PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 143, Number 8, August 2015, Pages 3613–3618
http://dx.doi.org/10.1090/proc/12174
Article electronically published on April 22, 2015

A VOLUME STABILITY THEOREM ON TORIC MANIFOLDS WITH POSITIVE RICCI CURVATURE

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(Communicated by Lei Ni)

ABSTRACT. In this short note, we will prove a volume stability theorem which says that if an n-dimensional toric manifold M admits a $\mathbb{T}^n$ invariant Kähler metric ω with Ricci curvature no less than 1 and its volume is close to the volume of $\mathbb{CP}^n$, M is bi-holomorphic to $\mathbb{CP}^n$.

1. INTRODUCTION

Understanding the geometry of manifolds under various curvature conditions is fundamental. In Riemannian geometry, we have Bishop-Gromov's volume comparison if the Ricci curvature of the manifold is bounded from below. Using this theorem and some techniques in comparison geometry, Colding proved the following result ([5]):

Theorem 1.1. *Given $\epsilon > 0$, there exists $\delta = \delta(n, \epsilon) > 0$ such that, if an n-dimensional manifold M has $Ric_M \geq n-1$ and $Vol(M) > Vol(\mathbb{S}^n) - \delta$, then $d_{GH}(M, \mathbb{S}^n) < \epsilon$.*

Here d_{GH} denotes the Gromov-Hausdorff distance between Riemannian manifolds. By another theorem of Colding (see the appendix in [4]), we know that M is in fact diffeomorphic to $\mathbb{S}^n$.

Natural questions are how to get a more useful version of Bishop-Gromov's volume comparison theorem in Kähler geometry and how to state a theorem analogous to the one above. Because we have more structures on the manifold, the volume comparison with space form is not sharp: see [8] for an improvement of the local volume comparison for Kähler manifolds with Ricci curvature bounded from below. More recently, Berman and Berndtsson considered toric manifolds with positive Ricci curvature in [2] and [3], and they proved:

Theorem 1.2. *Suppose that (M, ω) is a smooth n-dimensional toric variety with $\mathbb{T}^n$ invariant Kähler form ω such that $Ric\ \omega \geq \omega$; then we have*

$$Vol(M) \leq Vol(\mathbb{CP}^n). \tag{1.1}$$

In fact, their theorem holds if the manifold admits a $\mathbb{C}^*$ action with finite fixed points and the metric is $\mathbb{S}^1$ invariant (see [3]). The theorem of Berman and Berndtsson partially answered a conjecture in [10].

Received by the editors November 3, 2012 and, in revised form, February 19, 2013.
2010 *Mathematics Subject Classification.* Primary 53C23; Secondary 53C55.
Key words and phrases. Differential geometry.

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Conjecture. *Any n-dimensional toric Fano manifold X that admits a Kähler-Einstein metric has anticanonical degree* $(-K_X)^n \leq (n+1)^n$*, with equality only for* $\mathbb{CP}^n$*.*

In this short note, we will determine when the equality holds in Theorem 1.2. So the above conjecture is completely solved. More precisely, we can prove a rigidity and stability theorem as follows:

Theorem 1.3. *The equality in Theorem* 1.2 *holds if and only if* (M, ω) *is isometric to* $(\mathbb{CP}^n, \omega_{FS})$*. Moreover, there exists a positive number* ϵ *which depends only on* n *such that if* M *is a toric manifold with a* $\mathbb{T}^n$ *invariant metric* ω *satisfying Ric* $\omega \geq \omega$ *and*

$$Vol(M) \geq (1-\epsilon)\, Vol(\mathbb{CP}^n), \tag{1.2}$$

M *is bi-holomorphic to* $\mathbb{CP}^n$*.*

In [3], Berman and Berndtsson applied a Moser-Trudinger typed inequality established in [1] to prove Theorem 1.2 . But so far we can't prove the rigidity using this analytic method. Inspired by the combinatoric proof by Bo'az Klartag for the Kähler-Einstein case in [3], we will apply the Grunbaum's inequality ([7]) to prove our theorem. In order to use this inequality we should know the position of the barycenter of the moment polytope of (M, ω). We will use the Ricci curvature condition to achieve this. More detailed analysis gives us the rigidity and stability.

2. Preliminaries

At first, we give some basic materials of toric manifolds which are used in our proof. Here a toric manifold means a Kähler manifold (M, ω) containing $(\mathbb{C}^*)^n$ as a dense subset such that the standard action of $(\mathbb{C}^*)^n$ on itself extends to a holomorphic action on M. In general we suppose that the metric is $\mathbb{T}^n$ invariant and we can consider the moment map of $(M, \omega, \mathbb{T}^n)$.

Definition 2.1. A polytope $P \subseteq \mathbb{R}^n$ is called a Delzant polytope if each vertex is contained in exactly n facets, and the normals of the n facets containing a given vertex form an integral basis of $\mathbb{Z}^n$.

The image of the moment map above should be a Delzant polytope according to a theorem of Delzant ([6]):

Theorem 2.2. *Each Delzant polytope gives rise to a symplectic manifold* (M, ω) *with an action of* $\mathbb{T}^n$ *that preserves* ω*, and all such symplectic manifolds arise this way.*

In fact, congruent polytopes correspond to isomorphic toric symplectic manifolds.

Using the embedding of $(\mathbb{C}^*)^n$ in M, we set:

$$\iota : (\mathbb{C}^*)^n \to M, \iota^*\omega = \sqrt{-1}\partial\bar{\partial}u. \tag{2.1}$$

In toric coordinates: $\exp(x_i) = |z_i|^2$(z_i are holomorphic coordinates in $(\mathbb{C}^*)^n$), the invariance of ω means u is a function of x_i. Then the image of $\nabla u = (\frac{\partial u}{\partial x_i})_{1 \leq i \leq n}$ will be a moment map of (M, ω).

Given a toric manifold (M,ω), two moment maps may differ by a constant vector. When we choose a basis of group $(\mathbb{C}^*)^n$, these two moment polytopes differ by a translation. A change of basis of group $(\mathbb{C}^*)^n$ corresponds to a change of the integral basis of $\mathbb{Z}^n$, so it transforms Delzant polytopes to Delzant polytopes. The polytope also changes if we choose another $\mathbb{T}^n$ invariant Kähler metric on M with the same complex structure; i.e., we choose another symplectic form compatible with the fixed complex structure. This can be described in the following way: we denote the moment polytope by

$$P = \{x | \langle l_i, x\rangle \geq \lambda_i, 1 \leq i \leq N, x \in \mathbb{R}^n, l_i \in \mathbb{Z}^n, \lambda_i \in \mathbb{R}\}. \tag{2.2}$$

Then only λ_i $(1 \leq i \leq n)$ change while l_i $(1 \leq i \leq n)$ remain the same since they are just related to the complex structure (see [9], [11]). Using the description above, changing the symplectic form corresponds to changing the potential function u on $\mathbb{R}^n$.

When the manifold is a Fano variety with $\omega \in 2\pi c_1(M)$, we can get a moment polytope P such that λ_i are all equal to -1. This can be realized in the following way (see [9]): choose a potential u of ω such that

$$|\ln \det u_{ij} + u| \text{ is bounded in } \mathbb{R}^n; \tag{2.3}$$

then the image of ∇u will be such a polytope P with $\lambda_i = -1 (1 \leq i \leq n)$.

Because the normal vectors of the facets passing any point form an integral basis, we can do a coordinate transformation to change these vectors to the standard basis $e_k = (0, 0, \ldots, 1, 0, \ldots, 0)$ with 1 placed at position k. We can write this transformation as follows: choosing a vertex $p \in P$ with l_i $(1 \leq i \leq n)$ as normal vectors of the facets passing p, we can form an affine map:

$$x \mapsto (\langle l_i, x\rangle)_{1 \leq i \leq n}, \tag{2.4}$$

which transforms p to $(-1, -1, \ldots, -1)$ and the polytope to

$$\tilde{P} = \{x | \langle \tilde{l}_i, x\rangle \geq -1, 1 \leq i \leq N, x \in \mathbb{R}^n, \tilde{l}_i \in \mathbb{Z}^n, \tilde{l}_k = e_k, 1 \leq k \leq n\}. \tag{2.5}$$

There are only a finite many such polytopes in a given dimension.

According to Mabuchi's theorem ([9]), we know that for a Kähler-Einstein manifold, the origin is the barycenter of P. We will prove a similar property of the barycenter of the moment map of a toric manifold admitting ω with Ric $\omega \geq \omega$.

3. Proof of Theorem 1.3

At first we give a lemma which deals with the volume of some specific kind of polytopes. Let Q be the simplex spanned by

$$(n+1, 0, 0, \ldots, 0), (0, n+1, 0, 0, \ldots, 0), \ldots, (0, 0, \ldots, 0, n+1). \tag{3.1}$$

Recall that Grunbaum's inequality ([7]) says that if P is a convex body, and K denotes the intersection of P with an affine half-space defined by one side of a hyperplane H passing through the barycenter of P, then

$$\mathrm{Vol}(P) \leq \left(\frac{n+1}{n}\right)^n \mathrm{Vol}(K). \tag{3.2}$$

Let F denote the simplex spanned by

$$(n, 0, 0, \ldots, 0), (0, n, 0, 0, \ldots, 0), \ldots, (0, 0, \ldots, 0, n).$$

We have the following lemma.

Lemma 3.1. *If the barycenter of a polytope P in the first quadrant lies inside F,*

$$Vol(P) \leq Vol(Q). \tag{3.3}$$

Moreover, if the equality holds, P is coincident with Q.

Proof. The first statement can be seen from Grunbaum's theorem above: the corresponding $K \subseteq F$. For the second statement, let $X = P \setminus Q, Y = Q \setminus P$ and choose a coordinate system s_i with the barycenter as the origin and $(1, 1, \ldots, 1)$ as the first axis. Then we have

$$\int_P s_1 dV \leq \int_Q s_1 dV = 0, \qquad \int_X s_1 dV \leq \int_Y s_1 dv. \tag{3.4}$$

But since

$$s_1(x) \geq s_1(y) \text{ for } x \in X \text{ and } y \in Y, \tag{3.5}$$

both X and Y should be empty. □

In order to apply this lemma to the moment polytope P of (M, ω), we should know how to place P and where the barycenter is. We are going to use the toric structure on M and explore the Ricci curvature condition.

Under the conditions of Theorem 1.3, we can write Ric $\omega = \omega + \beta$ where β is a semi-positive 1-1 form. In $(\mathbb{C}^*)^n$, we can choose u such that $\omega = \sqrt{-1}\partial\bar{\partial}u$. In toric coordinates: $|z_i|^2 = \exp(x_i)$, we set:

$$v = -\ln \det u_{ij} - u. \tag{3.6}$$

Using the formula of Ricci curvature and $\frac{\partial^2 u}{\partial z_i \partial \bar{z_j}} = \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{1}{z_i} \frac{1}{\bar{z_j}}$, we see that

$$\sqrt{-1}\partial\bar{\partial}v = \text{Ric } \omega - \omega = \beta. \tag{3.7}$$

As β is semi-positive, v is a convex function.

From the following equalities:

$$\ln \det(u+v)_{ij} + u + v = \ln \det(u+v)_{ij} - \ln \det u_{ij} = \ln \frac{(\text{Ric } \omega)^n}{\omega^n}, \tag{3.8}$$

we know that $\ln \det(u+v)_{ij} + u + v$ is bounded, so $\nabla(u+v)$ will be a moment map of $(M, \text{Ric } \omega)$. Denote the image of $\nabla(u+v)$ by L. As illustrated in section 2, we can suppose that $(-1, -1, \ldots, -1)$ is a vertex of L and the facets passing it are parallel to coordinate hyperplanes, respectively:

$$L = \{y | \langle l_i, y\rangle \geq -1, 1 \leq i \leq N, y \in \mathbb{R}^n, l_i \in \mathbb{Z}^n, l_k = e_k, 1 \leq k \leq n\}. \tag{3.9}$$

The gradient of u will be a moment of (M, ω). We denote the image of ∇u by P. Without changing $u + v$, we can add a linear function to u and subtract the same one from v. This corresponds to a translation of P. As we have said above, P can be obtained from L by parallel movement of the facets. So we can translate P so that $(-1, -1, \ldots, -1)$ is a vertex of P and the facets passing this vertex are parallel to coordinate hyperplanes like L:

(3.10)

$$P = \{y | \langle l_i, y\rangle \geq \lambda_i, 1 \leq i \leq N, y \in \mathbb{R}^n, l_i \in \mathbb{Z}^n, l_k = e_k, \lambda_k = -1, 1 \leq k \leq n\}.$$

Such a pair of polytopes (P, L) is called an adapted pair of (M, ω).

Lemma 3.2. *For an adapted pair (P, L), the coordinates of the barycenter of P are all nonpositive.*

Proof.

$$\lim_{x_i\to-\infty}\frac{\partial v}{\partial x_i}=\lim_{x_i\to-\infty}\frac{\partial(u+v)}{\partial x_i}-\lim_{x_i\to-\infty}\frac{\partial u}{\partial x_i}=(-1)-(-1)=0 \tag{3.11}$$

for any i and fixed x_j ($1\leq j\leq n$, $j\neq i$). Because v is a convex function we know that all the partial derivatives of v are nonnegative. Denoting the coordinates of the barycenter by a_i, we have

$$\det u_{ij}=exp(-u-v), \frac{\partial v}{\partial x_i}\geq 0, \tag{3.12}$$

$$a_i=\int_P y_i dV=\int_{R^n}\frac{\partial u}{\partial x_i}\det u_{ij}dx\leq\int_{R^n}\frac{\partial(u+v)}{\partial x_i}exp(-u-v)dx=0. \tag{3.13}$$

The last inequality is the statement of the lemma. □

Proof of Theorem 1.3. Using the notations above, we do a translation which moves $(-1,-1,\dots,-1)$ to the origin. Then P will be a polytope inside the first quadrant with barycenter inside F by the second lemma. The rigidity follows from this together with the assumption that $\mathrm{Vol}(P)=\mathrm{Vol}(Q)$ by the first lemma.

Now we consider the stability. Suppose the statement doesn't hold; then there is a sequence of manifolds (M_i,ω_i) $(i=1,2,3,\dots)$ with volume converging to $\mathrm{Vol}(\mathbb{CP}^n)$ and with none holomorphic to $\mathbb{CP}^n$.

Construct adapted pairs (P_i, L_i) of (M_i,ω_i) $(i=1,2,3,\dots)$. Because there are only finitely many such L, one of them appears infinitely times. We denote it by B and select these P_i corresponding to B. These P_i as moment polytopes of different symplectic classes can be obtained from B by parallel movement of B's facets towards the interior. So P_i can be determined by N real numbers λ_i such that n of them are always -1. This gives us a correspondence:

$$P_i\leftrightarrow\lambda^{(i)}\in\mathbb{R}^{N-n}. \tag{3.14}$$

Because P_i are inside B, these vectors in $\mathbb{R}^{N-n}$ are bounded. We can choose a convergent subsequence, and the limit corresponds to a polytope P_∞. $\mathrm{Vol}(P_\infty)=\mathrm{Vol}(Q)$ and the coordinates of the barycenter of P_∞ are all nonpositive. According to the first lemma, P_∞ should be isomorphic to Q by a translation. We are going to show that $B=P_\infty$: Since $P_i\subseteq B$, we have $P_\infty\subseteq B$. If $P_\infty\neq B$, the integral points in the interior of the facet of P_∞ opposite $(-1,-1,\dots,-1)$ will be contained in the interior of B. But there is only one integral point in the interior of B, so we must have $B=P_\infty$.

We assumed that M_i are not holomorphic to $\mathbb{CP}^n$, but now B just differs from Q by a translation. Because congruent polytopes give rise to isomorphic toric varieties, these M_i are all holomorphic to $\mathbb{CP}^n$. This is a contradiction, so our theorem is proved. □

Acknowledgement

The author is grateful to his advisor, Professor Zhu Xiaohua, for many helpful conversations.

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