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ON THE LERAY-SCHAUDER DEGREE OF THE TODA SYSTEM ON COMPACT SURFACES

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Abstract. In this paper we consider the following *Toda system* of equations on a compact surface:

$$\begin{cases} -\Delta u_1 = 2\rho_1 \left(h_1 e^{u_1} - 1\right) - \rho_2 \left(h_2 e^{u_2} - 1\right), \\ -\Delta u_2 = 2\rho_2 \left(h_2 e^{u_2} - 1\right) - \rho_1 \left(h_1 e^{u_1} - 1\right). \end{cases}$$

Here h_1, h_2 are smooth positive functions and ρ_1, ρ_2 two positive parameters.

In this note we compute the Leray-Schauder degree mod $\mathbb{Z}_2$ of the problem for $\rho_i \in (4\pi k, 4\pi(k+1))$ $(k \in \mathbb{N})$. Our main tool is a theorem of Krasnoselskii and Zabreiko on the degree of maps symmetric with respect to a subspace. This result yields new existence results as well as a new proof of previous results in the literature.

1. Introduction

In this paper we consider the following problem on a compact orientable surface Σ:

$$\begin{cases} -\Delta u_1 = 2\rho_1 \left(h_1 e^{u_1} - 1\right) - \rho_2 \left(h_2 e^{u_2} - 1\right), \\ -\Delta u_2 = 2\rho_2 \left(h_2 e^{u_2} - 1\right) - \rho_1 \left(h_1 e^{u_1} - 1\right). \end{cases} \tag{1}$$

Here h_1, h_2 are smooth positive functions and Δ is the Laplace-Beltrami operator.

Equation (1) is known as the *Toda system*, and has been extensively studied in the literature. This problem has a close relationship to geometry, since it describes the integrability of Frenet frames for holomorphic curves in $\mathbb{CP}^2$; see [6]. Moreover, it arises in the study of non-abelian Chern-Simons theory in the self-dual case, when a scalar Higgs field is coupled to a gauge potential; see [5, 19, 21].

Let us first discuss the scalar counterpart of (1), namely a Liouville equation in the form

$$-\Delta u = \rho \left(h\, e^u - 1\right), \tag{2}$$

where $\rho \in \mathbb{R}$ and $h(x) > 0$. Equation (2) arises in the prescribed Gaussian curvature problem under conformal deformation of the metric, and also describes the abelian counterpart of (1) from thephysical point of view. There are several contributions

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to the analysis of the latter problem, originating with the works [3, 10]. There are by now many results regarding existence, compactness of solutions, bubbling behavior, etc. We refer the interested reader to the reviews [16, 20].

Problem (2) presents a lack of compactness, as its solutions might blow up. Indeed, take a blowing-up sequence u_n of (1) with $\rho_n \in \mathbb{R}$ bounded. Then it was proved in [2, 13, 14] that, up to a subsequence,

$$\rho_n \to 8k\pi, \ k \in \mathbb{N}.$$

Moreover, e^{u_n} behaves like the conformal factor of the stereographic projection from S^2 onto $\mathbb{R}^2$, composed with a dilation, and located at a finite number of points. A complete construction of blow-up sequences of solutions in this fashion was made in [4].

In particular, one can define the Leray-Schauder degree associated to problem (1) and $\rho \in (8k\pi, 8(k+1)\pi)$. By the homotopy property of the degree, the latter is independent of the metric g and the function h, and will only vary with k and the topology of Σ. The computation of the degree was accomplished in [4], where the following formula is given:

$$(3) \ \ d_{LS} = \frac{1}{k!}(-\chi(\Sigma)+1)\cdots(-\chi(\Sigma)+k). \ \ (\chi(\Sigma) \text{ is the Euler characteristic of } \Sigma.)$$

In order to obtain this formula, in [4] a detailed study of all blowing-up solutions and their local degree was performed; see also [15] for a different approach.

Coming back to system (1), it was proved in [7, 9] that the set of solutions is compact for $(\rho_1, \rho_2) \notin (4\pi\mathbb{N} \times \mathbb{R}) \cup (\mathbb{R} \times 4\pi\mathbb{N})$. In other words, if blowing up occurs, at least one of the components is quantized.

Therefore, as above, the degree for (1) is well defined for (ρ_1, ρ_2) away from that set. It is easy to observe that this degree is equal to 1 if both ρ_i are smaller than 4π (one can deform the parameters to $\rho_1 = \rho_2 = 0$). Apart from that, there exists no formula for the Leray-Schauder degree for system (1) yet.

Because of that, most of the existence results for problem (1) have used variational methods so far. Indeed, it was proved that there exists at least one solution in the following cases:

(1) for both $\rho_i < 4\pi$ (see [8]);
(2) for any $\rho_1 < 4\pi$, $\rho_2 \in (4k\pi, 4(k+1)\pi)$, $k \in \mathbb{N}$ (see [17]);
(3) for both $\rho_i \in (4\pi, 8\pi)$ (see [18]);
(4) for $\rho_1 \in (4k\pi, 4(k+1)\pi), \rho_2 \in (4m\pi, 4(m+1)\pi)$, $k, \ m \in \mathbb{N}$ and Σ with positive genus (see [1]).

In (1), it was proved that the associated energy functional is coercive and hence a minimum is found. The rest of the results use min-max theory, as the functional is no longer bounded from below.

In this note we discuss the parity of the degree for $\rho_i \in (4n\pi, 4(n+1)\pi)$; see Proposition 2.2. Our result is a consequence of a general theorem (recalled in the next section) concerning the degree of maps symmetric with respect to a subspace; see [11]. We will show that the degree of the Toda system has the same parity as the degree of the scalar case with $\rho = \rho_i$, which is given by (3).

In particular, the degree is always odd for $\rho_i \in (4n\pi, 4(n+1)\pi)$ if $n = 1,\ 2,\ 3$. The case $n = 1$ implies a new, simpler, proof of the existence result of [18]. The cases $n = 2$ or 3 yield a new existence result:

Theorem 1.1. *Assume that Σ is homeomorphic to $\mathbb{S}^2$, and $\rho_i \in (4n\pi, 4(n+1)\pi)$, with $n = 2$ or 3. Then there exists a solution to* (1).

A final comment regarding the physical motivation of the problem. Vortices would imply that the functions h_i vanish at the vortex point; therefore, our equation models a problem without such vortices. Of course the case with vortices would be of more interest, but in that case a degree formula is not even known for the single equation (2) (related to the abelian case).

2. The parity of the Leray-Schauder degree

The main abstract tool we are going to use is the following one, which is a version of Theorem 21.12 of [11, page 115]:

Theorem 2.1. *Let P be a continuous linear projection from a Banach space E onto a (closed) subspace $E^0 \subseteq E$, and define $U = -Id + 2P$ the reflection with respect to E^0, which is assumed to be an isometry. Let $A : E \to E$ be a compact operator equivariant with respect to U, that is*

$$AU(x) = UA(x) \quad \forall x \in E.$$

Observe that in particular $A(E^0) \subset E^0$. Finally, assume that $\Phi x = x - Ax$ does not vanish on the boundary of $B_R = B(0, R)$. Then, $\deg(\Phi, B_R, 0)$ *and* $\deg(\Phi|_{E^0}, B_R \cap E^0, 0)$ *have the same parity.*

Remark 2.1. Theorem 2.1 is easy to understand if we assume that Φ is C^1 and that all its zeroes are non-degenerate. In such a case, all zeroes have index ± 1, and the total degree is the sum of the indexes of all zeroes. Observe now that if x is a zero of Φ, then also Ux is a zero. Moreover $x = Ux$ if and only if $x \in E^0$. In other words, the zeroes outside E^0 come in pairs and give an even contribution to the total degree.

Furthermore, the index of $x \in E^0$ as a zero of Φ could be different from its index as a zero of $\Phi|_{E^0}$, but in both cases it is ± 1. So the difference is even.

The proof of [11] is topological and does not use these arguments.

Let us assume, for simplicity, that $Vol_g(\Sigma) = 1$. For $\alpha \in (0, 1)$ we consider the functional framework

$$E = C_0^{2,\alpha}(\Sigma) \times C_0^{2,\alpha}(\Sigma),$$

where $C_0^{2,\alpha}(\Sigma)$ stands for the class of $C^{2,\alpha}$ functions with zero average. Now define the operator $A = A_{\rho_1,\rho_2}^{h_1,h_2} : E \to E$ as

$$A\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} (-\Delta)^{-1}\left[2\rho_1\left(h_1\frac{e^{u_1}}{\int_\Sigma h_1 e^{u_1}\,dV_g} - 1\right) - \rho_2\left(h_2\frac{e^{u_2}}{\int_\Sigma h_2 e^{u_2}\,dV_g} - 1\right)\right] \\ (-\Delta)^{-1}\left[2\rho_2\left(h_2\frac{e^{u_2}}{\int_\Sigma h_2 e^{u_2}\,dV_g} - 1\right) - \rho_1\left(h_1\frac{e^{u_1}}{\int_\Sigma h_1 e^{u_1}\,dV_g} - 1\right)\right] \end{pmatrix}.$$

Here, by $(-\Delta)^{-1}f$, $f \in C^\alpha(\Sigma)$ with zero average, we denote the unique solution u of $-\Delta u = f$ with zero average. In the above formula, solutions exist by Fredholm's theory. Notice also that zeroes of $\Phi = Id - A$ give rise to solutions of (1). Indeed, it suffices to add proper constants to u_1, u_2 in order to have $\int_\Sigma h_i e^{u_i}\,dV_g = 1$.

By elliptic regularity theory, the operator A is compact. Moreover, if $n \in \mathbb{N}$ and ρ_1, $\rho_2 \in (4n\pi, 4(n+1)\pi)$, the solutions are a priori bounded; see [7]. Therefore, for such values of ρ_1, ρ_2 and for R sufficiently large, the degree $\deg(\Phi, B_R, 0)$ is well defined. The main result of this paper is the following:

Proposition 2.2. *Let $n \in \mathbb{N}$ and let ρ_1, $\rho_2 \in (4n\pi, 4(n+1)\pi)$. Then, for sufficiently large R, $\deg(\Phi, B_R, 0)$ has the same parity as*

$$d_k = \frac{1}{k!}(-\chi(\Sigma)+1)\cdots(-\chi(\Sigma)+k),$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ and $k = [n/2]$.

Proof. Let us consider the homotopy

$$(1-s)(\rho_1,\rho_2) + s(\rho,\rho), \qquad \rho = \frac{1}{2}(\rho_1+\rho_2),$$

$$(1-s)(h_1,h_2) + s(h,h), \qquad h = \frac{1}{2}(h_1+h_2).$$

By the homotopy invariance of the degree, we deduce that (for R sufficiently large)

$$\deg(\Phi^{h_1,h_2}_{\rho_1,\rho_2}, B_R, 0) = \deg(\Phi^{h,h}_{\rho,\rho}, B_R, 0). \tag{4}$$

Because of that, it suffices to study the degree for $h_1 = h_2 = h$, $\rho_1 = \rho_2 = \rho$.

We choose E^0 to be the couples of identical functions in E, namely

$$E^0 = \left\{ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in E \ : \ u_1 = u_2 \right\},$$

and we define the projection $P : E \to E^0$ as

$$P\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} u_1+u_2 \\ u_1+u_2 \end{pmatrix}.$$

Observe that the reflection U is given by

$$U\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_2 \\ u_1 \end{pmatrix}.$$

With these definitions, we are in a position to apply Theorem 2.1, and hence $\deg(\Phi, B_R, 0)$ has the same parity as $\deg(\Phi|_{E_0}, B_R \cap E_0, 0)$.

We now define

$$T : C^{2,\alpha}_0(\Sigma) \to E_0, \ T(u) = \begin{pmatrix} u \\ u \end{pmatrix}.$$

Clearly T is an homeomorphism, which implies that

$$\deg(\Phi|_{E_0}, B_R \cap E_0, 0) = \deg(T^{-1} \circ \Phi|_{E_0} \circ T, \tilde{B}_R, 0),$$

where $\tilde{B}_R = B(0,R) \subset C^{2,\alpha}_0(\Sigma)$.

Observe now that

$$\tilde{\Phi}(u) := T^{-1} \circ \Phi|_{E_0} \circ T(u) = u - (-\Delta)^{-1}\left[\rho\left(h\frac{e^u}{\int_\Sigma he^u dV_g} - 1\right)\right].$$

Moreover, $\rho \in (4n\pi, 4(n+1)\pi) \subset (8k\pi, 8(k+1)\pi)$ for $k = [n/2]$. Finally, $\deg(\tilde{\Phi}, \tilde{B}_R, 0)$ has been computed in [4] and it is given by the formula (3). Notice that the Leray-Schauder degree in [4] has been computed in the H^1 setting, but using elliptic regularity theory one can prove that this coincides with the degree in the $C^{2,\alpha}$ setting; see Theorem B.1 in [12]. This concludes the proof. $\square$

Observe that if $n = 1$, then $k = 0$ and hence the degree is odd. In this way we obtain the existence result of [18] with an alternative approach. There are other cases in which the degree is odd, so we recover some of the results of [1]. For instance, in the physically relevant case of the thorus, $d_k = 1$ and the degree is odd for any $n \in \mathbb{N}$.

Moreover, if $n = 2, 3$, then $k = 1$, and the degree is odd for any compact orientable Σ. As a consequence, we obtain Theorem 1.1, which gives a new existence result.

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