

## NON-REMOVABLE TERM ERGODIC ACTION SEMIGROUPS/GROUPS

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**ABSTRACT.** In this work, we introduce the concept of term ergodicity for action semigroups and construct semigroups on two-dimensional manifolds which are  $C^{1+\alpha}$ -robustly term ergodic. Moreover, we illustrate term ergodicity by some exciting examples.

Finally, we study a problem in the context of circle packing which is concerned with term ergodicity.

### 1. INTRODUCTION

As is well known, in ordinary dynamical system  $(M, f)$ , where  $M$  is a compact metric space and  $f$  is a map from  $M$  to itself, an invariant probability measure  $\mu$  is said to be ergodic if every invariant measurable set is either of zero or full  $\mu$ -measure. The definition of ergodicity for action semigroups/groups generated by  $\mathcal{G} = \{g_1, \dots, g_s\}$  on  $M$  is naturally extended to quasi-invariant measure.<sup>1</sup> Recall that a quasi-invariant measure is ergodic if every measurable invariant set with respect to  $\mathcal{G}$  is either of zero or full measure. In view of topological theory, the counterpart to ergodicity is minimality. More precisely, an action semigroup/group is said to be minimal if each closed invariant subset is either empty or coincides with the whole of space. Minimal systems have been studied extensively by many authors; see for instance [1, 2, 4, 5, 7, 9]. Gorodetskiĭ and Il'yashenko, in [5], provided an example of an action semigroup/group generated by two circle diffeomorphisms that is robustly minimal in the  $C^1$ -topology. In [4], this one-dimensional example is generalized to an action semigroup/group on higher-dimensional manifolds which is also  $C^1$ -robustly minimal. Recently, in [7], the authors provided an action semigroup generated by two diffeomorphisms on any compact manifold that is  $C^1$ -robustly minimal. In typical papers, finding a local invariant set with non-empty interior plays a key role.

Now, we are going to concentrate on some relations between ergodicity and minimality in action semigroups/groups. Notice that there are some examples of ergodic action semigroups/groups having global fixed points. So, in general, ergodicity does not imply minimality. Thus, in the opposite direction, a natural

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<sup>1</sup>A measure  $\mu$  is said to be quasi-invariant if  $(g_i)_*\mu$  is absolutely continuous with respect to  $\mu$  for every element  $g_i$  in  $\mathcal{G}$ .

question arises: which system having the minimal property is ergodic? To answer this question one can refer to an earlier result contained in [2] which allows us to solve the following conjecture concerning the one-dimensional case in the affirmative under some additional assumptions, although the conjecture and the question are apparently far from each other.

**Conjecture 1.** *Every minimal smooth action of a finitely generated group on the circle is ergodic with respect to the Lebesgue measure.*

In this note, we prove the above mentioned conjecture under some assumption which is not unusual. Actually, by this assumption, we insure that our result does not have any contradiction with results by Furstenberg [3].

On the other hand, if there exists a relation between minimality and ergodicity, it is natural to get, as a corollary, robustness of ergodic systems. By these results, we provide an example of some action semigroup/group which is both robustly minimal and robustly term ergodic.

Finally, as an application of this note, we study the branch of mathematics generally known as *circle packing*. The packing problem is concerned with how to pack a number of objects, each with given shape and size, into a bounded region without overlap (see more details on this context [6]; also see <http://hydra.nat.uni-magdeburg.de/packing/cci/#Overview>). Here, we consider a problem of packing circles with unequal radii and some additional property into a given circle. In fact, by benefit of the concept of term ergodicity, we give a negative answer to this problem of the circle packing in general.

## 2. MAIN RESULTS

We begin by introducing some definitions and notation and then formulate our main results. Throughout this paper,  $M$  stands for a smooth compact Riemannian manifold and  $\text{vol}$  is normalized volume. Also, consider the space  $\text{Diff}^1(M)$  of  $C^1$ -diffeomorphisms of  $M$ , endowed with the  $C^1$ -topology. A point  $x \in M$  is a Lebesgue density point of measurable set  $A \subseteq M$  if

$$\lim_{r \rightarrow 0} \text{vol}(A : B(x, r)) = \frac{\text{vol}(A \cap B(x, r))}{\text{vol}(B(x, r))} = 1,$$

where  $B(x, r)$  is the geodesic ball of radius  $r$  centered about  $x$ . Denote by  $DP(A)$  the set of Lebesgue density points of a measurable set  $A$ . By the Lebesgue density point theorem, for every measurable set  $A$ ,

$$\text{vol}(A \Delta DP(A)) = 0.$$

Now, consider a collection of diffeomorphisms  $\mathcal{G} = \{g_1, \dots, g_s\}$  on  $M$ . Write  $\mathcal{G}^{-1} = \{g_1^{-1}, \dots, g_s^{-1}\}$ . The action semigroup  $\langle \mathcal{G} \rangle^+$  generated by  $\mathcal{G}$  is given by

$$\langle \mathcal{G} \rangle^+ = \{h : M \rightarrow M : h = g_{i_n} \circ \dots \circ g_{i_1}, i_j \in \{1, \dots, s\}\} \cup \{\text{id} : M \rightarrow M\}.$$

Furthermore, the action group  $\langle \mathcal{G} \rangle$  generated by  $\mathcal{G}$  is defined by  $\langle \mathcal{G} \cup \mathcal{G}^{-1} \rangle^+$ . So, every action group is no more than an action semigroup.

Notice that  $\lim_{r \rightarrow \infty} g_{\omega}^r \notin \langle \mathcal{G} \rangle^+$  where  $g_{\omega_1, \dots, \omega_r}^r = g_{\omega_r} \circ g_{\omega_1, \dots, \omega_{r-1}}^{r-1}$ . Also, we denote the reverse iteration by  $\hat{g}_{\omega_1, \dots, \omega_r}^r = g_{\omega_1} \circ \hat{g}_{\omega_2, \dots, \omega_r}^{r-1}$ .

Let us recall that the subset  $\mathcal{K}$  of  $M$  is invariant with respect to an action semigroup  $\Gamma$  generated by  $\mathcal{G} = \{g_1, \dots, g_s\}$  if

$$\mathcal{K} = \bigcup_{i=1}^s g_i(\mathcal{K}).$$

Also,  $\Gamma$  is said to be *minimal* if each closed invariant subset  $A$  of  $M$  with respect to  $\Gamma$  is empty or coincides with  $M$ .

Observe that for ordinary dynamical system  $(M, f)$ , the minimality of  $f$  is equivalent to that of  $f^{-1}$ . This is not the case for dynamical systems with several maps: there exists a minimal action semigroup  $\langle f_1, \dots, f_s \rangle^+$  on the circle such that  $\langle f_1^{-1}, \dots, f_s^{-1} \rangle^+$  is not minimal [9].

**Definition 2.1.** Let  $\Gamma$  be an action semigroup generated by  $\mathcal{G} = \{g_1, \dots, g_s\}$  and a probability measure space  $(M, \mathcal{M}, \mu)$  where  $\mu$  is quasi-invariant with respect to  $\Gamma$ . The measure  $\mu$  is called *term ergodic* if for every measurable set  $\mathcal{B} \in \mathcal{M}$  with

$$g_i^{-1}(\mathcal{B}) = \mathcal{B} \quad \forall i = 1, \dots, s,$$

we have that either  $\mu(\mathcal{B}) = 0$  or  $\mu(\mathcal{B}) = 1$ .

Here, to obtain term ergodic results for an action semigroup/group, we need just  $C^{1+\alpha}$ -regularity. In this regard, we begin by stating the term ergodic result for an action semigroup.

**Theorem A.** *Every boundaryless compact two-dimensional manifold  $M$  admits a finite set of  $C^{1+\alpha}$ -diffeomorphisms that generates a  $C^{1+\alpha}$ -robustly term ergodic action semigroup with respect to volume measure.*

We recall that a property  $\mathcal{P}$  holds  $C^r$ -robustly for action semigroup  $\Gamma$  generated by  $\mathcal{G} = \{g_1, \dots, g_s\}$  if it holds for action semigroup  $\hat{\Gamma}$  generated by  $\mathcal{F} = \{f_1, \dots, f_s\}$  whose elements are  $C^r$ -perturbations of elements of  $\mathcal{G}$ .

Since every action group is an action semigroup, the following corollary is an immediate consequence of Theorem A.

**Corollary A'.** *Theorem A is valid for action groups.*

### 3. SOME NEW RESULTS ABOUT MINIMALITY

First of all, we state some results about minimality on compact manifolds in any dimension.

**Lemma 3.1.** *Let  $M$  be a compact manifold and let the action semigroup generated by homeomorphisms  $\mathcal{G} = \{g_1, \dots, g_s\}$  be minimal. Then every proper invariant set with respect to each  $g_i$  for  $i = 1, \dots, s$  and its complement are dense in the whole space.*

*Proof.* Suppose that

$$g_i(\mathcal{B}) = \mathcal{B} \quad \forall i = 1, \dots, s.$$

So we have

$$g_i(\overline{\mathcal{B}}) = \overline{\mathcal{B}} \quad \forall i = 1, \dots, s.$$

If  $\overline{\mathcal{B}}$  is closed and  $\langle \mathcal{G} \rangle^+$  is minimal on  $M$ , then  $\overline{\mathcal{B}} = M$ . On the other hand,  $g_i(\mathcal{B}^c) = \mathcal{B}^c$  for  $i = 1, \dots, s$ , where  $\mathcal{B}^c$  is the complement of  $\mathcal{B}$ . By minimality of  $\langle \mathcal{G} \rangle^+$  and invariance of the subset  $\mathcal{B}^c$  with respect to each generator, one can have density of this subset in  $M$ ; that is,  $\overline{\mathcal{B}^c} = M$ .  $\square$

It follows that

$$\mathcal{B} \cap B(x, r) \neq \emptyset \quad \text{and} \quad \mathcal{B}^c \cap B(x, r) \neq \emptyset$$

for each real number  $r > 0$  and  $x \in M$ .

**Lemma 3.2.** *Let  $M$  be a compact Riemannian manifold and let the action semigroup generated by diffeomorphisms  $\mathcal{G} = \{g_1, \dots, g_s\}$  be minimal. Then density points of every proper invariant set  $\mathcal{B}$  with respect to each  $g_i$  for  $i = 1, \dots, s$ , with positive volume is dense in the whole space.*

*Proof.* Suppose that  $0 < \text{vol}(\mathcal{B})$  and

$$g_i(\mathcal{B}) = \mathcal{B} \quad \forall i = 1, \dots, s.$$

If  $DP(\mathcal{B})$  is not dense, then there exists a neighborhood  $B(x_0, r_0)$  for some point  $x_0$  of  $M$  so that  $B(x_0, r_0) \cap DP(\mathcal{B}) = \emptyset$ . By the Lebesgue density point theorem, one can have  $\text{vol}(B(x_0, r_0) \cap \mathcal{B}) = 0$ ; equivalently  $\text{vol}(B(x_0, r_0) \cap \mathcal{B}^c) = \text{vol}(B(x_0, r_0))$ . We remark that the volume measure is a quasi-invariant for any  $C^1$ -diffeomorphism. So, the union of iterates  $B(x_0, r_0) \cap \mathcal{B}$  under  $\mathcal{G}$  has zero volume measure. Now, the assumption of minimality  $\langle \mathcal{G} \rangle^+$  and invariance of  $\mathcal{B}^c$  under  $\mathcal{G}$  yields a contradiction with  $0 < \text{vol}(\mathcal{B})$ . Hence  $\overline{DP(\mathcal{B})} = M$ .  $\square$

Notice that, similarly, when  $0 < \text{vol}(\mathcal{B}) < 1$  both the subsets  $DP(\mathcal{B})$  and  $DP(\mathcal{B}^c)$  are dense.

**Lemma 3.3.** *Let  $M$  be a compact connected two-dimensional manifold. Then there exist a set  $\mathcal{H} = \{h_1, \dots, h_k\}$  of  $C^1$ -diffeomorphisms on  $M$  and an invariant set  $\Delta$  with respect to  $\mathcal{H}$  and non-empty interior so that  $\langle \mathcal{H} \rangle^+$  is minimal on  $\Delta$  and for every  $x \in \Delta$  and  $i = 1, \dots, k$ ,  $Dh_i$  at  $x$  has two complex eigenvalues.*

*Proof.* Let  $A$  be rotation matrix by angle  $\theta = 179^\circ$ . Define a linear map  $T$  as

$$T(x, y) = \kappa \cdot A(x, y) \quad \forall (x, y) \in \mathbb{R}^2$$

where  $3/4 < \kappa < 1$ . The choice of  $\theta$  insures that each eigenvalue of  $T$  at each point is complex. Put  $V = B(0, \delta)$ . Clearly,  $T(\overline{V}) \subset V$  and for every  $x \in \overline{V}$ ,  $DT$  at  $x$  has two complex eigenvalues  $\lambda, \bar{\lambda}$  with  $|\lambda| = |\bar{\lambda}| > 3/4$ . For every  $y \in W$ , we define

$$S_y(x) = T(x - y) + y \quad \forall x \in V$$

where  $W = \{x \in V : |x| = \frac{3}{4}\delta\}$ . Observe that by construction of  $T$  and  $S_y$  the set

$$T(V) \cup \left[ \bigcup_{y \in W} S_y(V) \right]$$

is a cover for  $\overline{V}$ . Therefore, it has a finite subcover as  $\overline{V} \subset T(V) \cup [\bigcup_{i=1}^{k-1} S_i(V)]$ . Take  $\mathcal{H}_0 = \{T, S_1, \dots, S_{k-1}\}$ . Since every element of  $\mathcal{H}_0$  is a contraction, there is a ball  $U$  that is mapped into itself by  $T$  and  $S_1, \dots, S_{k-1}$ ; i.e.,  $\mathcal{H}_0(U) = T(U) \cup [\bigcup_{i=1}^{k-1} S_i(U)] \subset U$  (when  $\kappa$  is very close to  $3/4$ , one can take  $U = B(0, 16\delta)$ ) to have this property). Thus,

$$\lim_{n \rightarrow \infty} \mathcal{H}_0^n(U) = \lim_{n \rightarrow \infty} \mathcal{H}_0^{n-1}[\mathcal{H}_0(U)] = \Delta$$

is the unique non-empty invariant set with respect to  $\mathcal{H}_0$  whose interior is non-empty. Notice that the semigroup generated by  $\mathcal{H}_0$  on  $\Delta$  is minimal.

Take gradient Morse-Smale vector fields  $\dot{x} = \nabla F_i(x)$  on  $M$  with unique hyperbolic attracting equilibrium  $p_i$  for  $i = 1, \dots, k$  (see e.g. Theorem 3.35 of [10] for

the existence of Morse functions  $F$  with unique extrema) and let  $h_i$  be its time-1 map. We may assume that each  $p_i$  belongs to an open neighborhood  $\hat{V}$  and each eigenvalue of  $Dh_i(p_i)$  is complex.

Now, working in a coordinate chart on a small open neighborhood  $\hat{V} \subset \mathbb{R}^2$  and  $U \subset \hat{V}$ , one can assume that  $h_1 = T$  and  $h_i = S_i$  on  $U$  for  $i = 2, \dots, k$ . The action semigroup generated by  $\mathcal{H} = \{h_1, \dots, h_k\}$  is minimal on the set  $\Delta$ . □

Since the construction used in Lemma 3.3 is  $C^1$ -robust, by an argument similar to that used in [4] and [7], we can have a finite extension  $\mathcal{G}$  of  $\mathcal{H}$  so that  $\langle \mathcal{G} \rangle^+$  is  $C^1$ -robust minimal on  $M$ .

**Corollary 3.4.** *There is a finite extension of  $\mathcal{H}$  to finite set  $\mathcal{G}$  such that  $\langle \mathcal{G} \rangle^+$  is  $C^1$ -robustly minimal.*

#### 4. ROBUSTLY TERM ERGODIC: PROOF OF THEOREM A

We use the following lemma to prove our main theorem.

**Lemma 4.1** (Bounded distortion in the Hutchinson attractor). *Consider a finite family  $\mathcal{H} = \{h_1, \dots, h_k\} \subset \text{Hom}(M)$  where each  $h_i$  is a contracting  $C^{1+\alpha}$ -map of the closure of an open set  $D \subset M$ . Then, there exists  $L_{\mathcal{H}} > 0$  such that for every  $n \in \mathbb{N}$  and  $\omega \in \Sigma_k^+$ ,*

$$L_{\mathcal{H}}^{-1} < \left| \frac{\det(D\hat{h}_{\omega}^n(x))}{\det(D\hat{h}_{\omega}^n(y))} \right| < L_{\mathcal{H}} \quad \forall \quad x, y \in \Delta_{\mathcal{H}}$$

where  $\Delta_{\mathcal{H}}$  is the Hutchinson attractor of IFS( $\mathcal{H}$ ) in  $\bar{D}$ .

*Proof.* Define  $\Phi : \text{GL}(\dim(M), \mathbb{R}) \rightarrow \mathbb{R}$  by  $\Phi(A) = \log |\det(A)|$ , and  $F_i(x) = \Phi(Dh_i(x))$ , for any  $i = 1, \dots, k$ . Note that by assumption,  $\log |\det Dh_i|$  is  $\alpha$ -Hölder and thus for every  $x, y \in \Delta_{\mathcal{H}}$  and  $1 \leq i \leq k$ ,

$$|F_i(x) - F_i(y)| \leq C \|x - y\|^\alpha,$$

for some constants  $C > 0$ . On the other hand, for every  $\omega = \omega_1 \omega_2 \dots \in \Sigma_k^+$ ,

$$\begin{aligned} \|\hat{h}_{\omega}^n(x) - \hat{h}_{\omega}^n(y)\| &\leq \|Dh_{\omega_1}\| \|\hat{h}_{\omega_2, \dots, \omega_n}^{n-1}(x) - \hat{h}_{\omega_2, \dots, \omega_n}^{n-1}(y)\| \leq \xi^n \|x - y\| \\ &\leq \xi^n \text{diam}(\Delta_{\mathcal{H}}), \end{aligned}$$

where

$$\xi = \sup_{x \in \Delta_{\mathcal{H}}, 1 \leq i \leq k} \|Dh_i(x)\| < 1.$$

Hence,

$$\begin{aligned} \log \left| \frac{\det(D\hat{h}_{\omega}^n(x))}{\det(D\hat{h}_{\omega}^n(y))} \right| &= \sum_{i=0}^{n-1} |F_{\omega_i}(\hat{h}_{\omega}^i(x)) - F_{\omega_i}(\hat{h}_{\omega}^i(y))| \leq C \sum_{i=0}^{n-1} \|\hat{h}_{\omega}^i(x) - \hat{h}_{\omega}^i(y)\|^\alpha \\ &\leq C \sum_{i=0}^{n-1} \{\xi^i \text{diam}(\Delta_{\mathcal{H}})\}^\alpha \leq CM(\text{diam}(\Delta_{\mathcal{H}}))^\alpha \sum_{i=0}^{\infty} (\xi^\alpha)^i. \end{aligned}$$

Taking  $L_{\mathcal{H}} = \exp\{C\xi^\alpha(\text{diam}(\Delta_{\mathcal{H}}))^\alpha/(1 - \xi^\alpha)\}$  the desired inequality holds. □

Now, we will prove Theorem A.

*Proof of Theorem A.* Suppose that  $M$  is a two-dimensional manifold. Consider Lemma 3.3 for  $C^{1+\alpha}$ -diffeomorphisms  $\mathcal{H} = \{h_1, \dots, h_k\}$ , which will provide an invariant set  $\Delta_{\mathcal{H}}$  with non-empty interior with respect to action semigroup  $\langle \mathcal{H} \rangle^+$  so that the  $Dh_i(x)$  has complex eigenvalues for each  $i = 1, \dots, k$  and  $x \in \Delta_{\mathcal{H}}^\circ$ . Take the subset  $U$  of  $M$  so that it maps into itself by  $h_i$ , i.e.,  $\mathcal{H}(U) = \bigcup_{i=1}^k h_i(U) \subset U$ , and also take  $\langle \mathcal{G} \rangle^+$  as a finite  $C^{1+\alpha}$ -extension of  $\mathcal{H}$  which is  $C^{1+\alpha}$ -robustly minimal on  $M$  (see Corollary 3.4).

We claim that  $\langle \mathcal{G}^{-1} \rangle^+$  is term ergodic. To this end, suppose that  $0 < \text{vol}(\mathcal{B}) < 1$  and

$$g_i(\mathcal{B}) = \mathcal{B} \quad i = 1, \dots, s.$$

Now, let  $p \in \Delta_{\mathcal{H}}^\circ$  and  $J = B(p, \hat{\delta}) \subset \Delta_{\mathcal{H}}^\circ$ . Since  $Dh_i(x)$  has two complex eigenvalues  $\lambda_i, \bar{\lambda}_i$  for every  $x \in U$  and  $i = 1, \dots, k$ , the image of an open ball under the iteration of  $h_i$  is a ball, too.

On the other hand, there is  $\omega \in \Sigma_k^+$  so that  $\lim_{n \rightarrow \infty} \text{diam}(\hat{h}_\omega^r(U)) = 0$  and  $\bigcap_r \hat{h}_\omega^r(U) = \{p\}$ . So,  $\hat{h}_\omega^r(U)$  is a ball for each  $r$  and  $\text{diam}(\hat{h}_\omega^r(U)) \rightarrow 0$  as  $r \rightarrow \infty$ . Define

$$r_0 = \min\{r : \text{diam}(\hat{h}_\omega^r(U)) < \delta\}.$$

Clearly,  $p \in \hat{h}_\omega^{r_0}(U) \subset B(p, \delta)$ . Since  $\hat{h}_\omega^{r_0}(U)$  is a ball with  $\text{diam}(\hat{h}_\omega^{r_0}(U)) < \delta$  and  $\text{diam}(\hat{h}_\omega^{r_0-1}(U)) \geq \delta$ , then

$$\text{diam}(h_i(B(p, \delta/2))) \leq \text{diam}(\hat{h}_\omega^{r_0}(U)) < \delta$$

and

$$\frac{\text{vol}(h_i(B(p, \delta/2)))}{\text{vol}(J)} \leq \frac{\text{vol}(\hat{h}_\omega^{r_0}(U))}{\text{vol}(J)}.$$

Notice that  $\hat{h}_\omega^{r_0}(U) \subset J$  and  $h_i(\mathcal{B}^c) = \mathcal{B}^c$  for every  $i = 1, \dots, k$  when  $\mathcal{H}$  is a subset of  $\mathcal{G}$ . So, one can have  $\hat{h}_\omega^{r_0}(\mathcal{B}^c \cap U) \subseteq \mathcal{B}^c \cap \hat{h}_\omega^{r_0}(U)$ .

Hence, we have

$$\begin{aligned} \frac{\text{vol}(\mathcal{B}^c \cap J)}{\text{vol}(J)} &\geq \frac{\text{vol}(\mathcal{B}^c \cap \hat{h}_\omega^{r_0}(J))}{\text{vol}(J)} \\ &\geq \frac{\text{vol}(\mathcal{B}^c \cap \hat{h}_\omega^{r_0}(J))}{\text{vol}(\hat{h}_\omega^{r_0}(J))} \cdot \frac{\text{vol}(\hat{h}_\omega^{r_0}(J))}{\text{vol}(J)} \\ &\geq \frac{\text{vol}(\hat{h}_\omega^{r_0}(\mathcal{B}^c \cap J))}{\text{vol}(\hat{h}_\omega^{r_0}(J))} \cdot \frac{\text{vol}(h_i(B(p, \delta/2)))}{\text{vol}(J)}. \end{aligned}$$

Indeed, the last inequality is implied by the bounded distortion result, Lemma 4.1, as follows:

$$\begin{aligned} \text{vol}(\hat{h}_\omega^{r_0}(\mathcal{B}^c \cap J)) &= \int_{\mathcal{B}^c \cap J} |\det D\hat{h}_\omega^{r_0}| \, d\text{vol} \geq \text{vol}(\mathcal{B}^c \cap J) \inf_{x \in \Delta_{\mathcal{H}}} |\det D\hat{h}_\omega^{r_0}(x)|, \\ \text{vol}(\hat{h}_\omega^{r_0}(J)) &= \int_J |\det \hat{h}_\omega^{r_0}| \, d\text{vol} \leq \text{vol}(J) \sup_{x \in \Delta_{\mathcal{H}}} |\det D\hat{h}_\omega^{r_0}(x)|. \end{aligned}$$

This means that  $\frac{\text{vol}(\mathcal{B}^c \cap J)}{\text{vol}(J)}$  is bounded from below for every neighborhood  $J$  of  $p$ .

So,  $p \notin DP(\mathcal{B})$ . Similarly, one can prove that  $p \notin DP(\mathcal{B}^c)$ . Hence

$$\Delta_{\mathcal{H}}^{\circ} \cap (DP(\mathcal{B}) \cup DP(\mathcal{B}^c)) = \emptyset,$$

which is a contradiction with Lemma 3.2 and the proof is completed. □

Observe that ergodicity may be removed from some ordinary dynamical systems under a perturbation; an irrational translation is such a system. But the following example, containing an irrational translation, is robust term ergodic, based on the exciting example of Gorodetskiĭ and Il'yashenko [5]. Moreover, it is shown that sufficiently close to such system in the  $C^{1+\alpha}$ - topology, there is a term ergodic action semigroup, each of whose generators is not ergodic.

**Example 4.2.** Suppose  $f_1$  is a north-south pole  $C^2$ -diffeomorphism of the circle  $\mathbb{S}^1$  possessing an attracting fixed point  $p$  as a north pole and a repelling fixed point  $q$  as a south pole with multipliers

$$1/2 < f_1'(p) < 1 \quad \text{and} \quad 1/2 < (f_1^{-1})'(q) < 1.$$

Consider the map  $f_2 = R_\lambda$ , where  $R_\lambda$  is the rotation by irrational angle  $\lambda$  on the circle. Observe that both systems  $\langle f_1, f_2 \rangle^+$  and  $\langle f_1^{-1}, f_2^{-1} \rangle^+$  are  $C^{1+\alpha}$ - robust minimal [5]. Thus the systems  $\langle f_1, f_2 \rangle^+$  and  $\langle f_1^{-1}, f_2^{-1} \rangle^+$  are  $C^{1+\alpha}$ -robust term ergodic. Moreover, take a rational number  $\gamma$  sufficiently close to  $\lambda$  so that the system  $\langle f_1, R_\gamma \rangle^+$  is minimal, too. Clearly, none of  $f_1, f_1^{-1}, R_\gamma, R_\gamma^{-1}$  is either ergodic or minimal, but  $\langle f_1, R_\gamma \rangle^+$  is both minimal and term ergodic.

### 5. A PROBLEM ON CIRCLE PACKING CONCERNING ERGODICITY

Let  $M$  be a compact Riemannian manifold and  $\Gamma = \langle \mathcal{G} \cup \mathcal{G}^{-1} \rangle^+$  be a minimal action group generated by homeomorphisms  $\mathcal{G} = \{g_1, \dots, g_s\}$ . Suppose that  $\mathcal{B}$  is an invariant set with respect to each generator of  $\Gamma$ , that is,

$$g_i(\mathcal{B}) = \mathcal{B} \quad \forall i = 1, \dots, s.$$

Assume that  $0 < \text{vol}(\mathcal{B}) < 1$ . By Lemmas 3.1 and 3.2, we insure that  $\mathcal{B}, \mathcal{B}^c, DP(\mathcal{B}), DP(\mathcal{B}^c)$  are dense in  $M$ .

Now, let  $y$  be an arbitrary point of  $DP(\mathcal{B})$ . By definition of density point, one can find  $\delta > 0$  so that

$$\text{vol}(\mathcal{B} \cap B(y, \delta)) > \frac{3}{4} \text{vol}(B(y, \delta)).$$

One kind of circle packing problem, with respect to dynamical systems, may be as follows.

**Problem 5.1.** Under the above assumptions, is there a family  $\{B(p, \delta_p)\}_{p \in \mathcal{P}}$  with  $\mathcal{P} \subset DP(\mathcal{B}^c) \cap B(y, \delta)$  so that it satisfies the following conditions:

- (i)  $B(p, \delta_p) \subseteq B(y, \delta)$ ;
- (ii)  $\forall p, q \in \mathcal{P}$  with  $p \neq q, B(p, \delta_p) \cap B(q, \delta_q) = \emptyset$ ;
- (iii)  $\frac{2}{3} \text{vol}(B(y, \delta)) < \text{vol}(\bigcup_{p \in \mathcal{P}} B(p, \delta_p))$ ; and
- (iv)  $\text{vol}(\mathcal{B}^c \cap B(p, \delta_p)) > \frac{1}{2} \text{vol}(B(p, \delta_p))$ .

In general and without any additional assumption, the answer to the problem is negative. Actually, one can prove that the minimal action group  $\Gamma$  is term ergodic

with respect to volume measure if there is a family  $\{B(p, \delta_p)\}_{p \in \mathcal{P}}$  which satisfies the properties (i)-(iv), for some point  $y$ . Indeed, one can have

$$\begin{aligned} \text{vol}(\mathcal{B}^c \cap B(y, \delta)) &= \text{vol}\left[\bigcup_{p \in \mathcal{P}} (\mathcal{B}^c \cap B(p, \delta_p))\right] \\ &= \sum_{p \in \mathcal{P}} \text{vol}(\mathcal{B}^c \cap B(p, \delta_p)) \\ &> \sum_{p \in \mathcal{P}} \frac{1}{2} \text{vol}(B(p, \delta_p)) \\ &= \frac{1}{2} \text{vol}\left(\bigcup_{p \in \mathcal{P}} B(p, \delta_p)\right) \\ &> \frac{1}{3} \text{vol}(B(y, \delta)), \end{aligned}$$

which is a contradiction and so  $\text{vol}(\mathcal{B}) \in \{0, 1\}$ . Therefore, minimality of  $C^1$ -action group  $\langle \mathcal{G} \cup \mathcal{G}^{-1} \rangle^+$  implies term ergodicity, which leads to the following counterexample in dimension two.

**Example 5.2.** In [3], Furstenberg constructed an analytic diffeomorphism  $T$  of torus which preserves Haar measure and is minimal but not ergodic. So,  $\langle T, T^{-1} \rangle^+$  is a minimal action group which is not term ergodic. This is a contradiction.

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