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REALIZATION OF AN EQUIVARIANT HOLOMORPHIC HERMITIAN LINE BUNDLE AS A QUILLEN DETERMINANT BUNDLE

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Abstract. Let M be an irreducible smooth complex projective variety equipped with an action of a compact Lie group G, and let $(\mathcal{L}\,, h)$ be a G–equivariant holomorphic Hermitian line bundle on M. Given a compact connected Riemann surface X, we construct a G–equivariant holomorphic Hermitian line bundle $(L\,,H)$ on $X \times M$ (the action of G on X is trivial) such that the corresponding Quillen determinant line bundle $(\mathcal{Q}\,,h_Q)$, which is a G–equivariant holomorphic Hermitian line bundle on M, is isomorphic to the given G–equivariant holomorphic Hermitian line bundle $(\mathcal{L}\,,h)$. This proves a conjecture by Dey and Mathai (2013).

1. Introduction

This work was inspired by [DM], where the following result is proved. Let M be an irreducible smooth complex projective variety and $\mathcal{L}$ an ample line bundle on M equipped with a Hermitian structure h of positive curvature. There is a natural family of Cauchy–Riemann operators on $\mathbb{CP}^1$, parametrized by M, such that the corresponding Quillen determinant line bundle, which is a holomorphic Hermitian line bundle on M, is holomorphically isomorphic to a positive tensor power of $(\mathcal{L}\,,h)$ [DM, p. 785, Theorem 1.1]. It is conjectured in [DM] that an equivariant version also holds (see [DM, p. 793, §5]).

Let M be as before. Assume that it is equipped with a C^∞ action of a compact Lie group G via holomorphic automorphisms of M. Let $(\mathcal{L}\,,h)$ be any G–equivariant holomorphic Hermitian line bundle on M.

Let X be a compact connected Riemann surface equipped with a Kähler form ω_X. Let L_0 be a holomorphic line bundle on X of degree genus$(X)-2$ such that $H^0(X,\, L_0) = 0$. Fix a Hermitian structure h_0 on L_0.

The action of G on M and the trivial action of G on X together produce an action of G on $X \times M$. Let p_1 and p_2 be the projections of $X \times M$ on X and M respectively. Consider the Hermitian structure $H := (p_1^*h_0)\otimes(p_2^*h)$ on the holomorphic line bundle

$$L := (p_1^*L_0)\otimes(p_2^*\mathcal{L})$$

over $X\times M$. The action of G on $\mathcal{L}$ and the trivial action of G on L_0 together produce an action of G on L, thus making L a G–equivariant holomorphic Hermitian line bundle on $X \times M$. We will consider $(L\,,H)$ as a family of holomorphic Hermitian

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line bundles on X. Let $(\mathcal{Q}, h_Q)$ be the Quillen determinant line bundle associated to the triple (L, H, ω_X). It is a G–equivariant holomorphic Hermitian line bundle on M.

We prove the following (see Theorem 2.4):

The two G–equivariant holomorphic Hermitian line bundle on M, namely $(\mathcal{L}, h)$ and $(\mathcal{Q}, h_Q)$, are isomorphic.

We note that this proves the earlier mentioned conjecture in [DM].

2. A holomorphic family and its determinant bundle

Let X be a compact connected Riemann surface. Let g be the genus of X. Fix a holomorphic line bundle L_0 on X of degree $g-2$ such that

$$H^0(X, L_0) = 0 . \tag{2.1}$$

We note that such a line bundle exists. Indeed, if $g \leq 1$, then any holomorphic line bundle of degree $g-2$ works; if $g = 2$, then any nontrivial holomorphic line bundle of degree zero works; if $g \geq 2$, then any point outside the image of the Abel-Jacobi map $\mathrm{Sym}^{g-2}(X) \longrightarrow \mathrm{Pic}^{g-2}(X)$ works. From Riemann–Roch it follows that

$$\dim H^1(X, L_0) = 1 . \tag{2.2}$$

Fix a C^∞ Hermitian structure h_0 on L_0. Also, fix a Kähler form ω_X on X.

Let M be a connected complex projective manifold (meaning a connected smooth complex projective variety). Assume that a compact Lie group G acts smoothly on M via holomorphic automorphisms. Let $(\mathcal{L}, h)$ be a G–equivariant holomorphic Hermitian line bundle on M. This means that the holomorphic line bundle L is equipped with an action of G such that

(1) for each element $z \in G$, the action of z on $\mathcal{L}$ is a holomorphic automorphism of the line bundle $\mathcal{L}$ over the automorphism of M given by the action of z on M,
(2) the action of G on $\mathcal{L}$ is C^∞ and it preserves h.

Let p_1 and p_2 be the projections on $X \times M$ to X and M respectively. Consider the holomorphic line bundle

$$L := (p_1^* L_0) \otimes (p_2^* \mathcal{L}) \longrightarrow X \times M .$$

It is equipped with the Hermitian structure

$$H := (p_1^* h_0) \otimes (p_2^* h) . \tag{2.3}$$

The action of G on M and the trivial action of G on X together define an action of G on $X \times M$. Similarly, the action of G on $\mathcal{L}$ and the trivial action of G on L_0 together define an action of G on L. This action of G on L clearly preserves H.

Consider (L, H) as a family of holomorphic Hermitian line bundles on X parametrized by M. Let

$$(\mathcal{Q}, h_Q) \longrightarrow M$$

be the Quillen determinant line bundle associated to (L, H, ω_X) [Qu]. For any point $y \in M$, let (L^y, H^y) be the holomorphic Hermitian line bundle on X obtained by restricting (L, H) to $X \times \{y\}$. Note that (L^y, H^y) is isomorphic to the holomorphic Hermitian line bundle (L_0, h_0). We recall that the fiber $\mathcal{Q}_y$ is identified with the complex line $\bigwedge^{\mathrm{top}} H^0(X, L^y)^* \otimes \bigwedge^{\mathrm{top}} H^1(X, L^y)$ [Qu]. In view of (2.1) and (2.2), the fiber $\mathcal{Q}_y$ is identified with $H^1(X, L^y)$.

The action of G on L produces an action of G on $\mathcal{Q}$. The action of any $z \in G$ on $\mathcal{Q}$ is a holomorphic automorphism of the line bundle $\mathcal{Q}$ over the automorphism of M given by z. The action of G on $\mathcal{Q}$ preserves the Hermitian structure h_Q on $\mathcal{Q}$ because the action of G on L preserves H and the trivial action on X preserves ω_X.

Let

$$\xi := M \times H^1(X,\, L_0) \longrightarrow M \tag{2.4}$$

be the holomorphically trivial line bundle with fiber $H^1(X,\, L_0)$ (see (2.2)). The trivial action of G on $H^1(X,\, L_0)$ and the action of G on M together define an action of G on ξ. The actions of G on $\mathcal{L}$ and ξ together produce an action of G on $\mathcal{L} \otimes \xi$ that is a lift of the action of G on M.

Lemma 2.1. *The holomorphic line bundle $\mathcal{Q}$ over M is identified with $\mathcal{L} \otimes \xi$. This identification is G–equivariant.*

Proof. From (2.1) it follows that

$$R^0 p_{2*} L \,=\, 0$$

(recall that L^y is isomorphic to L_0). By the projection formula [Ha, p. 253, Ex. 8.3], we have

$$R^1 p_{2*} L \,=\, \mathcal{L} \otimes R^1 p_{2*}(p_1^* L_0)\,.$$

But $R^1 p_{2*}(p_1^* L_0) \,=\, \xi$. Therefore, we get an isomorphism

$$\tau \,:\, \mathcal{Q} := \mathrm{Det}(L) \,=\, R^1 p_{2*} L \xrightarrow{\sim} \mathcal{L} \otimes \xi\,. \tag{2.5}$$

From the construction of τ it follows immediately that the isomorphism intertwines the actions of G on $\mathcal{Q}$ and $\mathcal{L} \otimes \xi$. □

Let ∇^Q be the Chern connection on $\mathcal{Q}$ for the Hermitian structure h_Q. The curvature of ∇^Q will be denoted by $\mathcal{K}(\nabla^Q)$. The curvature $\mathcal{K}(\nabla^Q)$ can be computed using [Qu], [BGS].

Let $\nabla^{\mathcal{L}}$ denote the Chern connection for $(\mathcal{L}\,, h)$. Its curvature will be denoted by $\mathcal{K}(\nabla^{\mathcal{L}})$.

Proposition 2.2. *The two $(1,1)$–forms $\mathcal{K}(\nabla^Q)$ and $\mathcal{K}(\nabla^{\mathcal{L}})$ on M coincide.*

Proof. The Chern connection on the holomorphic Hermitian line bundle $(L_0\,, h_0)$ (respectively, $(L\,, H)$) will be denoted by ∇^{L_0} (respectively, ∇^L). Let $\mathcal{K}(\nabla^{L_0})$ (respectively, $\mathcal{K}(\nabla^L)$) be the curvature of ∇^{L_0} (respectively, ∇^L). From the definition of H (see (2.3)) it follows immediately that

$$\mathcal{K}(\nabla^L) \,=\, p_1^*\mathcal{K}(\nabla^{L_0}) + p_2^*\mathcal{K}(\nabla^{\mathcal{L}})\,. \tag{2.6}$$

Let $\mathcal{K}(\omega_X) \in C^\infty(X;\, \Omega_X^{1,1})$ be the curvature of TX for the Kähler form ω_X.

A theorem due to Quillen and Bismut–Gilet–Soulé says that $\mathcal{K}(\nabla^Q)$ is given by the following fiber integral along X:

(2.7)
$$\mathcal{K}(\nabla^Q) \,=\, -\frac{1}{2\pi\sqrt{-1}}\left(\int_{(X\times M)/M} (\mathcal{K}(\nabla^L) + \frac{1}{2}\mathcal{K}(\nabla^L)^2) \wedge (1 + \frac{1}{2}p_1^*\mathcal{K}(\omega_X))\right)_2$$

[BGS, p. 51, Theorem 0.1], [Qu], where $(\beta)_2$ denotes the component of the differential form β of degree two; note that $\frac{1}{4\pi\sqrt{-1}}\mathcal{K}(\omega_X)$ is the Todd form on X for the Kähler form ω_X that represents the Todd class $\frac{1}{2}c_1(TX)$. Using (2.6), the expression in (2.7) reduces to

$$2\pi\sqrt{-1}\cdot\mathcal{K}(\nabla^{Q}) \;=\; -\mathcal{K}(\nabla^{\mathcal{L}})\cdot\int_X(\mathcal{K}(\nabla^{L_0})+\frac{1}{2}\mathcal{K}(\omega_X))\,. \tag{2.8}$$

Now note that

$$\frac{1}{2\pi\sqrt{-1}}\int_X(\mathcal{K}(\nabla^{L_0})+\frac{1}{2}\mathcal{K}(\omega_X)) \;=\; \text{degree}(L_0)+\frac{1}{2}\text{degree}(TX) \;=\; g-2+1-g \;=\; -1\,.$$

Using this, from (2.8) we conclude that $\mathcal{K}(\nabla^{Q}) \;=\; \mathcal{K}(\nabla^{\mathcal{L}})$. □

The Hermitian structure L_0 and the Kähler form ω_X together produce an inner product on $H^1(X,\,L_0)$. This inner product defines a Hermitian structure h_ξ on the holomorphic line bundle ξ in (2.4). Note that the Chern connection on ξ for h_ξ is flat.

The Hermitian structure h on $\mathcal{L}$ and the Hermitian structure h_ξ on ξ together produce a Hermitian structure $\widetilde{h}$ on $\mathcal{L}\otimes\xi$.

Proposition 2.3. *For the isomorphism τ in* (2.5)*, there is a positive real number t such that $\tau^*\widetilde{h} \;=\; t\cdot h_Q$.*

Proof. There is a real valued C^∞ function f on M such that

$$\tau^*\widetilde{h} \;=\; \exp(f)\cdot h_Q\,.$$

From Proposition 2.2 it follows the two holomorphic Hermitian line bundles $(\mathcal{Q}\,,h_Q)$ and $(\mathcal{Q}\,,\exp(f)\cdot h_Q)$ have the same curvature. This implies that f is a harmonic function. Since M is compact and connected, any harmonic function on it is a constant one. □

Theorem 2.4. *The two G–equivariant holomorphic Hermitian line bundles $(\mathcal{L}\,,h)$ and $(\mathcal{Q}\,,h_Q)$ are isomorphic.*

Proof. Consider the isomorphism

$$\frac{1}{\sqrt{t}}\cdot\tau\;:\;\mathcal{Q}\;\longrightarrow\;\mathcal{L}\otimes\xi\,,$$

where τ is the isomorphism in (2.5), and t is the constant in Proposition 2.3. From Lemma 2.1 and Proposition 2.3 it follows immediately that this is a G–equivariant holomorphic isomorphism that takes the Hermitian structure h_Q on $\mathcal{Q}$ to the Hermitian structure $\widetilde{h}$ on $\mathcal{L}\otimes\xi$.

The two G–equivariant holomorphic Hermitian line bundles $(\mathcal{L}\,,h)$ and $(\mathcal{L}\otimes\xi\,,\widetilde{h})$ are clearly isomorphic. Therefore, the two G–equivariant holomorphic Hermitian line bundles $(\mathcal{L}\,,h)$ and $(\mathcal{Q}\,,h_Q)$ are isomorphic. □

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