

## THE OPTIMAL FOURTH MOMENT THEOREM

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ABSTRACT. We compute the exact rates of convergence in total variation associated with the ‘fourth moment theorem’ by Nualart and Peccati (2005), stating that a sequence of random variables living in a fixed Wiener chaos verifies a central limit theorem (CLT) if and only if the sequence of the corresponding fourth cumulants converges to zero. We also provide an explicit illustration based on the Breuer-Major CLT for Gaussian-subordinated random sequences.

### 1. INTRODUCTION AND MAIN RESULT

On a suitable probability space  $(\Omega, \mathcal{F}, P)$ , let  $X = \{X(h) : h \in \mathfrak{H}\}$  be an isonormal Gaussian process over a real separable Hilbert space  $\mathfrak{H}$ , and let  $\{C_q : q = 0, 1, \dots\}$  be the sequence of Wiener chaoses associated with  $X$  (see Section 2.4 for details). The aim of the present work is to prove an optimal version of the following quantitative ‘fourth moment’ central limit theorem (CLT) which combines results from [12] and [20].

**Theorem 1.1** (Fourth moment theorem [12, 20]). *Fix an integer  $q \geq 2$  and let  $N \sim \mathcal{N}(0, 1)$  be a standard Gaussian random variable. Consider a sequence  $\{F_n : n \geq 1\}$  living in the  $q$ th Wiener chaos  $C_q$  of  $X$ , and assume that  $E[F_n^2] = 1$ . Then,  $F_n$  converges in distribution to  $N \sim \mathcal{N}(0, 1)$  if and only if  $E[F_n^4] \rightarrow 3 = E[N^4]$ . Also, the following upper bound on the total variation distance holds for every  $n$ :*

$$(1.1) \quad d_{TV}(F_n, N) \leq \sqrt{\frac{4q-4}{3q}} \sqrt{|E[F_n^4] - 3|}.$$

Recall that, given two real-valued random variables  $Y, Z$ , the *total variation distance* between the law of  $Y$  and  $Z$  is given by

$$(1.2) \quad d_{TV}(Y, Z) = \sup_{A \in \mathcal{B}(\mathbb{R})} |P[Y \in A] - P[Z \in A]|$$

$$(1.3) \quad = \frac{1}{2} \sup |E[g(Y)] - E[g(Z)]|,$$

where, in view e.g. of Lusin’s Theorem, the supremum is taken over the class of continuously differentiable functions  $g$  that are bounded by 1 and have compact

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support. Also, if the distributions of  $Y, Z$  have densities (say  $f_Y, f_Z$ ), then one has the integral representation

$$(1.4) \quad d_{TV}(Y, Z) = \frac{1}{2} \int_{\mathbb{R}} |f_Y(t) - f_Z(t)| dt.$$

It is a well-known fact that the topology induced by  $d_{TV}$  over the class of all probabilities on  $\mathbb{R}$  is strictly stronger than the topology of convergence in distribution (see e.g. [7]). It is also important to notice that bounds analogous to (1.1) hold for other distances, like, for instance, the Kolmogorov and 1-Wasserstein distances (see [15, Chapter 5]).

The first part of Theorem 1.1 was first proved in [20] by using tools of continuous-time stochastic calculus. The upper bound (1.1) comes from [12] and is obtained by combining the *Malliavin calculus of variations* (see e.g. [15, 19]) with *Stein's method* for normal approximations (see [5, 15]). The content of Theorem 1.1 has sparked a great amount of generalisations and applications, ranging from density estimates [18] to entropic CLTs [17], and from estimates for Gaussian polymers [21] to universality results [16]. The reader is referred to the monograph [15] for a self-contained introduction to the theoretical aspects of this direction of research, as well as to [8, 10] for applications, respectively, to the high-frequency analysis of fields defined on homogeneous spaces and to the power variation of stochastic processes related to fractional Brownian motion. One can also consult the constantly updated webpage

<http://sites.google.com/site/malliavinstein>

for literally hundreds of results related to Theorem 1.1 and its ramifications.

One challenging question is whether the upper bound (on the rate of convergence in total variation) provided by (1.1) is optimal or whether it can be ameliorated in some specific situations. Recall that a positive sequence  $\{\phi(n) : n \geq 1\}$  decreasing to zero yields an *optimal rate of convergence*, with respect to some suitable distance  $d(\cdot, \cdot)$ , if there exist two finite constants  $0 < c < C$  (independent of  $n$ ) such that

$$c \phi(n) \leq d(F_n, N) \leq C \phi(n), \quad \text{for all } n \geq 1.$$

There are indeed very few references studying optimal rates of convergence for CLTs on a Gaussian space. Our paper [13] provides some partial characterisation of optimal rates of convergence in the case where  $d = d_{Kol}$  (the Kolmogorov distance), whereas [4] contains similar findings for multidimensional CLTs and distances based on smooth mappings. Of particular interest for the present analysis is the work [2] that we cowrote with H. Biermé and A. Bonami, where it is proved that whenever the distance  $d$  is defined as the supremum over a class of *smooth enough* test functions (e.g. twice differentiable with a bounded second derivative), an optimal rate of convergence in Theorem 1.1 is given by the sequence

$$(1.5) \quad \mathbf{M}(F_n) := \max \{ |E[F_n^3]|, |E[F_n^4] - 3| \}, \quad n \geq 1.$$

In particular, if  $E[F_n^3] = 0$  (for instance, if  $q$  is odd), then the rate suggested by the estimate (1.1) is suboptimal by a whole square root factor. We observe that if  $F$  is a non-zero element of the  $q$ th Wiener chaos of  $X$ , for some  $q \geq 2$ , then  $E[F^4] > 3E[F^2]^2$ ; see e.g. [20] or [15, Chapter 5].

The statement of the subsequent Theorem 1.2, which is the main achievement of the present paper, provides a definitive characterisation of the optimal rate of convergence in the total variation distance for the fourth moment CLT appearing in

Theorem 1.1. It was somehow unexpected that our optimal rate relies only on the simple quantity (1.5) and also that we do not need to impose a further restriction other than being an element of a given Wiener chaos. One remarkable consequence of our findings is that this optimal rate exactly coincides with the one related to the smooth test functions considered in [2]. We will see that the proof relies on estimates taken from the paper [2], which we combine with a new fine analysis of the main upper bound proved in [12] (see Proposition 2.4).

**Theorem 1.2** (Optimal fourth moment theorem in total variation). *Fix  $q \geq 2$ . Let  $\{F_n : n \geq 1\}$  be a sequence of random variables living in the  $q$ th Wiener chaos of  $X$ , such that  $E[F_n^2] = 1$ . Then,  $F_n$  converges in distribution to  $N \sim \mathcal{N}(0, 1)$  if and only if  $E[F_n^4] \rightarrow 3 = E[N^4]$ . In this case, one has also that  $E[F_n^3] \rightarrow 0$  and there exist two finite constants  $0 < c < C$  (possibly depending on  $q$  and on the sequence  $\{F_n\}$ , but not on  $n$ ) such that the following estimate in total variation holds for every  $n$ :*

$$(1.6) \quad c\mathbf{M}(F_n) \leq d_{TV}(F_n, N) \leq C\mathbf{M}(F_n),$$

where the quantity  $\mathbf{M}(F_n)$  is defined according to (1.5).

*Remark 1.3.* It is important to point out the main differences between the techniques used in the proof of Theorem 1.2 and those adopted in [2], where we obtained a bound analogous to (1.6) for the quantity

$$d(F_n, N) = \sup_{h \in \mathcal{U}} |E[h(F_n)] - E[h(N)]|,$$

with  $\mathcal{U}$  indicating the collection of all test functions  $h$  that are twice continuously differentiable and such that  $\|h''\|_\infty \leq 1$ .

- (i) Concerning upper bounds, the crucial element both in [2] and in the present paper is to exploit the estimates on the iterated Gamma operators of Malliavin calculus, as in Proposition 2.5. In [2], this task could be accomplished by using the regularity of the solution to Stein's equation associated with any  $h \in \mathcal{U}$  in such a way that one could apply several times the integration by parts formulae of Malliavin calculus to the corresponding Stein-type bounds. One of the main technical advances introduced in the present paper is that, by a careful use of Lusin's Theorem, one can directly deal with very irregular test functions (such as the ones appearing in the definition of the total variation distance) without reducing the speed of convergence. As a general remark, it is usually acknowledged that, within the framework of probabilistic approximations, bounds in non-smooth distances are much harder to obtain, so that Theorem 1.2 can be regarded as a genuine breakthrough. In particular, we could not find in the literature any example of a CLT with explicit optimal rates of convergence in the total variation distance.
- (ii) The proofs of the lower bounds in [2] and the present paper are very similar, with only one noticeable difference: owing to the definition of the class  $\mathcal{U}$  introduced above, in [2] we were allowed to compute lower bounds by considering test functions with the form  $h(x) = t(x) + p(x)$ , where  $t$  is a trigonometric function and  $p$  is a polynomial of degree 2 (see e.g. [2, Lemma 5.1]). When working with the total variation distance, such a choice of  $h$  is of course not possible (since  $h$  is not necessarily bounded), so we

will have to adapt the arguments used in [2] in order to directly deal with trigonometric test functions.

The rest of the paper is organised as follows. Section 2 contains some notation and useful preliminaries. Our main result, Theorem 1.2, is proved in Section 3. Finally, in Section 4 we provide an explicit illustration based on the Breuer-Major CLT for Gaussian-subordinated random sequences.

2. NOTATION AND PRELIMINARIES

**2.1. Cumulants.** In what follows, the notion of *cumulant* is sometimes used. Recall that, given a random variable  $Y$  with finite moments of all orders and with characteristic function  $\psi_Y(t) = E[\exp(itY)]$  ( $t \in \mathbb{R}$ ), one defines the sequence of cumulants (sometimes known as *semi-invariants*) of  $Y$ , denoted  $\{\kappa_n(Y) : n \geq 1\}$ , as

$$(2.7) \quad \kappa_n(Y) = (-i)^n \frac{d^n}{dt^n} \log \psi_Y(t) |_{t=0}, \quad n \geq 1.$$

For instance,  $\kappa_1(Y) = E[Y]$ ,  $\kappa_2(Y) = E[(Y - E[Y])^2] = \text{Var}(Y)$ , and if  $E[Y] = 0$ ,

$$\kappa_3(Y) = E[Y^3] \quad \text{and} \quad \kappa_4(Y) = E[Y^4] - 3E[Y^2]^2.$$

**2.2. Hermite polynomials.** We will also denote by  $\{H_q : q = 0, 1, \dots\}$  the sequence of *Hermite polynomials* given by the recursive relation  $H_0 = 1$  and  $H_{q+1}(x) = xH_q(x) - H'_q(x)$  in such a way that  $H_1(x) = x$ ,  $H_2(x) = x^2 - 1$ ,  $H_3(x) = x^3 - 3x$  and  $H_4(x) = x^4 - 6x^2 + 3$ . We recall that Hermite polynomials constitute a complete orthogonal system of the space  $L^2(\mathbb{R}, \mathcal{B}(\mathbb{R}), (2\pi)^{-1/2}e^{-x^2/2}dx)$ .

**2.3. Stein's equations.** See [15, Chapter 3] for more details on the content of this section. Let  $N \sim \mathcal{N}(0, 1)$ . Given a bounded and continuous function  $g : \mathbb{R} \rightarrow \mathbb{R}$ , we define the *Stein's equation* associated with  $g$  to be the ordinary differential equation

$$(2.8) \quad f'(x) - xf(x) = g(x) - E[g(N)].$$

It is easily checked that every solution to (2.8) has the form  $ce^{x^2/2} + f_g(x)$ , where

$$(2.9) \quad f_g(x) = e^{x^2/2} \int_{-\infty}^x \{g(y) - E[g(N)]\} e^{-y^2/2} dy, \quad x \in \mathbb{R}.$$

Some relevant properties of  $f_g$  appear in the next statement (see e.g. [15, Section 3.3] for proofs).

**Proposition 2.1** (Stein's bounds). *Assume  $g : \mathbb{R} \rightarrow \mathbb{R}$  is continuous and bounded. Then  $f_g$  given by (2.9) is  $C^1$  and satisfies*

$$\|f_g\|_\infty \leq \sqrt{\pi/2} \|g - E[g(N)]\|_\infty \quad \text{and} \quad \|f'_g\|_\infty \leq 2 \|g - E[g(N)]\|_\infty.$$

**2.4. The language of Gaussian analysis and Malliavin calculus.** We now briefly recall some basic notation and results connected to Gaussian analysis and Malliavin calculus. The reader is referred to [15, 19] for details.

Let  $\mathfrak{H}$  be a real separable Hilbert space with inner product  $\langle \cdot, \cdot \rangle_{\mathfrak{H}}$ . Recall that an *isonormal Gaussian process* over  $\mathfrak{H}$  is a centered Gaussian family  $X = \{X(h) : h \in \mathfrak{H}\}$ , defined on an adequate probability space  $(\Omega, \mathcal{F}, P)$  and such that  $E[X(h)X(h')] = \langle h, h' \rangle_{\mathfrak{H}}$ , for every  $h, h' \in \mathfrak{H}$ . For every  $q = 0, 1, 2, \dots$ , we denote by  $C_q$  the  $q$ th *Wiener chaos* of  $X$ . We recall that  $C_0 = \mathbb{R}$  and, for  $q \geq 1$ ,  $C_q$  is the  $L^2$ -closed space composed of those random variables having the form  $I_q(f)$

where  $I_q$  indicates a *multiple Wiener-Itô* integral of order  $q$  and  $f \in \mathfrak{H}^{\odot q}$  (the  $q$ th symmetric tensor power of  $\mathfrak{H}$ ). Recall that  $L^2(\sigma(X), P) := L^2(P) = \bigoplus_{q=0}^{\infty} C_q$ ; that is: every square-integrable random variable  $F$  that is measurable with respect to  $\sigma(X)$  (the  $\sigma$ -field generated by  $X$ ) admits a unique decomposition of the type

$$(2.10) \quad F = E[F] + \sum_{q=1}^{\infty} I_q(f_q),$$

where the series converges in  $L^2(P)$ , and  $f_q \in \mathfrak{H}^{\odot q}$ , for  $q \geq 1$ . This last result is known as the *Wiener-Itô chaotic decomposition* of  $L^2(P)$ . When the kernels  $f_q$  in (2.10) all equal zero except for a finite number, we say that  $F$  has a *finite chaotic expansion*. According to a classical result (discussed e.g. in [15, Section 2.10]) the distribution of non-zero random variables with a finite chaotic expansion has necessarily a density with respect to the Lebesgue measure.

We will use some standard operators from Malliavin calculus. The *Malliavin derivative*  $D$  has domain  $\mathbb{D}^{1,2} \subset L^2(P)$  and takes values in the space  $L^2(P; \mathfrak{H})$  of square-integrable  $\mathfrak{H}$ -valued random variables that are measurable with respect to  $\sigma(X)$ . The *divergence operator*  $\delta$  is defined as the adjoint of  $D$ . In particular, denoting by  $\text{dom } \delta$  the domain of  $\delta$ , one has the so-called *integration by parts formula*: for every  $D \in \mathbb{D}^{1,2}$  and every  $u \in \text{dom } \delta$ ,

$$(2.11) \quad E[F\delta(u)] = E[\langle DF, u \rangle_{\mathfrak{H}}].$$

We will also need the so-called *generator of the Ornstein-Uhlenbeck semigroup*, written  $L$ , which is defined by the relation

$$L = -\delta D,$$

meaning that  $F$  is in  $\text{dom } L$  (the domain of  $L$ ) if and only if  $F \in \mathbb{D}^{1,2}$  and  $DF \in \text{dom } \delta$ , and in this case  $LF = -\delta DF$ . The *pseudo-inverse* of  $L$  is denoted by  $L^{-1}$ . It is important to note that  $L$  and  $L^{-1}$  are completely determined by the following relations, valid for every  $c \in \mathbb{R}$ , every  $q \geq 1$  and every  $f \in \mathfrak{H}^{\odot q}$ :

$$Lc = L^{-1}c = 0, \quad LI_q(f) = -qI_q(f), \quad L^{-1}I_q(f) = -\frac{1}{q}I_q(f).$$

We will exploit the following *chain rule* (see e.g. [15, Section 2.3]): for every  $F \in \mathbb{D}^{1,2}$  and every mapping  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  which is continuously differentiable and with a bounded derivative, one has that  $\varphi(F) \in \mathbb{D}^{1,2}$ , and moreover

$$(2.12) \quad D\varphi(F) = \varphi'(F)DF.$$

We conclude this section by recalling the definition of the operators  $\Gamma_j$ , as first defined in [14]; see [15, Chapter 8] for a discussion of recent developments.

**Definition 2.2** (Gamma operators). Let  $F$  be a random variable having a finite chaotic expansion. The sequence of random variables  $\{\Gamma_j(F) : j \geq 0\}$  is recursively defined as follows. Set  $\Gamma_0(F) = F$  and, for every  $j \geq 1$ ,

$$\Gamma_j(F) = \langle DF, -DL^{-1}\Gamma_{j-1}(F) \rangle_{\mathfrak{H}}.$$

In view of the product formulae for multiple integrals (see e.g. [15, Theorem 2.7.10]), each  $\Gamma_j(F)$  is a well-defined random variable having itself a finite chaotic expansion.

Finally, for further use let us state the following result taken from [14, Theorem 4.3].

**Theorem 2.3** (Computation of cumulants in terms of Gamma operators). *Let  $F$  be a random variable having a finite chaotic expansion. Then, for any integer  $m \geq 0$ ,*

$$\kappa_{m+1}(F) = m!E[\Gamma_m(F)].$$

**2.5. Some useful bounds.** The next bound contains the main result of [12] in a slightly more general form (whose proof can be found in [9]) not requiring that the random variable  $F$  have a density.

**Proposition 2.4** (General total variation bound). *Let  $F$  be a centered element of  $\mathbb{D}^{1,2}$ , and let  $N \sim \mathcal{N}(0, 1)$ . Then,*

$$(2.13) \quad d_{TV}(F, N) \leq 2E |E[1 - \langle DF, -DL^{-1}F \rangle_{\mathfrak{H}}]F| \leq 2E |1 - \langle DF, -DL^{-1}F \rangle_{\mathfrak{H}}|.$$

*In particular, if  $F = I_q(f)$  belongs to the  $q$ th Wiener chaos  $C_q$  and  $F$  has unit variance, one has that  $\langle DF, -DL^{-1}F \rangle_{\mathfrak{H}} = \frac{1}{q}\|DF\|_{\mathfrak{H}}^2$ ,  $E[F^4] > 3$ , and the following inequality holds:*

$$(2.14) \quad E \left| 1 - \frac{1}{q}\|DF\|_{\mathfrak{H}}^2 \right| \leq \sqrt{E \left[ \left( 1 - \frac{1}{q}\|DF\|_{\mathfrak{H}}^2 \right)^2 \right]} \leq \sqrt{\frac{q-1}{3q}(E[F^4] - 3)}.$$

Of course, relation (2.14) is equivalent to (1.1). In particular, using the language of cumulants, the estimates in the previous statement yield that, for  $F = I_q(f)$  ( $q \geq 2$ ) with unit variance,

$$(2.15) \quad d_{TV}(F, N) \leq \frac{2}{\sqrt{3}}\sqrt{\kappa_4(F)}.$$

The following result (which is taken from [2]) provides some useful bounds on the Gamma operators introduced in Definition 2.2.

**Proposition 2.5** (Estimates on Gamma operators). *For each integer  $q \geq 2$  there exist positive constants  $c_0, c_1, c_2$  (depending only on  $q$ ) such that, for all  $F = I_q(f)$  with  $f \in \mathfrak{H}^{\odot q}$  and  $E[F^2] = 1$ , one has*

$$(2.16) \quad \begin{aligned} E \left[ \left( \Gamma_2(F) - \frac{1}{2}\kappa_3(F) \right)^2 \right]^{1/2} &\leq c_0 \kappa_4(F)^{\frac{3}{4}}; \\ E[|\Gamma_3(F)|] &\leq c_1 \kappa_4(F); \\ E[|\Gamma_4(F)|] &\leq c_2 \kappa_4(F)^{\frac{5}{4}}. \end{aligned}$$

*Proof.* See [2, Proposition 4.3]. □

### 3. PROOF OF THEOREM 1.2

Since  $E[F_n^2] = 1$ , the fact that  $F_n$  converges in distribution to  $N \sim \mathcal{N}(0, 1)$  if and only if  $E[F_n^4] \rightarrow 3$  is a direct consequence of the main result of [20]. Whenever  $E[F_n^4] \rightarrow 3$ , one has that the collection of random variables  $\{F_n^3 : n \geq 1\}$  is uniformly integrable in such a way that, necessarily,  $E[F_n^3] \rightarrow E[N^3] = 0$ . The upper and lower bounds appearing in formula (1.6) will be proved separately.

(Upper bound) We first establish some preliminary estimates concerning a general centered random variable  $F \in \mathbb{D}^{1,2}$ . We have

$$\begin{aligned} \frac{1}{2}d_{TV}(F, N) &\leq E\left[|E[1 - \langle DF, -DL^{-1}F \rangle_{\mathfrak{H}}|F]|\right] \\ &= E\left[E[1 - \langle DF, -DL^{-1}F \rangle_{\mathfrak{H}}|F] \times \text{sign}\{E[1 - \langle DF, -DL^{-1}F \rangle_{\mathfrak{H}}|F]\}\right] \\ &\leq \sup_{g \in C_c^1: \|g\|_\infty \leq 1} E[g(F)(1 - \langle DF, -DL^{-1}F \rangle_{\mathfrak{H}})], \end{aligned}$$

where  $C_c^1$  denotes the class of all continuously differentiable functions with compact support, and we have implicitly applied Lusin's Theorem. Fix a test function  $g \in C_c^1$  bounded by 1, and consider a random variable  $N \sim \mathcal{N}(0, 1)$  independent of  $F$ . By virtue of Proposition 2.1, writing  $\varphi = f_g$  for the solution of the Steins equation associated with  $g$  (see (2.9)), one has that  $\|\varphi\|_\infty \leq 2\sqrt{\frac{\pi}{2}}$  and  $\|\varphi'\|_\infty \leq 4$ . Exploiting independence together with the fact that  $E[1 - \langle DF, -DL^{-1}F \rangle_{\mathfrak{H}}] = 1 - \text{Var}(F) = 0$ , we deduce that

$$\begin{aligned} &E[g(F)(1 - \langle DF, -DL^{-1}F \rangle_{\mathfrak{H}})] \\ &= E[(g(F) - E[g(N)])(1 - \langle DF, -DL^{-1}F \rangle_{\mathfrak{H}})] \\ &= E[(\varphi'(F) - F\varphi(F))(1 - \langle DF, -DL^{-1}F \rangle_{\mathfrak{H}})] \\ &= E[\varphi'(F)(1 - \langle DF, -DL^{-1}F \rangle_{\mathfrak{H}})] - E[\varphi'(F)\langle DF, -DL^{-1}F \rangle_{\mathfrak{H}}(1 - \langle DF, -DL^{-1}F \rangle_{\mathfrak{H}})] \\ &\quad + E[\varphi(F)\Gamma_2(F)] \\ &= E[\varphi'(F)(1 - \langle DF, -DL^{-1}F \rangle_{\mathfrak{H}})^2] + E[\varphi(F)\Gamma_2(F)], \end{aligned}$$

where we have used several times the integration by parts formula (2.11). In order to properly assess the term  $E[\varphi(F)\Gamma_2(F)]$ , we shall consider the function  $\psi = f_\varphi$ , corresponding to the solution of the Steins equation associated with  $\varphi$ . Using again Proposition 2.1, we deduce the estimates  $\|\psi\|_\infty \leq \frac{8}{\pi}$  and  $\|\psi'\|_\infty \leq 16$ , and moreover

$$\begin{aligned} &E[\varphi(F)\Gamma_2(F)] - \frac{1}{2}E[\varphi(N)]\kappa_3(F) \\ &= E[(\varphi(F) - E[\varphi(N)])\Gamma_2(F)] = E[(\psi'(F) - F\psi(F))\Gamma_2(F)] \\ &= E[\psi'(F)\Gamma_2(F)] - E[\psi'(F)\langle DF, -DL^{-1}F \rangle_{\mathfrak{H}}\Gamma_2(F)] + E[\psi(F)\Gamma_3(F)] \\ &= E[\psi'(F)(1 - \langle DF, -DL^{-1}F \rangle_{\mathfrak{H}})\Gamma_2(F)] + E[\psi(F)\Gamma_3(F)], \end{aligned}$$

where we used Theorem 2.3 and once again independence and integration by parts. Combining the previous bounds, one infers that

$$\begin{aligned} d_{TV}(F, N) &\leq \pi|\kappa_3(F)| + 8E[(1 - \langle DF, -DL^{-1}F \rangle_{\mathfrak{H}})^2] \\ &\quad + 32\sqrt{E[(1 - \langle DF, -DL^{-1}F \rangle_{\mathfrak{H}})^2]}\sqrt{E[\Gamma_2(F)^2]} + \frac{16}{\pi}E[|\Gamma_3(F)|]. \end{aligned}$$

To conclude, let us consider a sequence  $F_n = I_q(f_n)$ ,  $n \geq 1$ , living in the  $q$ th Wiener chaos of  $X$  and such that each  $F_n$  has variance 1. Assume that  $F_n$  converges in distribution to  $N$ . Then, for  $n$  large enough one has that  $|\kappa_3(F_n)| \leq 1$  and  $\kappa_4(F_n) \leq 1$ . Using (2.14) and Proposition 2.5, we immediately deduce that, for some universal constant  $c_q > 0$  (depending only on  $q$ ),

$$\begin{aligned} d_{TV}(F_n, N) &\leq c_q \left( |\kappa_3(F_n)| + \kappa_4(F_n) + \kappa_4(F_n)^{\frac{5}{4}} + \sqrt{\kappa_4(F_n)}|\kappa_3(F_n)| \right) \\ &\leq C \max(|\kappa_3(F_n)|, \kappa_4(F_n)), \end{aligned}$$

where  $C$  is the constant appearing in the statement.

(Lower bound) According to the representation (1.3), the distance  $d_{TV}(F_n, N)$  is bounded from below by the quantity

$$(3.17) \quad \frac{1}{2} \max \{ |E[\cos(F_n)] - E[\cos(N)]|, |E[\sin(F_n)] - E[\sin(N)]| \}.$$

Combining [2, Corollary 3.12] with Proposition 2.5, one can write

$$\left| E[\sin(F_n)] - E[\sin(N)] - \frac{1}{2} E[f''_{\sin}(F_n)] \kappa_3(F_n) - \frac{1}{6} E[f'''_{\sin}(F_n)] \kappa_4(F_n) \right| \leq 2E|\Gamma_4(F_n)| \leq C \kappa_4(F_n)^{\frac{5}{4}}.$$

Here,  $C$  denotes a positive constant which is independent of  $n$  and whose value can change from line to line, whereas  $f_{\sin}$  stands for the solution of the Steins equation associated with the sine function, as given in (2.9). From [2, formula (5.2)], one has that  $E[f''_{\sin}(N)] = -\frac{1}{3} E[\sin(N)H_3(N)] = \frac{1}{\sqrt{e}}$ . Similarly,  $E[f'''_{\sin}(N)] = -\frac{1}{4} E[\sin(N)H_4(N)] = 0$ . Moreover, from [6, Theorem 1.1] one sees that  $f''_{\sin}$  and  $f'''_{\sin}$  are both bounded by 2. Finally, from (2.15) one has that  $d_{TV}(F_n, N) \leq C\sqrt{\kappa_4(F_n)}$ . Combining all these facts leads to

$$\begin{aligned} \frac{1}{4} |E[f''_{\sin}(F_n)] - E[f''_{\sin}(N)]| &\leq d_{TV}(F_n, N) \leq C\sqrt{\kappa_4(F_n)}; \\ \frac{1}{4} |E[f'''_{\sin}(F_n)] - E[f'''_{\sin}(N)]| &\leq d_{TV}(F_n, N) \leq C\sqrt{\kappa_4(F_n)} \end{aligned}$$

so that

$$\left| E[\sin(F_n)] - E[\sin(N)] - \frac{1}{2\sqrt{e}} \kappa_3(F_n) \right| \leq C \max \{ |\kappa_3(F_n)|, \kappa_4(F_n) \} \times \kappa_4(F_n)^{\frac{1}{4}}.$$

Similarly, one shows that

$$\left| E[\cos(F_n)] - E[\cos(N)] + \frac{1}{4\sqrt{e}} \kappa_4(F_n) \right| \leq C \max \{ |\kappa_3(F_n)|, \kappa_4(F_n) \} \times \kappa_4(F_n)^{\frac{1}{4}}.$$

As a consequence, exploiting the lower bound (3.17) we deduce that

$$d_{TV}(F_n, N) \geq \left( \frac{1}{4\sqrt{e}} - C\kappa_4(F_n)^{\frac{1}{4}} \right) \max (|\kappa_3(F_n)|, \kappa_4(F_n)),$$

so that the proof of the theorem is concluded.

#### 4. APPLICATION TO THE HERMITE VARIATIONS OF THE DISCRETE-TIME FRACTIONAL BROWNIAN MOTION

We now discuss an application of Theorem 1.2 to non-linear functionals of a fractional Brownian motion. Consider a fractional Brownian motion with Hurst index  $H \in (0, 1)$ . We recall that  $B_H = \{B_H(t) : t \in \mathbb{R}\}$  is a centered Gaussian process with continuous paths such that

$$E[B_H(t)B_H(s)] = \frac{1}{2} (|t|^{2H} + |s|^{2H} - |t - s|^{2H}), \quad s, t \in \mathbb{R}.$$

The process  $B_H$  is self-similar with stationary increments. We refer the reader to Nourdin [10] for a self-contained introduction to its main properties.

In this section, we shall rather work with the so-called *fractional Gaussian noise* associated with  $B_H$ , which is the Gaussian sequence given by

$$(4.18) \quad X_k = B_H(k + 1) - B_H(k), \quad k \in \mathbb{Z}.$$

Note that the family  $\{X_k : k \in \mathbb{Z}\}$  constitutes a centered stationary Gaussian family with covariance

$$\rho(k) = E[X_r X_{r+k}] = \frac{1}{2} (|k+1|^{2H} - 2|k|^{2H} + |k-1|^{2H}), \quad r, k \in \mathbb{Z}.$$

Also, it is readily checked that  $\rho(k)$  behaves asymptotically as

$$\rho(k) \sim H(2H - 1)|k|^{2H-2} \quad \text{as } |k| \rightarrow \infty.$$

Now fix an integer  $q \geq 2$ , consider the  $q$ th Hermite polynomial  $H_q$ , and set

$$F_n := \frac{1}{\sqrt{n} v_n} \sum_{k=0}^{n-1} H_q(X_k),$$

with  $v_n > 0$  chosen so that  $E[F_n^2] = 1$ . An important problem in modern Gaussian analysis is to characterise those values of  $H, q$  such that the sequence  $F_n$  verifies a CLT. The following statement is the celebrated *Breuer-Major CLT*, first proved in [3] (see [15, Chapter 7] for a modern proof and for an overview of its many ramifications).

**Theorem 4.1** (Breuer-Major CLT [3]). *Let the previous notation and assumptions prevail. Then, one has that*

$$(4.19) \quad F_n \xrightarrow{\text{Law}} N \sim \mathcal{N}(0, 1), \quad \text{as } n \rightarrow \infty,$$

*if and only if  $H$  belongs to the interval  $(0, 1 - \frac{1}{2q}]$ .*

Deducing explicit estimates on the speed of convergence in the CLT (4.19) is a difficult problem that has generated a large amount of research (see [10, 15] for an overview of the available literature, as well as [1, 2, 11] for recent developments). Using our Theorem 1.2 together with the forthcoming Proposition 4.2 allows one to deduce exact rates of convergence in total variation for every value of  $q, H$  such that  $H \in (0, 1 - \frac{1}{2q})$ . We adopt the following convention for non-negative sequences  $(u_n)$  and  $(v_n)$ : we write  $v_n \propto u_n$  to indicate that  $0 < \liminf_{n \rightarrow \infty} v_n/u_n \leq \limsup_{n \rightarrow \infty} v_n/u_n < \infty$ .

**Proposition 4.2** (See [2]). *Let the above notation and assumptions prevail.*

- (1) *If  $q$  is odd, then  $\kappa_3(F_n) = E[F_n^3] = 0$  for every  $n$ .*
- (2) *For every even integer  $q \geq 2$ ,*

$$\kappa_3(F_n) \propto \begin{cases} n^{-\frac{1}{2}} & \text{if } 0 < H < 1 - \frac{2}{3q}, \\ n^{-\frac{1}{2}} \log^2 n & \text{if } H = 1 - \frac{2}{3q}, \\ n^{\frac{3}{2} - 3q + 3qH} & \text{if } 1 - \frac{2}{3q} < H < 1 - \frac{1}{2q}. \end{cases}$$

- (3) *For  $q \in \{2, 3\}$ , one has that*

$$(4.20) \quad \kappa_4(F_n) \propto \begin{cases} n^{-1} & \text{if } 0 < H < 1 - \frac{3}{4q}, \\ n^{-1} \log^3 n & \text{if } H = 1 - \frac{3}{4q}, \\ n^{4qH - 4q + 2} & \text{if } 1 - \frac{3}{4q} < H < 1 - \frac{1}{2q}. \end{cases}$$

(4) For every integer  $q > 3$ ,

$$(4.21) \quad \kappa_4(F_n) \propto \begin{cases} n^{-1} & \text{if } 0 < H < \frac{3}{4}, \\ n^{-1} \log(n) & \text{if } H = \frac{3}{4}, \\ n^{4H-4} & \text{if } \frac{3}{4} < H < 1 - \frac{1}{2q-2}, \\ n^{4H-4} \log^2 n & \text{if } H = 1 - \frac{1}{2q-2}, \\ n^{4qH-4q+2} & \text{if } 1 - \frac{1}{2q-2} < H < 1 - \frac{1}{2q}. \end{cases}$$

Considering for example the cases  $q = 2$  and  $q = 3$  yields the following exact asymptotics, which are outside the scope of any other available technique.

**Proposition 4.3.** (1) If  $q = 2$ , then

$$d_{TV}(F_n, N) \propto \begin{cases} n^{-\frac{1}{2}} & \text{if } 0 < H < \frac{2}{3}, \\ n^{-\frac{1}{2}} \log^2 n & \text{if } H = \frac{2}{3}, \\ n^{6H-\frac{9}{2}} & \text{if } \frac{2}{3} < H < \frac{3}{4}. \end{cases}$$

(2) If  $q = 3$ , then

$$d_{TV}(F_n, N) \propto \begin{cases} n^{-1} & \text{if } 0 < H < \frac{3}{4}, \\ n^{-1} \log^3 n & \text{if } H = \frac{3}{4}, \\ n^{12H-10} & \text{if } \frac{3}{4} < H < \frac{5}{6}. \end{cases}$$

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