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# INDIVISIBILITY OF CENTRAL VALUES OF $L$-FUNCTIONS FOR MODULAR FORMS

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Abstract. In this paper, we generalize works of Kohnen and Ono (in Invent. Math., 1999) and James and Ono (in Math. Ann., 1999) on indivisibility of (the algebraic part of) central critical values of $L$-functions to higher weight modular forms.

## 1. Introduction

In this article, we show an indivisibility result on central critical values of $L$-functions associated to quadratic twists of modular forms using the method of Kohnen-Ono [7] and James-Ono [5].

Let $f(z) = \sum_{n=1}^{\infty} a(n)q^n$ be a normalized newform of weight $2k$ for $\Gamma_0(N)$ with trivial character. For a fundamental discriminant $D$ with $(D, N)$=1, we define the $D$-th quadratic twist of $f$ by

$$f \otimes \chi_D = \sum_{n=1}^{\infty} a(n)\chi_D(n)q^n,$$

where $\chi_D$ is the quadratic character corresponding to the quadratic extension $\mathbb{Q}(\sqrt{D})/\mathbb{Q}$. Then $f \otimes \chi_D$ is a newform of weight $2k$ for $\Gamma_0(D^2N)$. Similarly, the $D$-th quadratic twist of the $L$-function $L(f, s)$ is given by

$$L(f \otimes \chi_D, s) = \sum_{n=1}^{\infty} \frac{a(n)\chi_D(n)}{n^s}.$$

Let $E$ be the number field generated by all Fourier coefficients of $f$ and $\mathbb{Q}$. Then it is known that there exists a period $\Omega_f \in \mathbb{C}^\times$ satisfying that $\frac{L(f\otimes\chi_D,k){D_0}^{k-1/2}}{\Omega_f}$ are integers in $E$ for all fundamental discriminants $D$ with $\delta(f) \cdot D > 0$, where $\delta(f) \in \{\pm 1\}$ is the sign defined in Ono-Skinner [10, p. 655] and $D_0$ is given by

$$D_0 = \begin{cases} |D| & \text{if } D \text{ is odd,} \\ |D|/4 & \text{if } D \text{ is even.} \end{cases}$$

We fix such a period $\Omega_f$.

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For convenience, we denote

$$S(X)=\{D\in\mathbb{Z}\,|\,|D|<X, D: \text{fundamental discriminant}\},$$

and if functions $f$, $g$ on $\mathbb{R}$ satisfy that there is a positive constant $c$ such that $f(X)\geq c\cdot g(X)$ for sufficiently large $X>0$, then we write $f(X)\gg g(X)$.

**Theorem 1.1.** *Let $f(z)=\sum_{n=1}^{\infty}a(n)q^n$ be a normalized newform of weight $2k$ for $\Gamma_0(N)$ with trivial character. Then, for all but finitely many primes $\lambda$ of $E$, we have*

$$\#\left\{D\in S(X)\,\middle|\,\delta(f)\cdot D>0, \lambda\nmid D \text{ and } \frac{L(f\otimes\chi_D,k)D_0^{k-\frac{1}{2}}}{\Omega_f}\not\equiv 0 \mod \lambda\right\}\gg_{f,\lambda}\frac{\sqrt{X}}{\log X}.$$

This result is a refinement of results of Bruinier [2] and Ono-Skinner [10]. The proof is based on a generalization of a method of Kohnen-Ono [7] and James-Ono [5]. In the above theorem, we do not assume that the Fourier coefficients of $f$ belong to $\mathbb{Z}$, therefore the surjectivity of the residual Galois representation associated to $f$ for almost all places in general does not hold. This creates some technical difficulty for the proof. To solve this problem, we may use a result of Ribet [12] on the image of Galois representations associated to modular forms. This is an ingredient in our proof. In the last section, we also consider an indivisibility result on non-central critical values of $L$-functions for higher weight modular forms using congruences of modular forms with different weights.

## 2. Modular forms of half-integral weight

We denote the space of modular forms of weight $k+1/2$, level $N$ with character $\chi$ by $M_{k+1/2}(N,\chi)$, and the space of cusp forms of weight $k+1/2$, level $N$ with character $\chi$ by $S_{k+1/2}(N,\chi)$. Then $M_{k+1/2}(N,\chi)$ and $S_{k+1/2}(N,\chi)$ are complex vector spaces.

For a modular form of half-integral weight

$$g(z)=\sum_{n=0}^{\infty}b(n)q^n\in M_{k+1/2}(N,\chi),$$

we define the action of Hecke operator $T_{p^2}$ by

$$T_{p^2}(g)(z)=\sum_{n=0}^{\infty}b'(n)q^n,$$

where $b'(n)$ are given by

$$b'(n)=b(p^2n)+\chi(p)\left(\frac{-1}{p}\right)^k\left(\frac{n}{p}\right)p^{k-1}b(n)+\chi(p^2)p^{2k-1}b(n/p^2)$$

and $b(n/p^2)$ are zero if $p^2\nmid n$.

Now we give a short review of the theory of the Shimura correspondence. Let $N$ be a positive integer which is divisible by four and $\chi$ a Dirichlet character mod $N$. Then we define a vector space $S^0_{3/2}(N,\chi)$ to be the subspace of $S_{3/2}(N,\chi)$ generated by

$$\left\{f(z)=\sum_{n=1}^{\infty}\psi(n)nq^{tm^2}\,\middle|\,4\text{cond}(\psi)^2\text{t/N}, \chi=\psi\chi_{-\text{t}} \quad\text{and}\quad \psi(-1)=-1\right\}$$

and denote the orthogonal complement by $S'_{3/2}(N,\chi)$. Then we assume

$$g(z)=\sum_{n=1}^{\infty} b(n)q^n \in S_{k+1/2}(N,\chi)$$

if $k\geq 2$, and

$$g(z)=\sum_{n=1}^{\infty} b(n)q^n \in S'_{3/2}(N,\chi)$$

if $k=1$. Let $t$ be a square-free positive integer. Define a number $A_t(n)$ to be

$$\sum_{n=1}^{\infty}\frac{A_t(n)}{n^s}=\left(\sum_{n=1}^{\infty}\frac{\chi(n)(\frac{-1}{n})^k\left(\frac{t}{n}\right)}{n^{s-k+1}}\right)\left(\sum_{n=1}^{\infty}\frac{b(tn^2)}{n^s}\right).$$

Then Shimura [14] proved that there is a positive integer $M$ such that $\mathrm{SH}_t(g(z))=f_t(z)=\sum_{n=1}^{\infty}A_t(n)q^n\in S_{2k}(M,\chi^2)$. (In fact, one can prove that $M=N/2$.) Furthermore for any $t,t'$, the difference between $\mathrm{SH}_t(g)$ and $\mathrm{SH}_{t'}$ is only constant multiple, so essentially this correspondence is independent of the choice of $t$. This correspondence between modular forms is called the Shimura correspondence. Moreover if $g$ is an eigenform for all Hecke operators $T_{p^2}$ with $(p,2N)=1$, then the image of $g$ under the Shimura correspondence is also an eigenform for all Hecke operators $T_p$ with $(p,2N)=1$ and the Hecke eigenvalue of $T_{p^2}$ for $g$ coincides with the Hecke eigenvalue for $T_p$ for $\mathrm{SH}_t(g)$.

We recall the following result which is a useful version of Waldspurger's formula ([17, Thèorém 1]) by Ono-Skinner. This formula gives a relation between the Fourier coefficients of modular forms of half-integral weight and the central values of twisted $L$-functions for modular forms.

**Theorem 2.1** (Ono-Skinner [9], (2a),(2b)). *Let $f(z)=\sum_{n=1}^{\infty}a(n)q^n$ be a normalized newform of weight $2k$, level $M$ with trivial character. Then there is $\delta(f)\in\{\pm1\}$, a positive integer $N$ with $4M\mid N$, a Dirichlet character $\chi$ modulo $N$, a period $\Omega_f\in\mathbb{C}^\times$ and a non-zero eigenform*

$$g(z)=\sum_{n=1}^{\infty} b(n)q^n \in S_{k+1/2}(N,\chi)$$

*with the property that $g(z)$ maps to a twist of $f$ under the Shimura correspondence and for all fundamental discriminants $D$ with $\delta(f)D/>0$ we have*

$$b(D_0)^2=\begin{cases}\alpha_D\dfrac{L(f\otimes\chi_D,k){D_0}^{k-1/2}}{\Omega_f} & \text{if } (D,N)=1,\\ 0 & \text{otherwise,}\end{cases}$$

*where $\alpha_D$ and $b(n)$ are algebraic integers in some finite extension of $\mathbb{Q}$. Moreover, there exists a finite set of primes $S$ such that if $D$ is a fundamental discriminant*

(1) $\delta(f)D>0$,
(2) $(D,N)=1$,

*then we have $|L(f\otimes\chi_D,k){D_0}^{k-1/2}/\Omega_f|_\lambda=|b(|D_0|)^2|_\lambda$ for $\lambda\notin S$.*

## 3. Some properties of Fourier coefficients of modular forms and Galois representations

In this section we generalize some results of Serre [13] and Swinnerton-Dyer [16] using a result of Ribet [12]. These results should be well-known for specialists. However we give a short review of them, since it does not seem to be available in the literature. Let $f = \sum_{n=1}^{\infty} a(n)q^n$ be a normalized newform of weight $2k$ for $\Gamma_0(N)$ with trivial character. Let $E$ be the subfield of $\mathbb{C}$ generated by the Fourier coefficients $a(n)$ of $f$. Then $E$ is a finite extension of $\mathbb{Q}$. Let $\mathcal{O}_E$ be the ring of integers of $E$. For each prime $\ell$, we let $\mathcal{O}_{E,\ell} = \mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ and $E_\ell = E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$.

**Theorem 3.1** (Deligne [3])**.** *For each prime $\ell$, there exists a continuous representation*

$$\rho_{f,\ell} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathcal{O}_{E,\ell}) \subset \mathrm{GL}_2(E_\ell)$$

*unramified at all primes $p \nmid N\ell$ such that* $\mathrm{trace}\rho_{f,\ell}(\mathrm{Frob}_p) = a(p)$ *and* $\det\rho_{f,\ell}(\mathrm{Frob}_p) = p^{2k-1}$ *for all primes $p \nmid N\ell$, where* $\mathrm{Frob}_p$ *is the arithmetic Frobenius at $p$.*

For each prime $\ell$, denote

$$A_\ell = \left\{ g \in \mathrm{GL}_2(\mathcal{O}_{E,\ell}) \,\middle|\, \det(g) \in \mathbb{Z}_\ell^{\times(2k-1)} \right\},$$

where $\mathbb{Z}_\ell^{\times(2k-1)}$ is the group of $(2k-1)$-th powers of elements in $\mathbb{Z}_\ell^\times$. Replacing $\rho_{f,\ell}$ by an isomorphic representation, we may assume that for almost all $\rho_{f,\ell}$ sends $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ to $A_\ell$. Then Ribet proved the following theorem.

**Theorem 3.2** (Ribet [12])**.** *Assume that $f$ has no complex multiplication. Then for almost all $\ell$, we have $\rho_{f,\ell}(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})) = A_\ell$.*

We call the set of primes $\ell$ with the property $\rho_{f,\ell}(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})) \neq A_\ell$ the exceptional primes for $f$. Let $S$ be the set of exceptional places for $f$. Let $\varepsilon_\ell : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Z}_\ell^\times$ be the $\ell$-adic cyclotomic character. Then by a similar argument to Swinnerton–Dyer [16], one can see that the image of

$$(\rho_{f,\ell}, \varepsilon_\ell) : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathcal{O}_{E,\ell}) \times \mathbb{Z}_\ell^\times$$

is $\left\{ (g,\alpha) \in \mathrm{GL}_2(\mathcal{O}_{E,\ell}) \times \mathbb{Z}_\ell^\times \,\middle|\, \det(g) = \alpha^{2k-1} \right\}$ if $\ell$ is not exceptional. Since $A_\ell$ contains an element with the form

$$\begin{pmatrix} \mathrm{trace}\rho_{f,\ell}(\sigma) & -1 \\ \det\rho_{f,\ell}(\sigma) & 0 \end{pmatrix},$$

the map $(\mathrm{trace}\rho_{f,\ell}, \varepsilon_\ell) : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathcal{O}_{E,\ell} \times \mathbb{Z}_\ell^\times$ is surjective. Moreover by a ramification argument, one can see that the map

$$\prod_{\ell \notin S} (\mathrm{trace}\rho_{f,\ell}, \varepsilon_\ell) : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \prod_{\ell \notin S} (\mathcal{O}_{E,\ell} \times \mathbb{Z}_\ell^\times)$$

is also surjective. Therefore we have the following result which is a generalization of a result of Serre [13, Thèorém 11] using the Chebotarev density theorem.

**Theorem 3.3.** *Assume that $f$ has no complex multiplication. Let $t$ be a positive integer and $\alpha$ a non-zero integer in $E$. Fix $\beta \in \mathcal{O}_E/\alpha\mathcal{O}_E$ and $r \in (\mathbb{Z}/t\mathbb{Z})^\times$. Suppose that $\alpha$ does not contain a prime divisor which divides an exceptional prime for $f$. Then the set of primes $p$ with the properties $a(p) \equiv \beta \mod \alpha$ and $p \equiv r \mod t$ has positive density.*

## 4. Indivisibility of Fourier coefficients of modular forms of half-integral weight

In this section, we give a result on modulo $\ell$ indivisibility of Fourier coefficients of half-integral weight modular forms using a method of Kohnen-Ono [7] and James-Ono [5]. Our result is a refinement of a result of Bruinier [2] and Ono-Skinner [10].

To consider the indivisibility of Fourier coefficients of half-integral weight modular forms, we will use the following results.

**Theorem 4.1** (Sturm [15])**.** *Let*

$$g(z)=\sum_{n=1}^{\infty} b(n)q^n \in M_k(N,\chi)$$

*be a half-integral or integral weight modular form for which the coefficients $b(m)$ are algebraic integers contained in a number field $E$. Let $v$ be a finite place of $E$ and let*

$$\mathrm{ord}_v(g)=\begin{cases} +\infty & \text{if } b(n)\equiv 0 \mod v \text{ for all } n,\\ \min\{\,n \mid b(n)\not\equiv 0 \mod v\} & \text{otherwise.}\end{cases}$$

*Moreover put*

$$\mu=\frac{k}{12}[\Gamma_0(1):\Gamma_0(N)]=\frac{kN}{12}\prod_{p\mid N}\frac{p+1}{p}.$$

*Assume that*

$$\mathrm{ord}_v(g)>\mu;$$

*then* $\mathrm{ord}_v(g)=+\infty$.

*Remark* 4.2 (cf. [5, Proposition 5]). In [15], Sturm proved this theorem for integral weight modular forms with trivial character, but the general case follows by taking an appropriate power of $g$.

**Lemma 4.3** (Shimura, [14, Section 1])**.** *Suppose*

$$g(z)=\sum_{n=1}^{\infty} b(n)q^n \in S_{k+1/2}(N,\chi)$$

*is a half-integral weight cusp form and $p$ is a prime. We define $(U_pg)(z)$, $(V_pg)(z)$ by*

$$\begin{aligned}(U_pg)(z) &= \sum_{n=1}^{\infty} u_p(n)q^n=\sum_{n=1}^{\infty} b(pn)q^n,\\ (V_pg)(z) &= \sum_{n=1}^{\infty} v_p(n)q^n=\sum_{n=1}^{\infty} b(n)q^{pn}.\end{aligned}$$

*Then*

$$(U_pg)(z),\ (V_pg)(z)\in S_{k+1/2}\left(Np,\chi\left(\frac{4p}{\cdot}\right)\right).$$

Let

$$f(z) = \sum_{n=1}^{\infty} a(n)q^n \in M_k(N, \chi)$$

be an integral weight modular form for which the coefficients $a(m)$ are algebraic integers in $E$. For a prime $\lambda$ of $E$ and positive integers $r, t$ with $(r, t) = 1$, define $T(r, t)$ and $T(\lambda, r, t)$ by

$$T(r, t) = \{p : \text{prime} \,|\, a(p) = 0,\ p \equiv r \bmod t\}$$

and

$$T(\lambda, r, t) = \{p : \text{prime} \,|\, a(p) \equiv 0 \bmod \lambda,\ p \equiv r \bmod t\}.$$

For a positive real number $X$, we also denote $T(r, t, X) = \{p \in T(r, t) \,|\, p \le X\}$ and $T(\lambda, r, t, X) = \{p \in T(\lambda, r, t) \,|\, p \le X\}$.

For $g = \sum_{n=1}^{\infty} b(n)q^n \in S_{k+1/2}(N, \chi) \cap \mathcal{O}_{E,\lambda}[[q]]$, denote $s_\lambda(g) = \min\{\mathrm{ord}_\lambda(b(n)) \,|\, n \in \mathbb{Z}_{>0}\}$. The following two lemmas give an estimate for indivisibility of Fourier coefficients of modular forms of half-integral weight.

**Lemma 4.4.** *Let $\ell$ be a prime greater than 3. Let $f(z) = \sum_{n=1}^{\infty} a(n)q^n$ be a normalized Hecke eigen newform of weight $2k$, level $M$ with trivial character and let*

$$g(z) = \sum_{n=1}^{\infty} b(n)q^n \in S_{k+1/2}(N, \chi)$$

*be the eigenform given in Theorem 2.1. Assume that $f$ has complex multiplication in the sense of Ribet [11] and let $\lambda$ be a prime in $E$ above $\ell$. If there exists an integer $D'$ such that $\delta(f)D' > 0$, $(D', N) = 1$, $\varepsilon = \left(\frac{D'}{\ell}\right) \neq 0$ and $\mathrm{ord}_\lambda(b(|D'|)) = s_\lambda(g)$, then*

$$\#\left\{D \in S(X) \,\middle|\, \left(\frac{D}{\ell}\right) = \varepsilon,\ \mathrm{ord}_\lambda(b(D)) = s_\lambda(g)\right\} \gg_{f,\ell} \frac{\sqrt{X}}{\log X}.$$

*Proof.* By dividing $g$ by $\lambda^{s_\lambda(g)}$, we may assume $s_\lambda(g) = 0$. If we put

$$b_0(n) = \begin{cases} b(n) & \text{if } (n, N\ell) = 1 \text{ and } \left(\frac{n}{\ell}\right) = \varepsilon, \\ 0 & \text{otherwise,} \end{cases}$$

then

$$g_0(z) = \sum_{n=1}^{\infty} b_0(n)q^n \in S_{k+1/2}(N\ell^2, \chi')$$

for a suitable character $\chi'$. Since $f$ has complex multiplication, then there exists an imaginary quadratic field $K$ such that for every prime $p$ satisfying $p \equiv 3 \mod 4$, $(p, N) = 1$ and $\left(\frac{\Delta_K}{p}\right) = -1$ we have $a(p) = 0$, where $\Delta_K$ is the discriminant of $K$. Therefore, for such $p$, using the formulae for the action of Hecke operator $T_{p^2}$, we find that

$$b(p^2 n) + \chi'(p)p^{k-1}\left(\frac{(-1)^k n}{p}\right) b(n) + \chi'(p^2)p^{2k-1} b(n/p^2) = 0.$$

Hence if $(r, t) = 1$, $4 \,|\, t$, $r \equiv 3 \mod 4$, then

$$\#T(r, t, X) = \#\{p \in T(r, t) \,|\, p \le X\} \gg_f \frac{X}{\log X}$$

and for any $p \in T(r,t)$ we have

$$b(p^2n) = -\chi'(p)p^{k-1}\left(\frac{(-1)^k n}{p}\right)b(n) - {\chi'}^2(p)p^{2k-1}b(n/p^2). \tag{4.1}$$

Put $\kappa = (k+\frac{1}{2})\frac{[\Gamma_0(1):\Gamma_0(N\ell^2)]}{12} + 1$. Now, we choose $(r_0, t_0)$ satisfying the following properties:

(1) $N\ell^2|t_0$, $(r_0,t_0)=1$, $\chi'(r_0)=1$ and $p \equiv 3 \mod 4$.
(2) If $p$ is a prime with $p \equiv r_0 \mod t_0$, then $\left(\frac{(-1)^k n}{p}\right) = -1$ for any $1 \le n \le \kappa$ with $(n, N\ell^2) = 1$.
(3) For each prime $p \equiv r_0 \mod t_0$ we have $\left(\frac{\Delta_K}{p}\right) = -1$.
(4) Each prime $p \equiv r_0 \mod t_0$ satisfies $\left|\chi'(p^2)p - \chi'(p)\left(\frac{(-1)^k|D'|}{p}\right)\right|_\lambda = 1$.

If $p \in T(r_0,t_0)$ is a sufficiently large prime, for all $1 \le n \le \kappa$

$$u_p(pn) = b_0(p^2n) = -\chi'(p)p^{k-1}\left(\frac{(-1)^k n}{p}\right)b_0(n) - p^{2k-1}{\chi'}^2(p)b_0(n/p^2).$$

Since $b_0(n/p^2) = 0$, we have $u_p(pn) = \chi'(p)p^{k-1}b_0(n) = p^{k-1}b_0(p) = p^{k-1}v_p(pn)$. By the relation (4.1),

$$v_p(p^3|D'|) = b_0(p^2|D'|) = -\chi'(p)p^{k-1}\left(\frac{(-1)^k|D'|}{p}\right)b_0(|D'|)$$

and

$$u_p(p^3|D'|) = b_0(p^4|D'|) = -p^{2k-1}\chi'(p^2)b_0(|D'|).$$

Therefore by the assumption and the choice of $(r_0,t_0)$,

$$\begin{aligned}&\left|u_p(p^3|D'|) - p^{k-1}v_p(p^3|D'|)\right|_\lambda \\ &\quad = \left|\left(\chi'(p^2)p^{2k-1} - \chi'(p)p^{2k-2}\left(\frac{(-1)^k|D'|}{p}\right)\right)b_0(|D'|)\right|_\lambda = 1.\end{aligned}$$

Hence

$$\mathrm{ord}_\lambda(U_p g_0 - p^{k-1}V_p g_0) < +\infty.$$

By Theorem 4.1 and Lemma 4.3, there exists an integer $n_p$ such that

$$1 \le n_p \le \left(k+\frac{1}{2}\right)\frac{[\Gamma_0(1):\Gamma_0(N\ell^2p)]}{12} = \kappa(p+1),\ (n_p,p)=1$$

and

$$b_0(n_pp) = u_p(n_p) \not\equiv p^{k-1}v_p(n_p) = 0 \mod \lambda.$$

Consequently, let $D_{\mathrm{sf}}$ be the square-free part of $D = n_pp$; then

$$|b_0(D_{\mathrm{sf}})|_\lambda = 1.$$

For convenience, let $p_i$ be the primes in $T(r_0,t_0)$ in increasing order, and let $D_i$ be the square-free part of $p_in_{p_i}$. If $r<s<t$ and $D_r = D_s = D_t$, then $p_rp_sp_t|D_r$. However this can only occur for finitely many $r,s$ and $t$ since $|D_i| < \kappa p_i(p_i+1)$. Therefore, the number of distinct $|D_i| < X$ is at least half the number of $p \in T(r_0,t_0)$ with $p \le \sqrt{X/\kappa}$. Therefore the lemma follows from $\#T(r_0,t_0,X) \gg_{f,\lambda} X/\log X$. □

**Lemma 4.5.** *Let $f(z) = \sum_{n=1}^{\infty} a(n)q^n$ be a normalized Hecke eigen newform of weight $2k$, level $M$ with trivial character. Denote $E = \mathbb{Q}(\{a(n)|n \geq 1\})$ and let*

$$g(z) = \sum_{n=1}^{\infty} b(n)q^n \in S_{k+1/2}(N, \chi)$$

*be the eigenform given in Theorem 2.1. We fix a prime number $\ell$ greater than 3 and let $\lambda$ be a prime in $E$ above $\ell$. Assume that $f$ does not have complex multiplication and the image of the Galois representation associated to $f$*

$$\rho_{f,\ell} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathcal{O}_{E,\ell})$$

*coincides with $A_\ell$. If there exists an integer $D'$ such that $\delta(f)D' > 0$, $(D', N) = 1$, $\varepsilon = \left(\frac{D'}{\ell}\right) \neq 0$ and $\mathrm{ord}_\lambda(b(|D'|)) = s_\lambda(g)$, then*

$$\#\left\{D \in S(X) \,\middle|\, \left(\frac{D}{\ell}\right) = \varepsilon,\ \mathrm{ord}_\lambda(b(D)) = s_\lambda(g)\right\} \gg_{f,\lambda} \frac{\sqrt{X}}{\log X}.$$

*Proof.* First, we may assume $\mathrm{ord}_\lambda(g) = 0$. If we put

$$b_0(n) = \begin{cases} b(n) & \text{if } (n, N\ell) = 1 \text{ and } \left(\frac{n}{\ell}\right) = \varepsilon, \\ 0 & \text{otherwise,} \end{cases}$$

then

$$g_0(z) = \sum_{n=1}^{\infty} b_0(n)q^n \in S_{k+1/2}(N\ell^2, \chi')$$

for a suitable character $\chi'$. If $a(p) \equiv 0 \mod \lambda$, by the formula for the action of Hecke operator $T_{p^2}$ we find that

$$b(p^2n) + \chi'(p)p^{k-1}\left(\frac{(-1)^k n}{p}\right) b(n) + {\chi'}^2(p)p^{2k-1}b(n/p^2) \equiv 0 \mod \lambda.$$

By the assumption, $\ell$ is not exceptional. Hence Theorem 3.3 implies

$$\#T(\lambda, r, t, X) = \#\{p \in T(\lambda, r, t) \,|\, p \leq X\} \gg_{f,\lambda} \frac{X}{\log X}$$

and for each $p \in T(\lambda, r, t)$

$$b(p^2n) \equiv -\chi'(p)p^{k-1}\left(\frac{(-1)^k n}{p}\right) b(n) - {\chi'}^2(p)p^{2k-1}b(n/p^2) \mod \lambda. \tag{4.2}$$

Let $\kappa$ be the number as in the proof of Lemma 3.4. Now, we choose $(r_0, t_0)$ satisfying the following properties:

(1) $N\ell^2 | t_0$, $(r_0, t_0) = 1$, $\chi'(r_0) = 1$.

(2) If $p$ is a prime with $p \equiv r_0 \mod t_0$, then $\left(\frac{(-1)^k n}{p}\right) = -1$ for any $1 \leq n \leq \kappa$ with $(n, N\ell^2) = 1$.

(3) For each prime $p \equiv r_0 \mod t_0$ we have $\left(\frac{(-1)^k |D'|}{p}\right) = -1$.

(4) Each prime $p \equiv r_0 \mod t_0$ has the property that $1 + p \not\equiv 0 \mod \lambda$.

If $p \in T(\lambda, r_0, t_0)$ is a sufficiently large prime, for all $1 \le n \le \kappa$ with $(n, N\ell^2) = 1$, then

$$\begin{aligned} u_p(pn) = b_0(p^2 n) &\equiv -p^{k-1}\left(\frac{(-1)^k n}{p}\right) b_0(n) - p^{2k-1} b_0(n/p^2) \\ &= p^{k-1} b_0(n) = p^{k-1} v_p(pn) \mod \lambda. \end{aligned}$$

By the relation (4.2), we have

$$v_p(p^3|D'|) = b_0(p^2|D'|) \equiv p^{k-1} b_0(|D'|) \mod \lambda,$$

also

$$u_p(p^3|D'|) = b_0(p^4|D'|) \equiv -p^{2k-1} b_0(|D'|) \mod \lambda.$$

Therefore by assumption and the choice of $(r_0, t_0)$,

$$p^{k-1} v_p(p^3|D'|) - u_p(p^3|D'|) \equiv p^{2k-2}(1+p) b_0(|D'|) \not\equiv 0 \mod \lambda.$$

Hence

$$\mathrm{ord}_\lambda(U_p g_0 - p^{k-1} V_p g_0) < +\infty.$$

By Theorem 4.1 and Lemma 4.3, there exists an integer $n_p$ such that

$$1 \le n_p \le (k + 1/2)[\Gamma_0(1) : \Gamma_0(N\ell^2 p)]/12 = \kappa(p+1),\ (n_p, p) = 1$$

and

$$b_0(n_p p) = u_p(n_p) \not\equiv v_p(n_p) = 0 \mod \lambda.$$

In particular, let $D_{\mathrm{sf}}$ be the square-free part of $D = n_p p$; then

$$|b_0(D_{\mathrm{sq}})|_\lambda = 1.$$

Now the lemma follows from the same argument as in the proof of the previous lemma. □

*Proof of Theorem 1.1.* Let

$$g(z) = \sum_{n=1}^{\infty} b(n) q^n \in S_{k+1/2}(N, \chi)$$

be the eigenform given in Theorem 2.1 for $f$.

By replacing $f$ by a suitable quadratic twist of $f$ if necessary, we may assume that $\varepsilon = \delta(f)$, where $\varepsilon$ is the sign of the functional equation of $L(f, s)$. By the result of Friedberg and Hoffstein [4], we can take an integer $D'$ such that $\delta(f)D' > 0$, $(D', 2N) = 1$ and $b(D') \neq 0$. In particular, for almost all finite places $\lambda$ of $E$ we have

$$|b(D')|_\lambda = 1.$$

Thus by Lemmas 4.4, 4.5, Theorem 2.1 and Theorem 3.3, for all but finitely many primes $\lambda$ we have

$$\#\left\{ D \in S(X) \,\middle|\, \delta(f) \cdot D > 0,\ (\ell, D) = 1 \text{ and } \left| \frac{L(f \otimes \chi_D, k) D_0^{k-1/2}}{\Omega_f} \right|_\lambda = 1 \right\} \gg_{f,\lambda} \frac{\sqrt{X}}{\log X}.$$

This completes the proof. □

## 5. Indivisibility for the non-central critical values

In this section, we consider a special case for non-central values of $L$-functions for modular forms. We fix a prime $\ell$ greater than 7 and let $f = \sum_{n=1}^{\infty} a(n)q^n$ be a normalized Hecke eigenform of weight $\ell + 1$ for $\mathrm{SL}_2(\mathbb{Z})$. Let $\lambda$ be a prime in a number field $E$. We assume that the integer ring of $E$ contains all Fourier coefficients of $f$ and choose a period $\Omega_f^{\pm}$ as in Ash-Stevens [1, Theorem 4.5]. Then for any Dirichlet character $\chi$, the quotient $\tau(\chi^{-1})\dfrac{L(f \otimes \chi, 1)}{(2\pi i)\Omega_f^{\pm}}$ is an integer in $E_\lambda(\chi)$ where $\tau$ is the Gauss sum and $\pm = \chi(-1)$.

**Theorem 5.1.** *Let $\lambda$ be a prime in $E$ above $\ell$. We assume the following conditions.*

(1) *There exists a unique eigenform $F$ of weight 2 for $\Gamma_0(\ell)$ such that*
$$F \equiv f \mod \lambda.$$
(2) *$\ell$ is not exceptional.*
(3) *There exists a square-free negative integer $d_0$ such that $(d_0, 2\ell) = 1$, $\left(\dfrac{d_0}{p}\right)\chi_{d_0}(\ell) = -\varepsilon(F)$, where $\varepsilon(F)$ is the sign of functional equation of $L(F, s)$ and*
$$\frac{L(f \otimes \chi_{d_0}, 1)\sqrt{d_0}}{(2\pi i)\Omega_f^{\pm}} \not\equiv 0 \mod \lambda.$$

*Then we have*
$$\#\left\{ D \in S(X) \,\middle|\, \frac{L(f \otimes \chi_D, 1)\sqrt{D}}{(2\pi i)\Omega_f^{\pm}} \not\equiv 0 \mod \lambda \right\} \gg_{f,\lambda} \frac{\sqrt{X}}{\log X}.$$

For the proof, we recall a result of Ash and Stevens.

**Theorem 5.2** (Ash-Stevens, [1])**.** *Let $k$ be a positive integer less than $\ell + 2$ and $f = \sum_{n=1}^{\infty} a(n)q^n \in S_k(\Gamma_0(1))$ an eigenform satisfying the assumptions of Theorem 5.1. We fix a prime $\lambda$ above $\ell$ in a number field $E$ which contains all Fourier coefficients of $f$. Assume that*

(1) *There exists a prime $q$ satisfying $a(q) \not\equiv q^{k-1} + 1 \mod \lambda$.*
(2) *There exists a unique eigenform $F \in S_2(\Gamma_1(\ell))$ such that $f \equiv F \mod \lambda$.*

*Then there exists a complex number $\Omega_F^{\pm}$ such that for any Dirichlet character $\chi$ satisfying $(\mathrm{cond}\,\chi, p) = 1$, we have*
$$\frac{\tau(\chi^{-1})L(f \otimes \chi, 1)}{(2\pi i)\Omega_f^{\pm}} \equiv \frac{\tau(\chi^{-1})L(F \otimes \chi, 1)}{(2\pi i)\Omega_F^{\pm}} \mod \lambda.$$

Now we prove Theorem 5.1. By the Kohnen-Zagier formula [6], there exists an eigenform
$$g(z) = \sum_{n=1}^{\infty} b(n)q^n \in S_{3/2}(\Gamma_0(4\ell))$$
such that for any negative square-free integer $D$ satisfying $\left(\dfrac{D}{\ell}\right) = -\varepsilon(F)$,
$$|b(|D|)|^2 = 2 \cdot \frac{\sqrt{D}}{\pi} \cdot \frac{\langle g, g\rangle}{\langle F, F\rangle} L(F \otimes \chi_D, 1),$$

where $\langle\cdot,\cdot\rangle$ is the Petersson inner product. We can normalize $g$ by the relation $\frac{\langle F,F\rangle}{\langle g,g\rangle} = \Omega_f^{\pm}$. Taking a linear combination of twists of $g$, one may assume $b(|D|) = 0$ if $\left(\frac{D}{\ell}\right) \neq -\varepsilon(F)$ and $D < 0$. From the assumptions of the theorem, $\ell$ is not exceptional. This implies the existence of a prime $q$ satisfying $a(q) \not\equiv q^{k-1} + 1 \mod \lambda$, therefore the assumptions of Theorem 5.1 imply the assumptions of Theorem 5.2. Since $\tau(\chi_D)^{-1} = \pm 1/\sqrt{D}$, one can see that

$$\frac{L(f\otimes\chi,1)\sqrt{D}}{(2\pi i)\Omega_f^{\pm}} \equiv \frac{L(F\otimes\chi,1)\sqrt{D}}{(2\pi i)\Omega_F^{\pm}} = |b(|D|)|^2\cdot c \mod \lambda$$

with a $\lambda$-adic unit $c$. By the assumption (3), we have

$$\mathrm{ord}_\lambda\left(\frac{L(f\otimes\chi_{d_0},1)\sqrt{d_0}}{(2\pi i)\Omega_f^{\pm}}\right) = 0$$

therefore $\mathrm{ord}_\lambda(b(d_0)) = \min\{\mathrm{ord}_\lambda(b(n))\,|\,n : \text{square-free}, \chi_{d_0}(\ell) = -\varepsilon(f)\}$. Hence Lemma 4.5 implies

$$\#\big\{D\in S(X)\,\big|\,\chi_D(\ell) = -\varepsilon(f),\ \mathrm{ord}_\lambda(b(D)) = s\big\} \gg_{f,\lambda} \frac{\sqrt{X}}{\log X},$$

thus we have

$$\#\left\{D\in S(X)\,\middle|\,\frac{L(f\otimes\chi_D,1)\sqrt{D}}{(2\pi i)\Omega_f^{\pm}} \not\equiv 0 \mod \lambda\right\} \gg_{f,\lambda} \frac{\sqrt{X}}{\log X}.$$

This completes the proof.

*Remark* 5.3. Lemma 4.5 is stated only for $g$ given in Theorem 2.1, but one can show a similar result for any eigenform $g\in S_{k+1/2}(N,\chi)$ if $k\geq 2$ ($S'_{\frac{3}{2}}(N,\chi)$ if $k=1$) corresponding to some eigenform $f\in S_{2k}(\Gamma_0(M))$ under the Shimura correspondence.

**Example 5.4.** Let

$$f = \Delta = q\prod_{n=1}^{\infty}(1-q^n)^{24} \in S_{12}(\Gamma_0(1))$$

and

$$F = q\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{11n})^2 \in S_2(\Gamma_0(11)).$$

Then it is well known that $f\equiv F \mod 11$, $\dim S_2(\Gamma_0(11)) = 1$ and the mod 11 Galois representation associated to $f$ is surjective. Moreover one can check that

$$\frac{L(\Delta\otimes\chi_{-3},1)}{\Omega^{+}_{\Delta\otimes\chi_{-3}}} = 36741600 \not\equiv 0 \mod 11$$

by using MAGMA. So the assumptions of Theorem 5.1 are satisfied for $f = \Delta$. Hence we have

$$\#\left\{D\in S(X)\,\middle|\,\frac{L(\Delta\otimes\chi_D,1)\sqrt{D}}{(2\pi i)\Omega_\Delta^{\pm}} \not\equiv 0 \mod 11\right\} \gg \frac{\sqrt{X}}{\log X}.$$

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## References

[1] Avner Ash and Glenn Stevens, *Modular forms in characteristic $l$ and special values of their $L$-functions*, Duke Math. J. **53** (1986), no. 3, 849–868, DOI 10.1215/S0012-7094-86-05346-9. MR860675 (88h:11036)

[2] Jan Hendrik Bruinier, *Nonvanishing modulo $l$ of Fourier coefficients of half-integral weight modular forms*, Duke Math. J. **98** (1999), no. 3, 595–611, DOI 10.1215/S0012-7094-99-09819-8. MR1695803 (2000d:11061)

[3] Pierre Deligne, *Formes modulaires et représentations $l$-adiques* (French), Séminaire Bourbaki. Vol. 1968/69: Exposés 347–363, Lecture Notes in Math., vol. 175, Springer, Berlin, 1971, pp. Exp. No. 355, 139–172. MR3077124

[4] Solomon Friedberg and Jeffrey Hoffstein, *Nonvanishing theorems for automorphic $L$-functions on* GL(2), Ann. of Math. (2) **142** (1995), no. 2, 385–423, DOI 10.2307/2118638. MR1343325 (96e:11072)

[5] Kevin James and Ken Ono, *Selmer groups of quadratic twists of elliptic curves*, Math. Ann. **314** (1999), no. 1, 1–17, DOI 10.1007/s002080050283. MR1689260 (2000j:11095)

[6] Winfried Kohnen, *Fourier coefficients of modular forms of half-integral weight*, Math. Ann. **271** (1985), no. 2, 237–268, DOI 10.1007/BF01455989. MR783554 (86i:11018)

[7] Winfried Kohnen and Ken Ono, *Indivisibility of class numbers of imaginary quadratic fields and orders of Tate-Shafarevich groups of elliptic curves with complex multiplication*, Invent. Math. **135** (1999), no. 2, 387–398, DOI 10.1007/s002220050290. MR1666783 (2000c:11087)

[8] B. Mazur, J. Tate, and J. Teitelbaum, *On $p$-adic analogues of the conjectures of Birch and Swinnerton-Dyer*, Invent. Math. **84** (1986), no. 1, 1–48, DOI 10.1007/BF01388731. MR830037 (87e:11076)

[9] Ken Ono and Christopher Skinner, *Non-vanishing of quadratic twists of modular $L$-functions*, Invent. Math. **134** (1998), no. 3, 651–660, DOI 10.1007/s002220050275. MR1660945 (2000a:11077)

[10] Ken Ono and Christopher Skinner, *Fourier coefficients of half-integral weight modular forms modulo $l$*, Ann. of Math. (2) **147** (1998), no. 2, 453–470, DOI 10.2307/121015. MR1626761 (99f:11059a)

[11] Kenneth A. Ribet, *Galois representations attached to eigenforms with Nebentypus*, Modular functions of one variable, V (Proc. Second Internat. Conf., Univ. Bonn, Bonn, 1976), Springer, Berlin, 1977, pp. 17–51. Lecture Notes in Math., Vol. 601. MR0453647 (56 #11907)

[12] Kenneth A. Ribet, *On $l$-adic representations attached to modular forms. II*, Glasgow Math. J. **27** (1985), 185–194, DOI 10.1017/S0017089500006170. MR819838 (88a:11041)

[13] Jean-Pierre Serre, *Congruences et formes modulaires [d'après H. P. F. Swinnerton-Dyer]* (French), Séminaire Bourbaki, 24e année (1971/1972), Exp. No. 416, Springer, Berlin, 1973, pp. 319–338. Lecture Notes in Math., Vol. 317. MR0466020 (57 #5904a)

[14] Goro Shimura, *On modular forms of half integral weight*, Ann. of Math. (2) **97** (1973), 440–481. MR0332663 (48 #10989)

[15] Jacob Sturm, *On the congruence of modular forms*, Number theory (New York, 1984), Lecture Notes in Math., vol. 1240, Springer, Berlin, 1987, pp. 275–280, DOI 10.1007/BFb0072985. MR894516 (88h:11031)

[16] H. P. F. Swinnerton-Dyer, *On $l$-adic representations and congruences for coefficients of modular forms*, Modular functions of one variable, III (Proc. Internat. Summer School, Univ. Antwerp, 1972), Springer, Berlin, 1973, pp. 1–55. Lecture Notes in Math., Vol. 350. MR0406931 (53 #10717a)

[17] J.-L. Waldspurger, *Sur les coefficients de Fourier des formes modulaires de poids demi-entier* (French), J. Math. Pures Appl. (9) **60** (1981), no. 4, 375–484. MR646366 (83h:10061)

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