (x - q_{r,p})\left(\frac{\phi_l(x) - \phi_l(q_{r,p})}{x - q_{r,p}} - \frac{\phi_l(q_{r,p+1}) - \phi_l(q_{r,p})}{q_{r,p+1} - q_{r,p}}\right) + \frac{2}{r} \\ &\ge (x - q_{r,p})(\phi_l'(q_{r,p}) - \phi_l'(q_{r,p+1})) + \frac{2}{r} \\ &\ge -(q_{r,p+1} - q_{r,p})T_r + \frac{2}{r} \\ &\ge \frac{1}{r}. \end{aligned}$$

To sum up, $\phi_l \ge s_{l,r}$. Now, we pick a symmetric mollifier function η on $\mathbb{R}$ that is positive on $(-1,1)$ and supported on $[-1,1]$. Define $\eta_r(x) = \varepsilon_r^{-1}\eta(x/\varepsilon_r)$ for $\varepsilon_r := (2rT_r)^{-1}$. We set

(A.4)
$$\phi_{l,r}(x) := \int \eta_r(x-u)s_{l,r}(u)du = \int \eta_r(u)s_{l,r}(x-u)du.$$

With this definition, one may see that $\phi_{l,r}$'s are twice differentiable functions and clearly they satisfy (1.4).

Now, we check that $(\phi_{1,r}, \phi_{2,r}) \in \mathcal{F}_i$. Let us consider $i=1$ first. Since $s_{1,r}$ and $s_{2,r}$ are convex, nondecreasing and $s_{1,r} \le s_{2,r}$, using the equation on the right-hand side of (A.4), it is easy to see that these properties are also preserved for $\phi_{1,r}, \phi_{2,r}$. Note that from our construction of $s_{l,r}$, $|s_{l,r}(x) - s_{l,r}(y)| \le T_r|x-y|$. This together with (A.3) and $\mathrm{supp}(\eta_r) = [-\varepsilon_r, \varepsilon_r]$ yields

$$\begin{aligned}\phi_l(x) - \phi_{l,r}(x) &= \phi_l(x) - s_{l,r}(x) + s_{l,r}(x) - \phi_{l,r}(x) \\ &= \phi_l(x) - s_{l,r}(x) + \int \eta_r(u)(s_{l,r}(x) - s_{l,r}(x-u))du \\ &\geq \frac{1}{r} - T_r \int \eta_r(u)|u|du \\ &\geq \frac{1}{r} - T_r\varepsilon_r \int \eta_r(u)du \\ &\geq \frac{1}{2r}.\end{aligned} \tag{A.5}$$

So $\phi_l \geq \phi_{l,r}$. Finally, it remains to check that $\phi'_{1,r} \leq \phi'_{2,r}$. To see this, observe that $s_{l,r}$ is differentiable everywhere except at $q_{r,p}$'s and is Lipschitz on $\mathbb{R}$. From the dominated convergence theorem,

$$\phi'_{1,r}(x) = \int \eta_r(u)s'_{1,r}(x-u)du \leq \int \eta_r(u)s'_{2,r}(x-u)du = \phi'_{2,r}(x) \tag{A.6}$$

for every $x \in \mathbb{R}$. So $\phi'_{1,r} \leq \phi'_{2,r}$ on $\mathbb{R}$. This completes our proof of $(\phi_{1,r}, \phi_{2,r}) \in \mathcal{F}_1$. As for the case that $(\phi_{1,r}, \phi_{2,r}) \in \mathcal{F}_2$ by assuming $(\phi_1, \phi_2) \in \mathcal{F}_2$, to show that $\phi_{1,r}, \phi_{2,r}$ are convex and $\phi_{1,r} \leq \phi_{2,r}$, it can be treated essentially in the same way as above. Moreover, using the additional assumption that η is symmetric gives that $\phi_{1,r}, \phi_{2,r}$ are even. As for $\phi'_{1,r} \leq \phi'_{2,r}$ on $[0, \infty)$, note that $s'_{2,r} \geq s'_{1,r}$ on $[0, \infty)$ and η_r is supported on $[-\varepsilon_r, \varepsilon_r]$. If $x \geq \varepsilon_r$, then we also have (A.6); if $x \in [0, \varepsilon_r)$, noting that $\varepsilon_r \leq q_{r,1}$ and $s_{r,l}(u) = A_l|u| + B_l$ with $A_l = (\phi_l(q_{r,1}) - \phi_l(q_{r,0}))(q_{r,1} - q_{r,0})^{-1}$ and $B_l = \phi_l(0)$, we compute

$$\begin{aligned}\phi'_{l,r}(x) &= \int \eta_r(u)s'_{l,r}(x-u)du \\ &= A_l\left(\int_{-\varepsilon_r}^{x} \eta_r(u)du - \int_x^{\varepsilon_r} \eta_r(u)du\right) \\ &= A_l\left(1 - 2\int_x^{\varepsilon_r} \eta_r(u)du\right).\end{aligned}$$

Since $\int_x^{\varepsilon_r} \eta_r(u)du \leq 1/2$ and $A_1 \leq A_2$, they imply $\phi'_{1,r}(x) \leq \phi'_{2,r}(x)$. Hence, $\phi'_{1,r} \leq \phi'_{2,r}$ on $[0, \infty)$. This completes the proof for the second case.

Next, we show that $\phi_{l,r}$ is linear on $(-\infty, -M_r] \cup [M_r, \infty)$ for some $M_r > 0$, $\phi_{l,r} \leq \phi_l$ and $\phi_{l,r} \to \phi_l$. From (A.5), $\phi_{l,r} \leq \phi_l$ holds. Also, from the second equality of (A.5), one can further see that

$$\begin{aligned}|\phi_l(x) - \phi_{l,r}(x)| &\leq \phi_l(x) - s_{l,r}(x) + J_{l,r} \int \eta_r(u)|u|du \\ &\leq \phi_l(x) - s_{l,r}(x) + \frac{1}{2r}.\end{aligned}$$

Since $s_{l,r} \uparrow \phi_l$, it follows that $\phi_{l,r} \to \phi_l$. Finally, recalling the definition of $s_{l,r}$ from (A.2), if we pick $M_r = r + \varepsilon_r$, then

$$
\begin{aligned}
&\phi_{l,r}(x)\\
&= \int \eta_r(u) s_{l,r}(x-u)du\\
&= \begin{cases} \big(\phi_l(r) - \frac{2}{r} - \phi_l'(r)\int \eta_r(u)udu\big) + \big(\phi_l'(r)\int \eta_r(u)du\big)x, & \text{if } x \geq M_r,\\[2ex] \big(\phi_l(-n) - \frac{2}{r} - \phi_l'(-r)\int \eta_r(u)udu\big) + \big(\phi_l'(-r)\int \eta_r(u)du\big)x, & \text{if } x \leq -M_r, \end{cases}
\end{aligned}
$$

which means that $\phi_{l,r}$ is linear on $(-\infty, -M_r] \cup [M_r, \infty)$. This completes our proof. □

Proof of Lemma 2.4. We will only show the first equality of the right-hand side of (2.15). For the second, it can be derived exactly in the same way. For notational convenience, we will use C to denote a positive constant independent of x, t. Let us emphasize that it might be different for each occurrence. Denote $z_{x,t} = x + \sqrt{t}z$. From the given assumption, we write

$$\mathbb{E} \exp m\phi(z_{x,t}) = I_1(x,t) + I_2(x,t) + I_3(x,t), \tag{A.7}$$

where

$$
\begin{aligned}
&I_1(x,t) := \mathbb{E}\exp m(Az_{x,t} + B),\\
&I_2(x,t) := \mathbb{E}[\exp m\phi(z_{x,t}); |z_{x,t}| \leq M],\\
&I_3(x,t) := \mathbb{E}[\exp m(A'z_{x,t} + B'); z_{x,t} \leq -M] - \mathbb{E}[\exp m(Az_{x,t} + B); z_{x,t} \leq M].
\end{aligned}
$$

Since ϕ is continuous, $|I_2| \leq C\mathbb{P}(|z_{x,t}| \leq M)$. From the Cauchy-Schwartz inequality,

$$
\begin{aligned}
|I_3| &\leq (\mathbb{E}\exp 2m(A'z_{x,t} + B'))^{\frac{1}{2}} \mathbb{P}(z_{x,t} \leq -M)^{\frac{1}{2}}\\
&\quad + (\mathbb{E}\exp 2m(Az_{x,t} + B))^{\frac{1}{2}} \mathbb{P}(z_{x,t} \leq M)^{\frac{1}{2}}\\
&= \exp m(A'x + B' + 2m(A')^2 t)\mathbb{P}(z_{x,t} \leq -M)^{\frac{1}{2}}\\
&\quad + \exp m(Ax + B + 2mA^2 t)\mathbb{P}(z_{x,t} \leq M)^{\frac{1}{2}}\\
&\leq C\exp(Cx)\mathbb{P}(z_{x,t} \leq M)^{\frac{1}{2}},
\end{aligned}
$$

where we used the fact that $\mathbb{E}\exp\beta z = \exp\beta^2/2$ for any $\beta \in \mathbb{R}$. Since $\mathbb{P}(z \geq L) \leq \exp(-L^2/2)$ whenever L is sufficiently large, we have

$$\mathbb{P}(z_{x,t} \leq M) = \mathbb{P}(x - M \leq \sqrt{t}z) \leq \exp\Big(-\frac{(x-M)^2}{2t}\Big) \leq C\exp\Big(-\frac{x^2}{Ct}\Big) \tag{A.8}$$

when x is sufficiently large. Therefore, $|I_2| + |I_3| \leq C\exp(-x^2/Ct)$. Finally, noting that $I_1 = \exp m(Ax + B + A^2 mt/2)$ gives

$$(|I_2| + |I_3|)/I_1 \leq C\exp(-x^2/Ct + m(Ax + B) + A^2m^2t/2).$$

Since the exponent is dominated by x^2/t, we get again

$$(|I_2| + |I_3|)/I_1 \leq C\exp(-x^2/Ct).$$

Hence, if we define $O = (1 + (I_2 + I_3)/I_1)^{1/m}$, then O converges to 1 uniformly over $0 \le t \le 1$ as x tends to infinity and

$$(\mathbb{E} \exp m\phi(z_{x,t}))^{1/m} = O(x,t) I_1(x,t)^{1/m} = O(x,t) \exp m(Ax + B + A^2 mt/2).$$

This gives the announced result. □

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