

Σ -PURE INJECTIVITY AND BROWN REPRESENTABILITY

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ABSTRACT. We prove that a right R -module M is Σ -pure injective if and only if $\text{Add}(M) \subseteq \text{Prod}(M)$. Consequently, if R is a unital ring, the homotopy category $\mathbf{K}(\text{Mod-}R)$ satisfies the Brown Representability Theorem if and only if the dual category has the same property. We also apply the main result to provide new characterizations for right pure-semisimple rings or to give a partial positive answer to a question of G. Bergman.

1. INTRODUCTION

It is well known that purity plays a central role in the study of module categories. In particular (Σ) -pure injective/projective modules are used in order to describe various properties for some subcategories of a module category. For instance the pure-semisimplicity (i.e. all modules are pure-injective) of $\text{Mod-}R$ can be characterized in many ways. However, there are many open problems related to these notions.

We approach in this note some of these problems. It was proved by L. Angeleri-Hügel, H. Krause and M. Saorin that $\text{Prod}(M) \subseteq \text{Add}(M)$ if and only if M is Σ -pure injective and product-rigid (see [2, Section 4.2] and [8, Section 3]), and this implies $\text{Prod}(M) = \text{Add}(M)$. We recall that $\text{Prod}(M)$ (resp. $\text{Add}(M)$) denotes the class of all direct summands in direct products (resp. direct sums) of copies of M . We will characterize the converse inclusion under the usual set theoretic assumption that there are no measurable cardinals. Here we use the terminology from [4], hence a cardinal $\lambda = |I|$ is measurable if it is uncountable and there exists a countably-additive, non-trivial, $\{0, 1\}$ -valued measure μ on the power set of I such that $\mu(I) = 1$ and $\mu(\{x\}) = 0$ for all $x \in I$.

Theorem 1.1 ($V = L$). *Let R be a unital ring and M be a right R -module. The following are equivalent:*

- (1) $\text{Add}(M) \subseteq \text{Prod}(M)$,
- (2) M is Σ -pure injective.

In particular, it is well known that a ring R is right pure-semisimple if and only if there exists a right R -module M such that $\text{Mod-}R = \text{Add}(M)$ (see [14, Proposition 2.6]). It is also important to know when we can write $\text{Mod-}R = \text{Prod}(M)$ for some

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right R -module M . One of the reasons comes from the theory of (co)homological functors on triangulated categories. After A. Neeman introduced Brown representability for the dual, [12], one of the main questions was to find examples of triangulated categories \mathcal{T} such that only one of the categories \mathcal{T} and \mathcal{T}^{op} satisfies the Brown Representability Theorem (BRT). For instance, the derived categories associated to some Grothendieck categories and their duals satisfy BRT, cf. [1] and [9]. One idea was to find a (non-pure-semisimple) ring R such that the homotopy category of complexes $\mathbf{K}(\text{Mod-}R)$ satisfies BRT for the dual, [11], i.e. to find a non-pure-semisimple ring such that $\text{Mod-}R = \text{Prod}(X)$, cf. [10]. By Theorem 1.1, this is not possible under the set theoretic hypothesis ($V = L$).

Corollary 1.2 ($V = L$). *Let R be a unital ring. The following are equivalent:*

- (1) R is right pure-semisimple (i.e. all right R -modules are pure-injective);
- (2) $\mathbf{K}(\text{Mod-}R)$ satisfies Brown Representability Theorem;
- (3) $\mathbf{K}(\text{Mod-}R)$ satisfies Brown Representability Theorem for the dual;
- (4) there exists $X \in \text{Mod-}R$ such that $\text{Mod-}R = \text{Add}(X)$;
- (5) there exists $X \in \text{Mod-}R$ such that $\text{Mod-}R = \text{Prod}(X)$.

2. THE PROOFS

Let R be a unital ring. The main tool used in the proof of our theorem was proved in [4]. We recall that a functor $P : \text{Mod-}R \rightarrow \text{Mod-}\mathbb{Z}$ is called a p -functor if it is a subfunctor of the identity functor which commutes with direct products. We refer to [7] for basic properties of these functors. We note here that every p -functor commutes with direct sums, since it is a subfunctor for the identity functor. From the same reason, it follows that if P is a p -functor and $f : M \rightarrow N$ is a homomorphism, then $P(f) : P(M) \rightarrow P(N)$ is in fact the restriction of f to $P(M)$. These functors represent very useful tools in the study of homomorphisms between direct products and direct sums, e.g. [16, Lemma 4] and [4, Theorem 2]. We start with a result which extends [4, Theorem 2].

Theorem 2.1. *Let $(U_i)_{i \in I}$ and $(V_j)_{j \in J}$ be two families of R -modules such that the set I is non-measurable. If $f : \prod_{i \in I} U_i \rightarrow \bigoplus_{j \in J} V_j$ is a homomorphism and $(P_n)_{n \in \mathbb{N}}$ is a descending chain of p -functors defined on $\text{Mod-}R$, then there exist two finite subsets $I' \subseteq I$, $J' \subseteq J$, and a positive integer n_0 such that*

$$f \left(P_{n_0} \left(\prod_{i \in I \setminus I'} U_i \right) \right) \subseteq \bigoplus_{j \in J'} V_j + \bigcap_{n \in \mathbb{N}} P_n \left(\bigoplus_{j \in J} V_j \right).$$

Proof. The proof presented in [4, Theorem 2] is valid for our case, once we prove that the theorem is valid for the case when I is countable.

If I is countable, we can apply [4, Lemma 1]. In order to do this we will prove that the topology defined on a direct product $U = \prod_{i \in \mathbb{N}} U_i$ by the set

$$\mathcal{B} = \left\{ \prod_{i \geq n} U_i \mid n \in \mathbb{N} \right\}$$

as a basis of neighborhoods of 0 is complete and Hausdorff. Note that all p -functors are compatible with this topology, cf. the second prototype described before [4, Lemma 1]. It is not hard to see that $\bigcap_{n \geq 0} \prod_{i \geq n} U_i = 0$, hence this topological space is Hausdorff.

For every $i \in \mathbb{N}$ we denote by $p_i : U \rightarrow U_i$ the canonical projection. Let $(x^k)_{k \in \mathbb{N}}$ be a Cauchy sequence in U . Then for every integer $n > 0$ there exists an integer

$u_n > n$ such that for all $v \geq u_n$ we have $x^v - x^{u_n} \in \prod_{i \geq n+1} U_i$, which means that $p_i(x^v) = p_i(x^{u_n})$ for all $i \leq n$. We can assume w.l.o.g. that the sequence $(u_n)_{n \in \mathbb{N}}$ is a strictly increasing sequence of positive integers. If we consider the element $x \in U$ defined by $p_i(x) = p_i(x^{u_i})$, it follows that for every $n > 0$ and for every $m > n$ we have $p_i(x^{u_m} - x) = 0$ for all $i \in \{1, \dots, n\}$. Then the subsequence $(x^{u_n})_{n \in \mathbb{N}}$ of the sequence $(x^k)_{k \in \mathbb{N}}$ converges to x . It follows that $(x^k)_{k \in \mathbb{N}}$ is convergent since it is a Cauchy sequence. Therefore U is complete.

For the general case we sketch the proof, which is exactly the same as in [4], for the reader's convenience. We consider the set \mathcal{I} of those subsets $T \subseteq I$ for which the restriction of f to $\prod_{i \in T} U_i$ satisfies our claim, and suppose that $I \notin \mathcal{I}$.

We can prove, as in [4], that \mathcal{I} has the following properties:

- (a) the set \mathcal{I} contains all finite subsets,
- (b) the set \mathcal{I} is closed under subsets and finite unions,
- (c) if $(J_n)_{n \in \mathbb{N}}$ is a family of pairwise disjoint subsets of I there exists $n_0 \in \mathbb{N}$ such that $\bigcup_{n > n_0} J_n \in \mathcal{I}$ (hence \mathcal{I} will be closed under countable unions).

Therefore, we can construct a countably additive $\{0, 1\}$ -valuated function on the power set of I , hence the cardinality of I is measurable. This leads to a contradiction, and it follows that $I \in \mathcal{I}$. □

We also need the following elementary lemma:

Lemma 2.2. *Let $V = \bigoplus_{j \in J} V_j$ be a right R -module. For every $j \in J$ we fix a family of submodules $Y_j^n \leq V_j$, $n \in \mathbb{N}$. Then*

$$\bigcap_{n \in \mathbb{N}} \left(\bigoplus_{j \in J} Y_j^n \right) = \bigoplus_{j \in J} \left(\bigcap_{n \in \mathbb{N}} Y_j^n \right).$$

Proof. If $x = (x_j) \in \bigoplus_{j \in J} V_j$ is in $\bigoplus_{j \in J} Y_j^n$ for all $n \in \mathbb{N}$ (with $x_j \in Y_j$ for all $j \in J$), then for all $j \in J$ and for all $n \in \mathbb{N}$ we have $x_j \in Y_j^n$ since the components x_j are unique with respect to the direct decomposition $\bigoplus_{j \in J} V_j$. Then for all $j \in J$ we have $x_j \in \bigcap_{n \in \mathbb{N}} Y_j^n$, hence $x \in \bigoplus_{j \in J} \left(\bigcap_{n \in \mathbb{N}} Y_j^n \right)$. Therefore $\bigcap_{n \in \mathbb{N}} \left(\bigoplus_{j \in J} Y_j^n \right) \subseteq \bigoplus_{j \in J} \left(\bigcap_{n \in \mathbb{N}} Y_j^n \right)$. The converse inclusion can be proved in the same way. □

Now we can prove the main result of this section:

Proposition 2.3. *Let $(U_i)_{i \in I}$ and $(V_j)_{j \in J}$ be two families of non-zero (right) R -modules such that $|I|$ is infinite and non-measurable and $|J| > |U_i|$ for all $i \in I$. Suppose that there exists an epimorphism $f : \prod_{i \in I} U_i \rightarrow \bigoplus_{j \in J} V_j$.*

If $(P_n)_{n \in \mathbb{N}}$ is a descending family of p -functors defined on $\text{Mod-}R$ such that the homomorphisms $P_n(f)$ are epimorphisms for all n , then there exists an infinite subset $L \subseteq J$ such that for every $j \in L$ the sequence $P_n(V_j)$ is stationary.

Proof. By Theorem 2.1, there exist two finite subsets $I' \subseteq I$, $J' \subseteq J$, and a positive integer n_0 such that

$$f \left(P_{n_0} \left(\prod_{i \in I \setminus I'} U_i \right) \right) \subseteq \bigoplus_{j \in J'} V_j + \bigcap_{n > 0} P_n \left(\bigoplus_{j \in J} V_j \right).$$

Since P_{n_0} is a subfunctor for the identity functor, $P_{n_0}(f)$ is the restriction of f to $P_{n_0} \left(\prod_{i \in I} U_i \right)$, hence $f \left(P_{n_0} \left(\prod_{i \in I} U_i \right) \right) \subseteq P_{n_0} \left(\bigoplus_{j \in J} V_j \right)$. Therefore,

$$f \left(P_{n_0} \left(\prod_{i \in I \setminus I'} U_i \right) \right) \subseteq \left[\left(\bigoplus_{j \in J'} V_j \right) + \bigcap_{n > 0} P_n \left(\bigoplus_{j \in J} V_j \right) \right] \cap P_{n_0} \left(\bigoplus_{j \in J} V_j \right).$$

Recall that the subgroup lattice of every abelian group is modular. Using this we obtain

$$\begin{aligned}
 f \left(P_{n_0} \left(\prod_{i \in I \setminus I'} U_i \right) \right) &\subseteq \left[\left(\bigoplus_{j \in J'} V_j \right) \cap P_{n_0} \left(\bigoplus_{j \in J} V_j \right) \right] + \left[\bigcap_{n > 0} P_n \left(\bigoplus_{j \in J} V_j \right) \right] \\
 &= P_{n_0} \left(\bigoplus_{j \in J'} V_j \right) + \left[\bigcap_{n > 0} P_n \left(\bigoplus_{j \in J} V_j \right) \right] \\
 &= P_{n_0} \left(\bigoplus_{j \in J'} V_j \right) + \left[\bigcap_{n > 0} \left(\bigoplus_{j \in J} P_n(V_j) \right) \right] \\
 &= P_{n_0} \left(\bigoplus_{j \in J'} V_j \right) + \left[\bigoplus_{j \in J} \left(\bigcap_{n > 0} P_n(V_j) \right) \right] \\
 &= P_{n_0} \left(\bigoplus_{j \in J'} V_j \right) + \left[\bigoplus_{j \in J \setminus J'} \left(\bigcap_{n > 0} P_n(V_j) \right) \right].
 \end{aligned}$$

Since $P_{n_0}(f) : P_{n_0} \left(\prod_{i \in I} U_i \right) \rightarrow P_{n_0} \left(\bigoplus_{j \in J} V_j \right)$ is an epimorphism, it follows that f induces an epimorphism of abelian groups

$$\bar{f} : \frac{P_{n_0} \left(\prod_{i \in I} U_i \right)}{P_{n_0} \left(\prod_{i \in I \setminus I'} U_i \right)} \rightarrow \frac{P_{n_0} \left(\bigoplus_{j \in J} V_j \right)}{P_{n_0} \left(\bigoplus_{j \in J'} V_j \right) + \left[\bigoplus_{j \in J \setminus J'} \left(\bigcap_{n > 0} P_n(V_j) \right) \right]}.$$

But

$$\begin{aligned}
 \frac{P_{n_0} \left(\bigoplus_{j \in J} V_j \right)}{P_{n_0} \left(\bigoplus_{j \in J'} V_j \right) + \left[\bigoplus_{j \in J \setminus J'} \left(\bigcap_{n > 0} P_n(V_j) \right) \right]} &\cong \frac{\bigoplus_{j \in J \setminus J'} P_{n_0}(V_j)}{\bigoplus_{j \in J \setminus J'} \left(\bigcap_{n > 0} P_n(V_j) \right)} \\
 &\cong \bigoplus_{j \in J \setminus J'} \frac{P_{n_0}(V_j)}{\bigcap_{n > 0} P_n(V_j)}.
 \end{aligned}$$

It follows that

$$\left| \bigoplus_{j \in J \setminus J'} \frac{P_{n_0}(V_j)}{\bigcap_{n > 0} P_n(V_j)} \right| \leq |P_{n_0} \left(\prod_{i \in I'} U_i \right)| = \left| \frac{P_{n_0} \left(\prod_{i \in I} U_i \right)}{P_{n_0} \left(\prod_{i \in I \setminus I'} U_i \right)} \right|,$$

hence the cardinality of the set

$$L = \{j \in J \setminus J' \mid P_{n_0}(V_j) = \bigcap_{n > 0} P_n(V_j)\}$$

is infinite since $|J| > |\prod_{i \in I'} U_i| \geq |P_{n_0} \left(\prod_{i \in I'} U_i \right)|$. □

Recall from [7, Observation 3] that the *finite matrix functors* are precisely the functors $\text{Hom}_R(Z, -)(z) : \text{Mod-}R \rightarrow \text{Mod-}\mathbb{Z}$ with finitely presented Z and $z \in Z$.

Corollary 2.4. *Let $(U_i)_{i \in I}$ and $(V_j)_{j \in J}$ be two families of (right) R -modules such that J is infinite, $|J| > |U_i|$ for all $i \in I$, and all modules V_j are isomorphic to a fixed module V .*

Suppose that I is non-measurable and there exists a pure epimorphism

$$f : \prod_{i \in I} U_i \rightarrow \bigoplus_{j \in J} V_j.$$

Then V is Σ -pure injective.

Proof. Since finitely presented modules are projective with respect to pure exact sequences, it is easy to see that all finite matrix functors map pure epimorphisms to epimorphisms. Therefore, every decreasing sequence of matrix functors is stationary on V . By [7, Theorem 6], V is Σ -pure injective. □

Remark 2.5. If the homomorphism f in the above results, Proposition 2.3 and Corollary 2.4, is split-epi, then the proofs are valid for all descending families of p -functors.

The proof for Theorem 1.1. Since $\text{Add}(M) \subseteq \text{Prod}(M)$, it follows that for all sets J the right R -module $M^{(J)}$ is a direct summand of a direct product of copies of M . If we take J of cardinality greater than the cardinality of M , the conclusion follows from Corollary 2.4 (or Remark 2.5) since all cardinals are non-measurable. \square

The proof for Corollary 1.2. (1) \Leftrightarrow (2) is proved in [14, Proposition 2.6], (2) \Leftrightarrow (4) and (3) \Rightarrow (5) follow from [11, Theorem 1 and Theorem 2]. (1) \Rightarrow (3) follows from [14, Theorem 5.2]; see also [10, Remark 11]. The implication (5) \Rightarrow (1) is a consequence of Theorem 1.1. \square

In [3, Question 12], G. Bergman asks if we can deduce that the canonical embedding $R^{(\omega)} \rightarrow R^\omega$ splits provided that $R^{(\omega)}$ is isomorphic to a direct summand in R^ω . Corollary 2.4 can be used to provide a partial positive answer to this question. In order to do this, let us recall from [15, Satz 3 and Satz 4] that if κ is a non-measurable cardinal, then 2^κ is also non-measurable.

Corollary 2.6. *Let M be a module of non-measurable cardinality κ . If $M^{(2^\kappa)}$ can be embedded as a direct summand in a direct product M^λ of copies of M such that λ is non-measurable (in particular in M^{2^κ}), then M is Σ -pure injective, thus the canonical embedding $M^{(\omega)} \rightarrow M^\omega$ splits.*

Finally, we mention that the condition $\text{Mod-}R = \text{Prod}(X)$ also appears as a characterization of rings which are copure-semisimple, i.e. all modules are copure injective. If M is a right R -module, we denote by $E(M)$ its injective envelope. We recall from [5], [6] and [17] that a module M is *finitely cogenerated* if its injective envelope $E(M)$ is isomorphic to $E(S_1) \oplus \dots \oplus E(S_n)$, where S_i are simple modules. The module M is *finitely copresented* if $E(M)$ and $E(M)/M$ are finitely cogenerated, and a short exact sequence is *copure* if every finitely copresented module is injective relative to this exact sequence. A module is *copure injective* if it is injective relative to all copure short exact sequences. We mention that direct products of copure injective modules are copure injective, [6, Proposition 3].

We obtain an interesting corollary concerning these rings:

Corollary 2.7 ($V = L$). *A ring R is right pure-semisimple if and only if all right R -modules are copure injective.*

Proof. Suppose that R is right pure-semisimple. Using [13, Theorem 4.5.4] we observe that all finitely presented modules are finitely copresented, hence they are copure injective. But all modules are direct sums of finitely presented modules. Moreover, since all pure exact sequences split, it follows that every canonical embedding $0 \rightarrow \bigoplus_{i \in I} U_i \rightarrow \prod_{i \in I} U_i$ splits, hence every module is a direct summand of a copure injective module. Therefore every module is copure injective.

Conversely, if every module is copure injective, then $\text{Mod-}R = \text{Prod } M$, where M is the direct product of a set of representatives of the isomorphism classes of all cofinitely related modules, [6, Theorem 7]. The conclusion follows from Corollary 1.2. \square

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