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NON-HYPERBOLIC MINIMAL SETS FOR TRIDIAGONAL COMPETITIVE-COOPERATIVE SYSTEMS

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Abstract. The dynamics on non-hyperbolic minimal sets is investigated for non-linear competitive-cooperative tridiagonal systems in time-recurrent structures including almost periodicity and almost automorphy. With the help of exponential separation of the Floquet bundles proved in a previous work of the present authors, we prove that the skew-product flow on a minimal set Y is topologically conjugate to a minimal flow in $\mathbb{R}^1 \times H(f)$ (where $H(f)$ is the hull of f), provided that the center-space associated with Y is one-dimensional. In particular, if Y is uniquely ergodic, then Y can be embedded into $\mathbb{R}^1 \times H(f)$. We further propose a conjecture in the case that the dimension of the center-space is greater than one.

1. Introduction

Dynamical systems that describe the interaction of $n \geq 2$ species have been the object of intense study ever since the seminal work by Lotka, Volterra and Kolmogorov [12,13,23]. The most commonly studied interactions are those of competition, cooperation and predator-prey interaction. In real ecosystems individuals do not interact with individuals of all the other species in the community. Often the species can be linearly ordered and each individual interacts only with its conspecifics and with individuals of the closest neighboring species. If the dynamics of the ecosystem is described by ordinary differential equations this assumption leads to a possibly non-autonomous system of the form

$$
\begin{aligned}
\dot{x}_1 &= f_1(t, x_1, x_2),\\
\dot{x}_i &= f_i(t, x_{i-1}, x_i, x_{i+1}), \quad 2 \leq i \leq n-1,\\
\dot{x}_n &= f_n(t, x_{n-1}, x_n).
\end{aligned}
\tag{1.1}
$$

Such systems are called *tridiagonal* [8,21]. The most commonly studied tridiagonal systems are those describing *food-chains* and competitive-cooperative systems. In food-chains individuals eat individuals of the closest species below them and are eaten by individuals of the closest species above them, whereas in competitive-cooperative systems individuals either compete or cooperate with their neighboring species.

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In this paper we focus on the dynamics of the non-autonomous tridiagonal system (1.1) in the case of competitive-cooperative interactions. Such systems have been investigated for instance in [3, 5, 21, 22, 24].

We assume that the non-linearity $f = (f_1, f_2, \cdots, f_n)$ is defined on $\mathbb{R} \times \mathbb{R}^n$ and C^1-*admissible*, i.e., f together with its first derivatives with respect to $x = (x_1, x_2, \cdots, x_n)$, is bounded and uniformly continuous on $\mathbb{R} \times K$ for any compact set $K \subset \mathbb{R}^n$. Moreover, we assume that there are $\varepsilon_0 > 0$ and $\delta_i \in \{-1, +1\}$, such that

$$\delta_i \frac{\partial f_i}{\partial x_{i+1}}(t,x) \geq \varepsilon_0, \quad \delta_i \frac{\partial f_{i+1}}{\partial x_i}(t,x) \geq \varepsilon_0, \qquad 1 \leq i \leq n-1,$$

for all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$. If $\delta_i = -1$ for all i, then (1.1) is called *strongly competitive*. If $\delta_i = 1$ for all i, then (1.1) is called *strongly cooperative*. Following [22], let $\hat{x}_i = \mu_i x_i, \mu_i \in \{+1, -1\}, 1 \leq i \leq n$, with $\mu_1 = 1, \mu_i = \delta_{i-1}\mu_{i-1}$. Then (1.1) transforms into a new system of the same type with new $\hat{\delta}_i = \mu_i \mu_{i+1} \delta_i = \mu_i^2 \delta_i^2 = 1$. Therefore one can always assume that (1.1) is in fact strongly cooperative, that is,

$$\frac{\partial f_i}{\partial x_{i+1}}(t,x) \geq \varepsilon_0, \ \frac{\partial f_{i+1}}{\partial x_i}(t,x) \geq \varepsilon_0, \quad 1 \leq i \leq n-1, \ (t,x) \in \mathbb{R} \times \mathbb{R}^n. \tag{1.2}$$

In order to investigate the qualitative properties of (1.1), one usually embeds it into the skew-product (local) flow $\Pi(t, \cdot, \cdot)$ on $\mathbb{R}^n \times H(f)$ defined by

$$\Pi(t, x_0, g) = (x(t, x_0, g), g \cdot t). \tag{1.3}$$

Here $x(t, x_0, g)$ is the solution of

$$\begin{aligned} \dot{x}_1 &= g_1(t, x_1, x_2), \\ \dot{x}_i &= g_i(t, x_{i-1}, x_i, x_{i+1}), \quad 2 \leq i \leq n-1, \\ \dot{x}_n &= g_n(t, x_{n-1}, x_n), \end{aligned} \tag{1.4}$$

with $x(0, x_0, g) = x_0 \in \mathbb{R}^n$, $g = (g_1, \cdots, g_n) \in H(f)$, and $(g \cdot t)(\cdot, \cdot) = g(t + \cdot, \cdot)$. The set $H(f)$ is the hull of f, that is, the closure of $\{f \cdot \tau | \tau \in \mathbb{R}\}$, with respect to the compact-open topology (see [16]). Since f is admissible, $H(f)$ is a compact metric space. We further assume that f is *time-recurrent*, or in other words, that the time-translation flow $g \cdot t$ on $H(f)$ is *minimal*. This means that $H(f)$ is the only non-empty compact subset of itself that is invariant under the flow $g \cdot t$. This is satisfied, for instance, when f is a uniformly almost periodic, or, more generally, a uniformly almost automorphic function (see [24]).

We say that a set $Y \subset \mathbb{R}^n \times H(f)$ is minimal for the skew-product flow Π if it is compact, invariant under Π, and has no proper non-empty compact invariant subsets.

The dynamics of the flow (1.3) on a minimal set $Y \subset \mathbb{R}^n \times H(f)$ has previously been investigated under different assumptions. For instance, if f is independent of t (in which case $H(f) = \{f\}$), Smillie [21] has shown that all bounded trajectories converge to equilibria. The transversality of stable and unstable manifolds of hyperbolic equilibria was then built up by Fusco and Oliva [4]. In the case that f is time periodic with period $T > 0$ (and hence $H(f) \sim S^1$), Y is a T-periodic minimal set in $\mathbb{R}^n \times H(f)$ (see [22]). When $H(f)$ is minimal, Wang [24] has shown that Y is an almost 1-cover of $H(f)$ (see Definition 4.1). By developing the theory of Floquet bundles (more delicate than Sacker-Sell spectral bundles; see Proposition 2.2) for the associated time-dependent linear systems of (1.1), the present authors [3] recently proved that Y is a 1-cover of $H(f)$ provided that it is hyperbolic.

In this article, which is motivated by the work of Shen and Yi [19, 20], we will focus on investigating the dynamics on a non-hyperbolic minimal set Y. More precisely, we show that the flow on $Y \subset \mathbb{R}^n \times H(f)$ is topologically conjugate to a minimal flow in $\mathbb{R}^1 \times H(f)$ (see Theorem 4.2) provided that the center-space of Y is one-dimensional. Together with the exponential separation of the Floquet bundles obtained in [3], this directly implies that any minimal set Y possessing a pure point spectrum (for instance, any Y that is uniquely ergodic) can be embedded into $\mathbb{R}^1 \times H(f)$. We are further led to the conjecture that the flow on $Y \subset \mathbb{R}^n \times H(f)$ is topologically conjugate to a subflow in $\mathbb{R}^m \times H(f)$ if the dimension of the center-space is $m \geq 2$.

We also would like to point out that (1.1) naturally reduces to a general two-dimensional competitive (or cooperative) system when $n = 2$. When such a competitive system is T-periodic, Hale and Somolinos [6, Theorem 4.2] have shown that all bounded solutions are asymptotic to T-periodic solutions. When the system is time almost-periodic, Hetzer and Shen [7, Theorem A] have proved the almost 1-cover lifting property of minimal sets for the associated skew-product flow. Moreover, the hyperbolic omega-limit set has recently been shown to be a 1-cover (see [3, Theorem 4.6]). The next step would be to understand the dynamics on the non-hyperbolic minimal sets. Our result (see Theorem 4.3) seems to be the first attempt to tackle this problem under certain additional assumptions (one-dimension center direction or unique ergodicity), which extends the above-mentioned results in [3, 7]. See [18] for other related extensions.

The paper is organized as follows. In Section 2 we agree on some notation and give relevant definitions and preliminaries. In particular, we introduce an integer-valued Lyapunov function σ and recall results from Floquet bundle theory (established in [3]) for time-dependent linearized systems of (1.1) that will play an important role in the proofs of our main results. In Section 3, we study the separating properties of the local invariant manifolds associated with the minimal set Y in terms of the values of σ. Finally, in Section 4, we investigate the dynamics on a minimal set Y whose center-space is one-dimensional.

2. Preliminaries

In this section, we consider the linearization of (1.4) along the orbit $y \cdot t \triangleq \Pi(t, x_0, g)$, $y = (x_0, g) \in Y$, that is, the equation

$$\dot{x} = B(y \cdot t)x, \quad t \in \mathbb{R},\ x \in \mathbb{R}^n,\ y \in Y, \tag{2.1}$$

where $B(y \cdot t) = Dg(t, x(t, x_0, g))$ is a matrix-valued function of cooperative tridiagonal form.

We denote by $\Phi(t, y)$ the principal fundamental matrix solution of (2.1). Then its associated linear skew-product flow $\pi : \mathbb{R} \times \mathbb{R}^n \times Y \to \mathbb{R}^n \times Y$ is

$$\pi(t, x, y) = (\Phi(t, y)x, y \cdot t). \tag{2.2}$$

For each $\lambda \in \mathbb{R}$, define $\pi_\lambda : \mathbb{R} \times \mathbb{R}^n \times Y \to \mathbb{R}^n \times Y$ by

$$\pi_\lambda(t, x, y) = (\Phi_\lambda(t, y)x, y \cdot t),$$

where $\Phi_\lambda(t, y) = e^{-\lambda t}\Phi(t, y)$. Clearly, π_λ is also a linear skew-product flow on $\mathbb{R}^n \times Y$. We say that π_λ admits an *exponential dichotomy over* Y if there is an invariant projection $Q : \mathbb{R}^n \times Y \to \mathbb{R}^n \times Y$, i.e., $\Phi_\lambda(t,y)Q(y) = Q(y \cdot t)\Phi_\lambda(t,y)$ and positive constants K and α such that for all $y \in Y$,

$$\begin{aligned} |\Phi_\lambda(t,y)(1 - Q(y))| \leq & K e^{-\alpha t}, \quad t \geq 0, \\ |\Phi_\lambda(t,y)Q(y)| \leq & K e^{\alpha t}, \quad t \leq 0. \end{aligned}$$

The set $\Sigma(Y) = \{\lambda \in \mathbb{R} : \pi_\lambda$ admits no exponential dichotomy over $Y\}$ is called the *Sacker-Sell spectrum* of (2.1) (or (2.2)) on Y. Furthermore, if Y is connected, then its Sacker-Sell spectrum $\Sigma(Y)$ is of the form: $\Sigma(Y) = \bigcup_{i=0}^{k-1}[a_i, b_i]$, where $[a_i, b_i]$ are spectral intervals ordered from right to left, that is, $a_{k-1} \leq b_{k-1} < a_{k-2} \leq b_{k-2} < \cdots < a_0 \leq b_0$ (see [15]). Throughout the rest of this paper the letter k will denote the number of intervals in this representation of $\Sigma(Y)$.

For any given i and j with $0 \leq i \leq j \leq k-1$, we denote by $V_{i,j}(Y)$ the *spectral bundle* associated with the spectral intervals $\bigcup_{m=i}^{j}[a_m, b_m]$, that is,

$$V_{i,j}(Y) = \bigcup_{y \in Y} V_{i,j}(y) \times \{y\}$$

with

$$V_{i,j}(y) = \Big\{ x \in \mathbb{R}^n : |\Phi(t,y)x| = o(e^{a_j^- t}) \text{ as } t \to -\infty, \\ |\Phi(t,y)x| = o(e^{b_i^+ t}) \text{ as } t \to \infty \Big\},$$

where a_j^-, b_i^+ are such that $b_{j+1} < a_j^- < a_j$ and $b_i < b_i^+ < a_{i-1}$. Here, we set $a_{-1} = +\infty$, $b_k = -\infty$. It follows from the results in [15] that the definition of $V_{i,j}(y)$ does not depend on the specific choice of a_j^- and b_i^+ as long as they belong to the specified intervals. For any $0 \leq i \leq k-1$, if we take $\lambda \in (b_i, a_{i-1})$, then

$$V_{0,i-1}(y) = R(Q(y)) \text{ and } V_{i,k-1}(y) = R(I - Q(y))$$

for any $y \in Y$. Here we define $V_{0,-1}(Y) = \{0\} \times Y$ and $V_{0,-1}(y) = \{0\} \times \{y\}$. For convenience, we hereafter write $V_{i,i}(Y)$ (resp. $V_{i,i}(y)$) as $V_i(Y)$ (resp. $V_i(y)$). By a result in the spectral theory of linear dynamical systems (see, e.g., [15, Theorems 3 and 4] or [10, Theorem 8.1]), for any $0 \leq i \leq k-1$, there holds

$$V_i(y) = \{x \in \mathbb{R}^n : \lambda_S^{\pm}(x,y), \lambda_I^{\pm}(x,y) \in [a_i, b_i]\} \cup \{0\}, \tag{2.3}$$

where $\lambda_S^{\pm}(x,y)$ and $\lambda_S^{\pm}(x,y)$ are four *Lyapunov characteristic exponents* of (x,y), which are defined by

$$\lambda_S^{\pm}(x,y) = \limsup_{t \to \pm\infty} \frac{\ln |\Phi(t,y)x|}{t}, \quad \lambda_I^{\pm}(x,y) = \liminf_{t \to \pm\infty} \frac{\ln |\Phi(t,y)x|}{t}.$$

Recall that $B(y)$ is strongly cooperative tridiagonal for each $y \in Y$. Following [21], we define a continuous map $\sigma : \Lambda \to \{0, 1, 2, \ldots, n-1\}$ on the open dense set $\Lambda = \{x \in \mathbb{R}^n : x_1 \neq 0, x_n \neq 0$ and if $x_i = 0$ for some $i,\ 2 \leq i \leq n-1$, then $x_{i-1}x_{i+1} < 0\}$ of $\mathbb{R}^n$ by

$$\sigma(x) = \#\{i : x_i = 0 \text{ or } x_i x_{i+1} < 0\},$$

where # denotes the cardinality of the set. Clearly, σ is locally constant on Λ.

For any fixed integers m and l satisfying $0 \leq m \leq l \leq n-1$ and $y \in Y$, we define the set

$$W_{m,l}(y) = \{x \in \mathbb{R}^n \setminus \{0\} : m \leq \sigma(\Phi(t,y)x) \leq l, \text{ whenever } \Phi(t,y)x \in \Lambda\} \cup \{0\}.$$

When $m = l$, we write $W_{m,l}(y)$ as $W_m(y)$ for brevity. The product metric in $\mathbb{R}^n \times H(f)$ is denoted by d. The following proposition was proved in [3]. It justifies the name *integer-valued Lyapunov function* of the function σ.

Proposition 2.1. *For $0 \le m \le l \le n-1$ and $y \in Y$, we have*

(i) *$\sigma(\Phi(t,y)x)$ is non-increasing as t increases with $\Phi(t,y)x \in \Lambda$. Moreover, if $\Phi(s,y)x \notin \Lambda$ for some $s \in \mathbb{R}$, then $\sigma(\Phi(s+,y)x) < \sigma(\Phi(s-,y)x)$;*

(ii) *$W_{m,l}(y)$ is a linear subspace of $\mathbb{R}^n$ with $\dim(W_{m,l}(y)) = l-m+1$. Moreover, $W_{m,l}(y)$ has the direct sum decomposition*

$$W_{m,l}(y) = \bigoplus_{k=m}^{l} W_k(y);$$

(iii) *$\Phi(t,y)W_{m,l}(y) = W_{m,l}(y\cdot t)$ for all $t \in \mathbb{R}$, and $W_{m,l}(y)$ varies continuously with $y \in Y$ as a subspace of $\mathbb{R}^n$.*

Proof. See [3, Lemma 2.1, and Proposition 3.1]. □

Whenever $0 \le m \le l \le n-1$, we write $W_{m,l}(Y) = \bigcup_{y\in Y} W_{m,l}(y) \times \{y\}$ and call them *Floquet bundles*. With this notation, the exponential separation of $W_{m,l}(Y)$ is given in the following proposition, in which we also present a more delicate decomposition of the spectral bundle $V_i(Y)$.

Proposition 2.2. (i) *For $0 \le m \le n-2$, one has*

$$\mathbb{R}^n \times Y = W_{0,m}(Y) \oplus W_{m+1,n-1}(Y).$$

Moreover, the pair of subbundles $(W_{0,m}(Y), W_{m+1,n-1}(Y))$ is exponentially separated in the sense that there exist positive numbers K and ν such that

$$\frac{|\Phi(t,y)x_2|}{|\Phi(t,y)x_1|} \le Ke^{-\nu t}, \quad t \ge 0, \tag{2.4}$$

for all $y \in Y$, $x_1 \in W_{0,m}(y)$, $x_2 \in W_{m+1,n-1}(y)$ with $|x_1| = |x_2| = 1$.

(ii) *For each $i \in \{0, \dots, k-1\}$ one has*

$$V_i(Y) = W_N(Y) \oplus \cdots \oplus W_{N+M-1}(Y),$$

where $N = \dim(V_{0,i-1}(Y))$ and $M = \dim V_i(Y)$.

Proof. See [3, Theorem 3.3, and Corollary 3.7]. □

Proposition 2.3. *Let $0 \le i \le j \le k-1$. Then for each $y \in Y$ and $x \in V_{i,j}(y) \cap \Lambda$, one has*

$$\dim V_{0,i-1}(y) \le \sigma(x) \le \dim V_{0,i-1}(y) + \dim V_{i,j}(y) - 1.$$

Proof. It follows from the definitions of $V_{i,j}(y)$ and $V_i(y)$ that $V_{i,j}(y) = V_i(y) \oplus \cdots \oplus V_j(y)$. Combining this with Proposition 2.2(ii), one obtains that

$$V_{i,j}(y) = W_m(y) \oplus W_{m+1}(y) \oplus \cdots \oplus W_l(y),$$

where $m = \dim V_{0,i-1}(y)$ and $l = \dim V_{0,i-1}(y) + \dim V_{i,j}(y) - 1$. Thus, the proposition follows directly from the definition of $W_{m,l}(y)$. □

We call the minimal set Y *hyperbolic* if $0 \notin \Sigma(Y)$. On the other hand, if Y is non-hyperbolic, then $0 \in [a_{i_0}, b_{i_0}] \subset \Sigma(Y)$ for precisely one $0 \le i_0 \le k-1$. We write $V^s(y) = V_{i_0+1,k-1}(y)$, $V^{cs}(y) = V_{i_0,k-1}(y)$, $V^c(y) = V_{i_0}(y)$, $V^{cu}(y) = V_{0,i_0}(y)$ and

$V^u(y) = V_{0,i_0-1}(y)$ and call them the *stable, center stable, center, center unstable, unstable subspace* of (2.1) at $y \in Y$, respectively.

Finally, we prove the following lemma which is a consequence of the continuity of $V_{i,j}(y)$ with respect to y. It will be useful in the next section.

Lemma 2.4. *Take any $y_1, y_2 \in Y$. If the distance $d(y_1, y_2)$ is sufficiently small, then*

$$V^{cs}(y_1) \oplus V^u(y_2) = \mathbb{R}^n,$$
$$V^{cu}(y_1) \oplus V^s(y_2) = \mathbb{R}^n.$$

Proof. We only prove $V^{cs}(y_1) \oplus V^u(y_2) = \mathbb{R}^n$, since the proof of $V^{cu}(y_1) \oplus V^s(y_2) = \mathbb{R}^n$ is completely analogous. Because $V^{cs}(y_1) \oplus V^u(y_1) = \mathbb{R}^n$ and $\dim V^u(y_2) = \dim V^u(y_1)$ for any $y_1, y_2 \in Y$, it suffices to show that $V^{cs}(y_1) \cap V^u(y_2) = \{0\}$ whenever $d(y_1, y_2)$ is sufficiently small. Suppose this is not true. Then one can find sequences $\{y_1^m\}, \{y_2^m\} \subset Y$ with $d(y_1^m, y_2^m) \to 0$ as $m \to \infty$ and unit vectors $\{w_m\}$ with $w_m \in V^{cs}(y_1^m) \cap V^u(y_2^m)$. Assume that $y_1^m \to y_*$ and $w_m \to w_*$ (otherwise, work with subsequences), with $|w_*| = 1$. Then the continuity of $V^{cs}(y)$ and $V^u(y)$ implies that $w_* \in V^{cs}(y_*) \cap V^u(y_*)$, which is a contradiction. ☐

3. Values of σ on invariant manifolds

In this section, we present an invariant manifold theorem associated with the linearized system (2.1), followed by a characterization of these invariant manifolds in terms of the values of σ.

Consider the system

$$\dot{z} = B(y \cdot t)z + F(z, y \cdot t), \quad t \in \mathbb{R},\ z \in \mathbb{R}^n,\ y \in Y, \tag{3.1}$$

where $B(y \cdot t)$ and Y are as in (2.1), F is continuous, together with its first derivatives in z, and $F(z, y) = O(|z|^2)$. Let $\Sigma(Y) = \bigcup_{i=0}^{k-1} [a_i, b_i]$ be the Sacker-Sell spectrum defined above. For fixed $y \in Y$, denote by $\Lambda_t(\cdot, y)$ the solution operator of (3.1). The following lemma can be proved by using the same arguments as in [1,2,9,17,20].

Lemma 3.1. *There is a $\delta_0 > 0$ such that for any $0 < \delta^* < \delta_0$ and $0 \le i \le j \le k-1$, the system (3.1) admits for each $y \in Y$ a local invariant manifold $W^{i,j}(y, \delta^*)$ with the following properties:*

(i) *There are $M > 0$ and bounded continuous functions $h^{i,j}$ with $h^{i,j}(\cdot, y) : V_{i,j}(y) \to V_{j+1,k}(y) \cup V_{1,i-1}(y)$ that are C^1 for each fixed $y \in Y$, and $h^{i,j}(x, y) = o(|x|)$, $|\partial h^{i,j}/\partial x(x, y)| \le M$ for all $y \in Y$, $x \in V_{i,j}(y)$ such that*

$$W^{i,j}(y, \delta^*) = \{x_0^{i,j} + h^{i,j}(x_0^{i,j}, y) :\ x_0^{i,j} \in V_{i,j}(y) \cap \{x \in \mathbb{R}^n :\ |x| < \delta^*\}\}.$$

Moreover, $W^{i,j}(y, \delta^*)$ is diffeomorphic to $V_{i,j}(y) \cap \{x \in \mathbb{R}^n :\ |x| < \delta^*\}$, and $W^{i,j}(y, \delta^*)$ is tangent to $V_{i,j}(y)$ at $0 \in \mathbb{R}^n$ for each $y \in Y$.

(ii) *$W^{i,j}(y, \delta^*)$ is locally invariant in the sense that there are $\delta_1^* < \delta_0$, $\tau > 0$ such that $\Lambda_t(W^{i,j}(y, \delta_1^*), y) \subset W^{i,j}(y \cdot t, \delta^*)$ for any $t \in \mathbb{R}^1$ with $|t| < \tau$.*

Remark. If $0 \in [a_{i_0}, b_{i_0}] \subset \Sigma(Y)$ for some $i_0 \in \{0, \dots, k-1\}$, we introduce the notation $W^s(y, \delta^*) = W^{i_0+1,k-1}(y, \delta^*)$, $W^{cs}(y, \delta^*) = W^{i_0,k-1}(y, \delta^*)$, $W^c(y, \delta^*) = W^{i_0,i_0}(y, \delta^*)$, $W^{cu}(y, \delta^*) = W^{0,i_0}(y, \delta^*)$, and $W^u(y, \delta^*) = W^{0,i_0-1}(y, \delta^*)$, and just call these manifolds local stable, center stable, center, center unstable and unstable manifolds of (3.1) at $y \in Y$, respectively. $W^s(y,\delta^*)$ and $W^u(y,\delta^*)$ are overflowing invariant in the sense that if δ^* is sufficiently small, then

$$\Lambda_t(W^s(y,\delta^*),y) \subset W^s(y\cdot t,\delta^*),$$

for t sufficiently positive, and

$$\Lambda_t(W^u(y,\delta^*),y) \subset W^u(y\cdot t,\delta^*),$$

for t sufficiently negative. Moreover, one can find constants $\alpha, C > 0$, such that for any $y \in Y$, and $x_s \in W^s(y,\delta^*), x_u \in W^u(y,\delta^*)$,

$$(3.2)\qquad \begin{aligned} |\Lambda_t(x_s,y)| &\le Ce^{-\frac{\alpha}{2}t}|x_s| \quad \text{for } t \ge 0,\\ |\Lambda_t(x_u,y)| &\le Ce^{\frac{\alpha}{2}t}|x_u| \quad \text{for } t \le 0. \end{aligned}$$

Let $Y \subset \mathbb{R}^n \times H(f)$ be a minimal set of (1.3). For any $y = (x_0,g) \in Y$, let $z = x - x(t,x_0,g)$. Then z satisfies the non-linear equation (3.1). As a consequence, for fixed $0 \le i \le j \le k-1$ and $y = (x_0,g) \in Y$, there exists a well-defined local invariant manifold $W^{i,j}(y,\delta^*)$ of (3.1). Define

$$M^{i,j}(y,\delta^*) = \{x \in \mathbb{R}^n : \ x - x_0 \in W^{i,j}(y,\delta^*)\}.$$

We call $M^{i,j}(y,\delta^*)$ a *local invariant manifold* of the system (1.1) at y. If $0 \in [a_{i_0},b_{i_0}] \subset \Sigma(Y)$, we introduce the notation $M^s(y,\delta^*) = M^{i_0+1,k-1}(y,\delta^*)$, $M^{cs}(y,\delta^*) = M^{i_0,k-1}(y,\delta^*)$, $M^c(y,\delta^*) = M^{i_0}(y,\delta^*)$, $M^{cu}(y,\delta^*) = M^{0,i_0}(y,\delta^*)$ and $M^u(y,\delta^*) = M^{0,i_0-1}(y,\delta^*)$, and call them local stable, center stable, center, center unstable and unstable manifolds of (1.1) at $y \in Y$, respectively.

The following proposition, which is the main result of this section, states a separating property of these local invariant manifolds in terms of the values of σ.

Proposition 3.2. *There exists a $\delta' < \delta_0$ such that, for all $0 \le i \le j \le k-1$, one has*

$$(3.3)\qquad N \le \sigma(x - x_0) \le N + M - 1,$$

for any $y = (x_0,g) \in Y$, $x \in M^{i,j}(y,\delta')$ with $x - x_0 \in \Lambda$. Here $N = \dim V_{0,i-1}(y)$ and $M = \dim V_{i,j}(y)$.

In particular, if $0 \in [a_{i_0},b_{i_0}] \subset \Sigma(Y)$, then there is a $\delta' < \delta^*$ such that

$$(3.4)\qquad \sigma(x - x_0) \le N^u + N^c - 1 \quad \text{for } x \in M^{cu}(y,\delta'), x - x_0 \in \Lambda,$$

$$(3.5)\qquad \sigma(x - x_0) \ge N^u \quad \text{for } x \in M^{cs}(y,\delta'), x - x_0 \in \Lambda,$$

where $N^c = \dim V_{i_0}(y)$ and $N^u = \dim V_{0,i_0-1}(y)$. Moreover,

$$N^u \le \sigma(x - x_0) \le N^u + N^c - 1 \quad \text{for } x \in M^c(y,\delta'), x - x_0 \in \Lambda.$$

Proof. It suffices to prove (3.3). Suppose that one can find some $0 \le i \le j \le k-1$ and sequences $\delta_m^* \to 0$, $y_m = (x_m,g_m) \in Y$ and $x'_m \in M^{i,j}(y_m,\delta_m^*)\backslash\{x_m\}$ such that

$$\sigma(x'_m - x_m) < N \text{ or } \sigma(x'_m - x_m) > N + M - 1.$$

Define $z_m = \frac{x'_m - x_m}{|x'_m - x_m|}$. Clearly, $\{z_m\} \subset S^{n-1}$. Since Y is compact, one can assume that $z_m \to z^*$ and $y_m \to y^* = (x^*,g^*)$ as $m \to \infty$. Moreover, by Lemma 3.1(i), $z^* \in V_{i,j}(y^*)$. For simplicity, we write $w_m(t) = x(t,x'_m,y_m) - x(t,x_m,y_m)$. Using Lemma 3.1(ii), one can find $0 < \tau < 1$ such that $w_m(t) \in V_{i,j}(y_m\cdot t)$, for $|t| < \tau$ and $m \in \mathbb{N}$. Then Lemma 3.1(i) implies that $w_m(t) = v_m(t) + h^{i,j}(v_m(t),y_m\cdot t)$, where $v_m(t) \in V_{i,j}(y_m\cdot t)$ and satisfies (2.1) with $y = y_m$ for $|t| < \tau$, with the initial condition $v_m(0)$. For each $|t| < \tau$, let $z^*(t) \triangleq \lim_{m\to\infty} \frac{w_m(t)}{|w_m(0)|}$. A direct calculation yields

$$\lim_{m\to\infty} \frac{w_m(t)}{|w_m(0)|} = \lim_{m\to\infty} \left(\frac{v_m(t)}{|w_m(0)|} + \frac{h^{i,j}(v_m(t), y_m \cdot t)}{|v_m(t)|} \frac{|v_m(t)|}{|v_m(0)|} \frac{|v_m(0)|}{|w_m(0)|} \right)$$
$$= \lim_{m\to\infty} \frac{v_m(t)}{|w_m(0)|}.$$

So $z^*(t) = \lim_{m\to\infty} \frac{v_m(t)}{|w_m(0)|}$, which implies that it satisfies (2.1) with $y = y^*$ for $|t| < \tau$ under the initial condition $x(0) = z^*$. Consequently, $z^*(t) \in V_{i,j}(y \cdot t)$ for all $|t| < \tau$. Now Proposition 2.3 implies that $N \le \sigma(z^*(t)) \le N + M - 1$, for $|t| < \tau$. Hence one can choose $t_1 > 0$, $t_2 < 0$, with $|t_1|, |t_2| < \tau$ such that $z^*(t_1), z^*(t_2) \in \Lambda$. Together with the fact that σ is locally constant on Λ, Proposition 2.1(i) implies that

$$N \le \sigma(x'_m(t_1) - x_m(t_1)) \le \sigma(x'_m - x_m) \le \sigma(x'_m(t_2) - x_m(t_2)) \le N + M - 1,$$

for all m large enough, a contradiction. This completes our proof. ☐

We close this section with the following lemma, which will be crucial in the proof of our main result.

Lemma 3.3. *For any $(x_1, g), (x_2, g) \in Y$ with $|x_1 - x_2|$ sufficiently small, one has*

$$M^{cs}(x_1, g) \cap M^u(x_2, g) \ne \emptyset,$$
$$M^{cu}(x_1, g) \cap M^s(x_2, g) \ne \emptyset.$$

Proof. We only prove that $M^{cs}(x_1, g) \cap M^u(x_2, g) \ne \emptyset$, since the proof that $M^{cu}(x_1, g) \cap M^s(x_2, g) \ne \emptyset$ is analogous. By Lemma 2.4, for $(x_1, g), (x_2, g) \in Y$ with $|x_1 - x_2|$ sufficiently small, one has $V^{cs}(x_1, g) \cap V^u(x_2, g) = \mathbb{R}^n$. Therefore $x_1 + V^{cs}(x_1, g) \cap x_2 + V^u(x_2, g) \ne \emptyset$. Since $W^{cs}(x_1, g, \delta^*)$ and $W^u(x_2, g, \delta^*)$ are tangent to $V^{cs}(x_1, g)$ and $V^u(x_2, g)$ at $0 \in \mathbb{R}^n$, by letting $|x_1 - x_2|$ small enough one can have $x_1 + W^{cs}(x_1, g) \cap x_2 + W^u(x_2, g) \ne \emptyset$, that is, $M^{cu}(x_1, g) \cap M^s(x_2, g) \ne \emptyset$. ☐

4. Central dimension one

In this section, we investigate the structure of the minimal sets of the skew-product flow (1.3) generated by the time-recurrent tridiagonal system (1.1) – (1.2).

Definition 4.1. Let $p : \mathbb{R}^n \times H(f) \to H(f), (x_0, g) \mapsto g$ be the natural flow homomorphism. A set $Y \subset \mathbb{R}^n \times H(f)$ is called an *almost* 1*-cover* (resp. 1*-cover*) of $H(f)$, if $p^{-1}(g) \cap Y$ is a singleton for at least one $g \in H(f)$ (resp. for any $g \in H(f)$).

The following lemma, adopted from [3,24], already gives a first result concerning the structure of minimal sets Y of the skew-product flow Π (1.3).

Lemma 4.1. *Let Y_1 and Y_2 be two different minimal subsets of $\mathbb{R}^n \times H(f)$ with respect to the skew-product flow Π. Then*

(i) *Y_1 is an almost 1-cover of $H(f)$; and moreover, Y_1 is a 1-cover of $H(f)$ if Y_1 is hyperbolic.*
(ii) *For any $(x_i, g) \in Y_i, i = 1, 2$, one has $\sigma(x(t, x_1, g) - x(t, x_2, g)) = \text{const}$ for all $t \in \mathbb{R}$.*

Proof. See [24, Lemma 4.2 and Theorem 3.6] and [3, Theorem 4.5]. ☐

Remark. If $Y_1 = Y_2$, the conclusion in Lemma 4.1(ii) is still unknown.

From now onwards we let Y be a minimal subset in ($\mathbb{R}^n \times H(f)$ with respect to Π).

Motivated by the work of Shen and Yi [19, 20], we will utilize the results obtained in Section 3 to investigate the dynamics on Y in the case when it is not hyperbolic. For this purpose, we assume $0 \in [a_{i_0}, b_{i_0}] \subset \Sigma(Y)$ for some $i_0 \in \{0, \dots, k-1\}$. Let $V^c(y) = V_{i_0}(y)$ be the center subspace of (2.1) at $y \in Y$ and $N^c = \dim V^c(y)$. We say that Y has *central dimension one* if $N^c = 1$.

We are now ready to state our main result.

Theorem 4.2. *Let Y be a minimal set of the skew-product flow* (1.3) *with central dimension one. Then the following assertions hold:*

(i) *For any $(x_i, g) \in Y, i = 1, 2$, one has $\sigma(x(t, x_1, g) - x(t, x_2, g)) = \text{const}$ for all $t \in \mathbb{R}$.*
(ii) *The flow (Y, Π) is topologically conjugate to a skew-product flow on some $\widetilde{Y} \subset \mathbb{R}^1 \times H(f)$.*

When $n = 2$, the system (1.1) naturally reduces to a two-dimensional strongly competitive (or cooperative) system. Then we have the following

Theorem 4.3. *Consider the* 2*-D time-recurrent competitive (resp. cooperative) system*

$$\begin{aligned} \dot{x}_1 &= f_1(t, x_1, x_2), \\ \dot{x}_2 &= f_2(t, x_1, x_2), \end{aligned} \tag{4.1}$$

with $\dfrac{\partial f_i}{\partial x_j}(t, x) \le -\varepsilon_0 < 0$ (resp. $\dfrac{\partial f_i}{\partial x_j}(t, x) \ge \varepsilon_0 > 0$), $1 \le i \ne j \le 2$, for all $(t, x) \in \mathbb{R} \times \mathbb{R}^2$. If Y is a minimal set of (4.1) *of central dimension one, then (Y, Π) is topologically conjugate to a skew-product subflow on $\mathbb{R}^1 \times H(f)$.*

Remark. When the competitive system (4.1) is time T-periodic, Hale and Somolinos [6, Theorem 4.2] have shown that all bounded solutions are asymptotic to T-periodic solutions. When such a competitive system is time almost-periodic, Hetzer and Shen [7, Theorem A] have proved the almost 1-cover lifting property of minimal sets for the associated skew-product flow. Recently, the present authors [3, Theorem 4.6] have shown that any hyperbolic omega-limit set is a 1-cover of the base flow. A natural question is to say something about the dynamics on the non-hyperbolic minimal sets. Our theorem seems to be the first attempt to answer this question under certain additional assumptions (one-dimensional center direction or unique ergodicity), which extends the above-mentioned results in [3, 7].

In order to prove Theorem 4.2, we need the following proposition.

Proposition 4.4. *Let Y be a minimal set of* (1.3)*. Then for any two distinct points $(x_i, g) \in Y, i = 1, 2$, $x_1 - x_2 \in \Lambda$,*

$$N^u \le \sigma(x_1 - x_2) \le N^u + N^c - 1, \tag{4.2}$$

where $N^u = \dim V^u(Y)$ and $N^c = \dim V^c(Y)$. In particular, if $N^c = 1$, then $\sigma(x_1 - x_2) = N^u$.

Proof. Write $y_1 = (x_1, g)$ and $y_2 = (x_2, g)$. By Lemma 4.1(i), there exist two sequences $t_m \to \infty$ and $s_m \to -\infty$ such that $x(t_m, x_1, g) - x(t_m, x_2, g) \to 0$ and $x(s_m, x_1, g) - x(s_m, x_2, g) \to 0$ as $n \to \infty$. Note that Proposition 2.1(i) holds for $x(t) = x(t, x_1, g) - x(t, x_2, g)$, where the element $B_{ij}(y \cdot t)$ of the matrix $B(y \cdot t)$ in (2.1) is $B_{ij}(y \cdot t) = \int_0^1 \frac{\partial g_i}{\partial x_j}(t, (1-\tau)x(t, x_1, g) + \tau x(t, x_2, g))d\tau$. Since Λ is an open subset of $\mathbb{R}^n$, there exists an $\varepsilon > 0$ such that for any $x \in \mathbb{R}^n$ with $|x| < \varepsilon$,

$$\sigma(x_1 - x_2 + x) = \sigma(x_1 - x_2). \tag{4.3}$$

By Lemma 3.3, $M^{cs}(y_1 \cdot t_m, \delta^*) \cap M^u(y_2 \cdot t_m, \delta^*) \neq \emptyset$ and $M^s(y_1 \cdot s_m, \delta^*) \cap M^{cu}(y_2 \cdot s_m, \delta^*) \neq \emptyset$ for large m. Choose $x_m^+ \in M^{cs}(y_1 \cdot t_m, \delta^*) \cap M^u(y_2 \cdot t_m, \delta^*)$ and $x_m^- \in M^s(y_1 \cdot s_m, \delta^*) \cap M^{cu}(y_2 \cdot s_m, \delta^*)$. By virtue of (3.2), one has

$$\begin{aligned} &|x(-t_m, x_m^+, g \cdot t_m) - x(-t_m, x(t_m, x_2, g), g \cdot t_m)| \\ \le &Ce^{-\frac{\alpha}{2}t_m}|x_m^+ - x(t_m, x_2, g)| \le Ce^{-\frac{\alpha}{2}t_m}\delta^*, \end{aligned}$$

which implies that

$$|x(-t_m, x_m^+, g \cdot t_m) - x_2| = |x(-t_m, x_m^+, g \cdot t_m) - x(-t_m, x(t_m, x_2, g), g \cdot t_m)| \to 0,$$

as $m \to \infty$. Similarly, one can obtain that $|x(-s_m, x_m^-, g \cdot s_m) - x_1| \to 0$ as $m \to \infty$. Therefore, by (4.3) and Proposition 2.1(i), for m large enough, one has

$$\begin{aligned} \sigma(x_1 - x_2) &= \sigma(x_1 - x_2 + x_2 - x(-t_m, x_m^+, g \cdot t_m)) \\ &= \sigma(x_1 - x(-t_m, x_m^+, g \cdot t_m)) \ge \sigma(x(t_m, x_1, g) - x_m^+) \ge N^u, \end{aligned} \tag{4.4}$$

where the last inequality is due to $x_m^+ \in M^{cs}(y_1 \cdot t_m, \delta^*)$ and (3.5). On the other hand, for s_m, we similarly obtain that

$$\begin{aligned} \sigma(x_1 - x_2) &= \sigma(x(-s_m, x_m^-, g \cdot s_m) - x_2 + x_1 - x(-s_m, x_m^-, g \cdot s_m)) \\ &= \sigma(x(-s_m, x_m^-, g \cdot s_m) - x_2) \\ &\le \sigma(x_m^- - x(s_m, x_2, g \cdot s_m)) \le N^u + N^c - 1, \end{aligned}$$

because $x_m^- \in M^{cu}(y_2 \cdot s_m, \delta^*)$ and (3.4). This contradicts (4.4). We have completed the proof. □

Proof of Theorem 4.2. (i) is a direct corollary of Proposition 4.4 and Proposition 2.1(i).

(ii) By Lemma 4.1 we need to consider only the case in which Y is only an almost 1-cover of $H(f)$. To this end, define the projection

$$h : Y \to \mathbb{R}^1 \times H(f), \quad (x, g) \longmapsto ((x)_1, g),$$

where $(x)_1$ is the first coordinate of x. Let $\widetilde{Y} = h(Y)$. Clearly, h is continuous and onto $\widetilde{Y}$. Recall that Y has central dimension one. It thus follows from Proposition 4.4 and Proposition 2.1(i) that, for any two distinct $(x_1, g), (x_2, g) \in Y$, $x_1 - x_2 \in \Lambda$. Hence the first coordinate $(x_1)_1 \neq (x_2)_1$, which implies that h is an injection. Since Y and $\widetilde{Y}$ are compact metric spaces, h is also a closed mapping. Hence h is a homeomorphism and the induced flow on $\widetilde{Y} : \widetilde{\Pi}(t, (x)_1, g) = ((x(t, x, g))_1, g \cdot t)$ is a skew-product flow. □

Conjecture. *If* $\dim V^c(Y) = N^c$, *then* (Y, Π) *is conjugate to a subflow of* $\mathbb{R}^{N^c} \times p(Y)$, *where* p *is the natural flow homomorphism as in Definition* 4.1.

Remark. Pujals and Sambarino [14] gave a complete description of the dynamics of a two-dimensional exponentially separated system (called dominated splitting in their paper). However, the corresponding issues for skew-product flows remain open.

Recall that a map is *uniquely ergodic* if it has a unique invariant measure.

Corollary 4.5. (i) *Assume that Y admits a pure point spectrum. Then the conclusions of Theorem* 4.2 *hold.*

(ii) *If Y is uniquely ergodic, then the conclusions of Theorem* 4.2 *hold.*

Proof. (i) Let $\Sigma(Y) = \{a_{k-1}, \cdots, a_0\}$, with $a_{k-1} < \cdots < a_0$. We claim that $k = n$ and that $\dim V_i(Y) = 1$ for all $0 \le i \le n-1$. Suppose to the contrary that there is $0 \le i \le k-1$ with $\dim V_i(Y) > 1$. By Proposition 2.2(ii), $V_i(Y) = W_N(Y) \oplus \cdots \oplus W_{N+M-1}(Y)$, with $M > 1$. It follows from [15, Theorem 3] (or (2.3)) that

$$\lim_{t\to\infty} \frac{\ln|\Phi(t,y)x_1|}{t} = \lim_{t\to\infty} \frac{\ln|\Phi(t,y)x_2|}{t} = a_i$$

for any $y \in Y$ and any $x_1 \in W_N(y)\backslash\{0\}$, $x_2 \in W_{N+M-1}(y)\backslash\{0\}$, which is incompatible with $W_N(Y)$ and $W_{N+M-1}(Y)$ being exponentially separated (see (2.4) and [11, Proposition A.3]).

If Y is a 1-cover of $H(f)$, then the conclusion is now obvious. According to Lemma 4.1 the only other possibility is that Y is an almost 1-cover. But then $0 \in \Sigma(Y)$ and $\dim V^c(Y) = 1$ and the conclusions of Theorem 4.2 hold.

(ii) is obvious from (i), because unique ergodicity (see, e.g., [10, Theorem 2.3]) implies that Y possesses a pure point spectrum. ☐

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