**P-ORDERINGS OF NONCOMMUTATIVE RINGS**

KEITH JOHNSON

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**Abstract.** Let $K$ be a local field with valuation $\nu$, $D$ a division algebra over $K$ to which $\nu$ extends, $R$ the maximal order in $D$ with respect to $\nu$ and $S$ a subset of $R$. If $D[x]$ denotes the ring of polynomials over $D$ with $x$ a central variable, then the set of integer valued polynomials on $S$ is $\text{Int}(S, R) = \{f(x) \in D[x] : f(S) \subseteq R\}$. If $D$ is commutative, then M. Bhargava showed how to construct a regular $R$-basis for this set by introducing the idea of a $P$-ordering of $S$. We show that this definition can be extended to the noncommutative case in such a way as to construct regular bases there also. We show how to extend methods developed to compute $P$-orderings in the commutative case and apply them to give a recursive formula for such an ordering for $D$ the rational quaternions and $S = R$ the Hurwitz quaternions localized at the prime $1 + i$.

1. Introduction

Let $R$ be a discrete valuation domain with field of fractions $K$, valuation $\nu$ and uniformizing element $P$, and let $S$ be a subset of $R$. A $P$-ordering of $S$, as introduced by M. Bhargava in [1], is a sequence $\{a_i : i = 0, 1, 2, \ldots\} \subseteq S$ with the property that for each $n > 0$ the element $a_n$ minimizes the quantity $\nu(\prod_{i=0}^{n-1} (a - a_i))$ over $a \in S$. The importance of such sequences stems from the fact that the sequence of polynomials $\{\prod_{i=0}^{n-1} (x - a_i) / (a_n - a_i) : n = 0, 1, 2, \ldots\}$ forms a regular basis for the $R$-algebra of integer valued polynomials $\text{Int}(S, R) = \{f(x) \in K[x] : f(S) \subseteq R\}$. The problem this paper addresses is extending this construction to certain noncommutative rings, specifically to the case of $R$ the maximal order in a division algebra over a local field. The particular case we will consider in some detail will be that of the Hurwitz integers in the quaternions localized at the prime $1 + i$.

Our definition is as follows:

**Definition 1.1.** If $K$ is a local field with valuation $\nu$, $D$ a division algebra over $K$ to which the valuation $\nu$ extends, $R$ the maximal order in $D$ and $S$ a subset of $R$, then a $\nu$-ordering of $S$ is a sequence $\{a_i : i = 0, 1, 2, \ldots\} \subseteq S$ with the property that for each $n > 0$ the element $a_n$ minimizes the quantity $\nu(f_n(a_0, \ldots, a_{n-1})(a))$ over $a \in S$ where $f_0 = 1$ and, for $n > 0$, $f_n(a_0, \ldots, a_{n-1})(x)$ is the minimal polynomial of the set $\{a_0, a_1, \ldots, a_{n-1}\}$. The sequence of valuations $\{\nu(f_n(a_0, \ldots, a_{n-1})(a_n)) : n = 0, 1, \ldots\}$ will be called the $\nu$-sequence of $S$.

The minimal, or Wedderburn, polynomial of a set $\{a_0, a_1, \ldots, a_{n-1}\}$, as introduced in [5], is the monic polynomial of least degree in $R[x]$ which vanishes on $\{a_0, a_1, \ldots, a_{n-1}\}$. In the commutative case this is, of course, $\prod_{i=0}^{n-1} (x - a_i)$; hence this definition reduces to the usual one.
The point of this definition is that it is related to the algebra of integer valued polynomials as in the commutative case:

**Proposition 1.2.** If \( K \) is a local field with valuation \( \nu \), \( D \) a division algebra over \( K \) to which the valuation \( \nu \) extends, \( R \) a maximal order in \( D \), \( P \in R \) a unimodular element, \( S \) a subset of \( R \), \( \{ a_i : i = 0, 1, 2, \ldots \} \subseteq S \) a \( \nu \)-ordering and \( f_n(a_0, \ldots, a_{n-1})(x) \) the minimal polynomial of \( \{a_0, a_1, \ldots, a_{n-1}\} \), then the sequence \( \{ \alpha_n = \nu(f_n(a_0, \ldots, a_{n-1})(a_0)) : n = 0, 1, 2, \ldots \} \) depends only on the set \( S \) and not on the choice of \( \nu \)-ordering and the sequence of polynomials

\[
\{ P^{-\alpha_n} \prod_{i=0}^{n-1} f_n(a_0, \ldots, a_{n-1})(x) : n = 0, 1, 2, \ldots \}
\]

forms a regular \( R \)-basis for the \( R \)-algebra of polynomials integer valued on \( S \).

To prove this result we connect polynomials integer valued on a set with the invariant factors of Vandermonde matrices of collections of elements of the set. This connection between invariant factors and \( P \)-orderings was noted in the commutative case in [2].

The value of this result depends on the computability of the orderings and \( \nu \)-sequences. In the commutative case effective tools for this are formulas for the orderings and \( \nu \)-sequences of translates, dilations and certain unions of sets, and we describe the extent to which these formulas extend to the noncommutative case. These formulas sometimes yield recursive formulas which determine the orderings and \( \nu \)-sequence completely. A particular case of this, which we describe in detail, is that of the Hurwitz quaternions localized at the prime \( 1+i \).

## 2. Noncommutative polynomials

We summarize here some basic results about polynomials over noncommutative rings which we will need. Most of this material is taken from [11] and [8]. Throughout this section \( R \) will denote a subring of a division algebra.

**Definition 2.1.** The left \( R \)-algebra of polynomials with coefficients in \( R \), which we will denote \( R[x] \), is the set of finite sums of the form \( \sum_{i=0}^{n} a_i x^i \) with \( a_i \in R \) for each \( i \) with sum and product

\[
\sum_{i=0}^{n} a_i x^i + \sum_{i=0}^{n} b_i x^i = \sum_{i=0}^{n} (a_i + b_i) x^i
\]

\[
\sum_{i=0}^{n} a_i x^i \sum_{j=0}^{n} b_j x^j = \sum_{k=0}^{n} \left( \sum_{i+j=k} a_i b_j \right) x^k.
\]

For \( a \in R \) the value of the polynomial \( \sum_{i=0}^{n} a_i x^i \) at \( a \) is \( \sum_{i=0}^{n} a_i a^i \).

Note that in this definition \( x \) is a central variable, i.e. a variable commuting with all elements of \( R \), and so evaluation may not act as a ring homomorphism unless either all of the coefficients of one of the polynomials are in the centre of \( R \) or the point at which the polynomials are being evaluated is.

**Notation 2.1.** For \( a, b \in R \) the conjugate of \( a \) by \( b \) is denoted \( a^b \) and equals \( bab^{-1} \).

**Lemma 2.2.** For \( a, b, c \in R \) we have \( (a^b)^c = a^{bc} \), and for \( k \) an integer, \( (a^k)^c = (a^c)^k \).
Lemma 2.3. If \( f, g \in R[x] \) and \( a \in R \), then the value of the product polynomial \( f(x)g(x) \) at \( a \) is \( f(a^g(a))g(a) \).

Notation 2.2. The minimal polynomial of the set \( S \subseteq R \) will be denoted \( f(S)(x) \) or, if the set is finite and indexed as in Definition 2.1,

\[
f(a_0, a_1, \ldots, a_n)(x).
\]

A subset \( S \) of \( R \) is polynomially independent in the sense of \([10]\), section 4, if for an \( a \in S \) the minimal polynomial of \( S \setminus \{ a \} \) does not vanish at \( a \). For such sets Lemma 2.3 and induction yield:

Proposition 2.4. If \( \{a_0, a_1, \ldots, a_n\} \) is a polynomially independent subset of \( R \), then the minimal polynomial is given by

\[
f(a_0)(x) = (x - a_0),
\]

\[
f(a_0, \ldots, a_n)(x) = (x - a_n^{f(a_0, \ldots, a_n-1)(a_n)})f(a_0, \ldots, a_{n-1})(x)
\]

\[
= (x - a_n^{f(a_0, \ldots, a_n-1)(a_n)}) \ldots (x - a_0).
\]

3. Vandermonde matrices

Notation 3.1. If \( \{a_i : i = 0, 1, \ldots, n\} \) is a subset of \( R \), then \( V(a_0, \ldots, a_n) \) will denote the \((n + 1) \times (n + 1)\) Vandermonde matrix whose \((i + 1, j + 1)\) entry is \( a_j^i \) and \( V_{i_0, \ldots, i_n}(a_0, \ldots, a_n) \) will denote the \((n + 1) \times (n + 1)\) generalized Vandermonde matrix whose \((k + 1, j + 1)\) entry is \( a_j^{i_k} \). Also \( \text{Diag}(v_0, \ldots, v_n) \) will denote the \((n + 1) \times (n + 1)\) diagonal matrix with diagonal entries \( v_0, \ldots, v_n \).

The familiar result that a matrix with entries in a principal ideal domain is equivalent to a diagonal matrix (the Smith normal form of the matrix) whose diagonal entries are the invariant factors of the matrix extends, according to \([4]\), section 98, to the case of a maximal order in a division algebra. The question of uniqueness of the invariant factors may be a delicate matter in general; however, in the case of an algebra over a local field, the valuations of the invariant factors are easily seen to be unique.

Proposition 3.1. If \( \{a_i : i = 0, 1, 2, \ldots\} \) is a \( \nu \)-ordering of \( S \subseteq R \), then the invariant factors of the Vandermonde matrix \( V(a_0, \ldots, a_n) \) are \( f_i(a_0, \ldots, a_{i-1})(a_i) \) for \( i = 0, 1, \ldots, n \).

Proof. It suffices to show that the Smith normal form of \( V(a_0, \ldots, a_n) \) is the diagonal matrix \( \text{Diag}(f_i(a_0, \ldots, a_{i-1})(a_i) : i = 0, \ldots, n) \). In the commutative case there is a direct proof of this using the fact that the product of the first \( k \) invariant factors of a matrix is the greatest common divisor of the determinants of the submatrices of size \( k \times k \). Since the submatrices of a Vandermonde matrix are all generalized Vandermonde matrices and in the commutative case \( \det(V_{i_0, \ldots, i_k}(b_0, \ldots, b_k)) \) is easily seen to be divisible by \( \det(V(b_0, \ldots, b_k)) \) for any strictly increasing sequence \( i_0, \ldots, i_k \), the result follows from the definition of a \( \nu \)-ordering. This proof is not available in the noncommutative case because the result concerning determinants of generalized Vandermonde matrices is lacking. Instead we construct the Smith normal form directly, following the algorithm described by Dickson in \([4]\).
Define a sequence of \( n + 1 \) matrices \( M[k] \) by \( M[0] = V(a_0, \ldots, a_n) \) and \( M[k] \) the matrix whose \((i + 1, j + 1)\) entry equals \( f_j(a_0, \ldots, a_{i-1})(a_i) \) if \( i = j < k \), equals 0 if either \( i < k \) or \( j < k \) and \( i \neq j \), and equals
\[
(a_j^i f_k(a_0, \ldots, a_{k-1})(a_j))_{i-k} f_k(a_0, \ldots, a_{k-1})(a_j)
\]
if \( i, j \geq k \). We claim that \( M[k+1] \) is obtained from \( M[k] \) by a sequence of operations each of which adds a left multiple of the \( k + 1 \)-st row or column to another row or column. Thus \( M[n] \), being diagonal, is the Smith normal form of \( M[0] \) as required. We verify this by induction on \( k \), first checking that \( \nu(M[k]_{i+1,j+1}) \) for \( i, j \geq k \) is minimized by \( \nu(M[k]_{k+1,k+1}) \):
\[
\nu(M[k]_{i+1,j+1}) = \nu((a_j^i f_k(a_0, \ldots, a_{k-1})(a_j))_{i-k}) + \nu(f_k(a_0, \ldots, a_{k-1})(a_j)) \\
\geq \nu(f_k(a_0, \ldots, a_{k-1})(a_j)) \\
= \nu(M[k]_{k+1,j+1})
\]
and
\[
\nu(M[k]_{k+1,j+1}) = \nu(f_k(a_0, \ldots, a_{k-1})(a_j)) \\
\geq \nu(f_k(a_0, \ldots, a_{k-1})(a_k)) \\
= \nu(M[k]_{k+1,k+1})
\]
since \( \{a_i : i = 0, 1, 2, \ldots\} \) is a \( \nu \)-ordering.

Given this it is clear that by a sequence of row additions of left multiples of rows the matrix \( M[k] \) can be converted to a matrix in which the only nonzero entry in the \( k + 1 \)-st column is \( f_k(a_0, \ldots, a_{k-1})(a_k) \) in position \((k + 1, k + 1)\) and that a sequence of column additions of multiples of this column will place 0’s in all entries of the \( k + 1 \)-st column except the \( k + 1 \) position. It remains to check that in doing this the resulting matrix is \( M[k + 1] \). To see this we calculate the effect of subtracting a left \( a_k^j f_k(a_0, \ldots, a_{k-1})(a_k) \) multiple of the \( i \)-th row from the \( i + 1 \)-st row for \( i = n, n - 1, \ldots, k + 1 \):
\[
M[k]_{i+1,j+1} - a_k^j f_k(a_0, \ldots, a_{k-1})(a_k) M[k]_{i,j+1} \\
= (a_j^i f_k(a_0, \ldots, a_{k-1})(a_j))_{i-k} f_k(a_0, \ldots, a_{k-1})(a_j) - a_k^j f_k(a_0, \ldots, a_{k-1})(a_j) f_k(a_0, \ldots, a_{k-1})(a_j) \\
= [a_j^i f_k(a_0, \ldots, a_{k-1})(a_j) - a_k^j f_k(a_0, \ldots, a_{k-1})(a_j)] \\
= [a_j^i f_k(a_0, \ldots, a_{k-1})(a_j)]_{i-k} f_k(a_0, \ldots, a_{k-1})(a_j) \\
= \left( (a_j^i f_k(a_0, \ldots, a_{k-1})(a_j))_{a_j^f k(a_0, \ldots, a_{k-1})(a_j) - f_k(a_0, \ldots, a_{k-1})(a_k)} \right)_{i-k} \\
= (a_j^i f_k(a_0, \ldots, a_{k-1})(a_j) - a_k^j f_k(a_0, \ldots, a_{k-1})(a_k))(a_j f_k(a_0, \ldots, a_{k-1})(a_j)) \\
= (a_j^i f_k(a_0, \ldots, a_{k-1})(a_j))_{i-(k+1)} f_k(a_0, \ldots, a_{k-1})(a_j),
\]
which equals \( M[k + 1]_{i+1,j+1} \) if \( i > k + 1 \) and \( j \geq k + 1 \), and equals 0 if \( j = k + 1 \) and \( i \geq k + 1 \).

4. Integer valued polynomials

The link between Vandermonde matrices and polynomials is, as in the commutative case.
Proposition 4.1. If \( f(x) = \sum_{i=0}^{n} c_i x^i \in R[x] \) or \( K[x] \) and \( [a_0, \ldots, a_n] \in R^{n+1} \) or \( K^{n+1} \), then \( \{ f(a_0), \ldots, f(a_n) \} = [c_0, \ldots, c_n]V(a_0, \ldots, a_n) \).

Proposition 4.2. If \( \{ a_i : i = 0, 1, 2, \ldots \} \) is a \( \nu \)-ordering of \( S \subseteq R \) and \( \alpha(n) = \nu(f_n(a_0, \ldots, a_{n-1}))(a_n) \) is the associated \( \nu \)-sequence of \( S \) and \( f(x) = \sum_{i=0}^{n} c_i x^i \in \text{Int}(S, R) \), then \( \nu(c_i) \geq -\alpha(n) \) for \( 0 \leq i \leq n \).

Proof. Let \( c = [c_0, \ldots, c_n] \) and \( b = [f(a_0), \ldots, f(a_n)] \) and \( M = V(a_0, \ldots, a_n) \).

Since \( f(x) \in \text{Int}(S, R) \) we have \( b \in R^{n+1} \) and, from the previous proposition, \( b = cM \). If \( N = \text{Diag}(a_0, \ldots, a_{n-1})(a_n) \), then there exist invertible matrices \( A \) and \( B \) with entries in \( R \) such that \( M = ANB \) and so
\[
bB^{-1} = cAN \in R^{n+1}.
\]

Let \( r = \min(\{-\nu(c_i) : i = 0, \ldots, n\}) \) and \( P \in R \) be a uniformizing element for \( \nu \).

By the minimality of \( r \) we have that \( P^r c \in R^{n+1} \) and at least one of its entries is of valuation 0. Since \( A \) is invertible over \( R \) the same is true of \( P^r c A \). Since the matrix \( N \) is diagonal it follows that \( r \) is bounded above by the valuation of one of the diagonal entries of \( N \). Since the valuations of these entries are in nondecreasing order down the diagonal it follows that \( r \) is bounded by \( \nu(f_n(a_0, \ldots, a_{n-1}))(a_n) = \alpha(n) \).

Corollary 4.3. If \( \{ a_i : i = 0, 1, 2, \ldots \} \) is a \( \nu \)-ordering of \( S \subseteq R \), then the polynomials \( \{ P^{-\alpha(n)} f_n(a_0, \ldots, a_{n-1})(x) : n = 0, 1, 2, \ldots \} \) form a regular basis for \( \text{Int}(S, R) \) as a left \( R \)-module.

From the definition of \( \nu \)-ordering we have that for any \( a \in S \) the valuation \( \nu(f_n(a_0, \ldots, a_{n-1}))(a) \) is at least \( \nu(f_n(a_0, \ldots, a_{n-1}))(a_n) = \alpha(n) \), and so it is the case that \( P^{-\alpha(n)} f_n(a_0, \ldots, a_{n-1})(x) \) is integer valued on \( S \). The definition also implies that \( \{ a_0, \ldots, a_{n-1} \} \) is a polynomially independent set and thus that the degree of \( P^{-\alpha(n)} f_n(a_0, \ldots, a_{n-1})(x) \) is \( n \) and these polynomials are linearly independent over \( R \).

Proposition 4.2 and downward induction on the degree show that any polynomial integer valued on \( S \) is an \( R \)-linear combination of the polynomials \( \{ P^{-\alpha(n)} f_n(a_0, \ldots, a_{n-1})(x) : n = 0, 1, 2, \ldots \} \).

5. Computing \( \nu \)-sequences

If \( R \) is a commutative ring the following lemma is of great use in computing \( \nu \)-sequences and constructing \( \nu \)-orderings:

Lemma 5.1. If \( R \) is commutative and \( S_1 \) and \( S_2 \) are disjoint subsets of \( S \) with the property that \( \nu(s_1 - s_2) = 0 \) for any \( s_1 \in S_1 \) and \( s_2 \in S_2 \) and if \( \{ a_i \} \) is a \( \nu \)-ordering of \( S_1 \cup S_2 \), then the subsequence of this ordering consisting of those elements in \( S_1 \) is a \( \nu \)-ordering of \( S_1 \) and similarly for \( S_2 \). Conversely if \( \{ b_i \} \) and \( \{ c_i \} \) are \( \nu \)-orderings of \( S_1 \) and \( S_2 \) respectively with associated \( \nu \)-sequences \( \{ \beta_i \} \) and \( \{ \gamma_i \} \), then the \( \nu \)-sequence of \( S_1 \cup S_2 \) is the shuffle of \( \{ \beta_i \} \) and \( \{ \gamma_i \} \) into nondecreasing order, and this shuffle applied to \( \{ b_i \} \) and \( \{ c_i \} \) gives a \( \nu \)-ordering of \( S_1 \cup S_2 \).

This lemma gives, for many sets of interest, a recursive formula for the \( \nu \)-sequence. See for example [6], [7]. The extension of this to the noncommutative case is as follows:

Lemma 5.2. If \( S_1 \) and \( S_2 \) are disjoint subsets of \( S \) with the property that there is a nonnegative integer \( k \) such that \( \nu(s_1 - s_2) = k \) for any \( s_1 \in S_1 \) and \( s_2 \in S_2 \) and if
$S_1$ and $S_2$ are each closed with respect to conjugation by elements of $R$ and if $\{a_i\}$ is a $\nu$-ordering of $S_1 \cup S_2$, then the subsequence of this ordering consisting of those elements in $S_1$ is a $\nu$-ordering of $S_1$ and similarly for $S_2$. Conversely if $\{b_i\}$ and $\{c_i\}$ are $\nu$-orderings of $S_1$ and $S_2$ respectively with associated $\nu$-sequences $\{\beta_i\}$ and $\{\gamma_i\}$, then the $\nu$-sequence of $S_1 \cup S_2$ is the sum of the sequence $\{ki : i = 0, 1, 2, \ldots\}$ with the shuffle of $\{\beta_i - ki\}$ and $\{\gamma_i - ki\}$ into nondecreasing order and this shuffle applied to $\{b_i\}$ and $\{c_i\}$ gives a $\nu$-ordering of $S_1 \cup S_2$.

Remark. If $R$ is commutative this lemma is no more general than the earlier one, since if $S_1$ and $S_2$ are as in the lemma for some $k > 0$, then $S_1 = P^kS'_i$ and $S_2 = P^kS'_i$ with $S'_i$ and $S'_2$ satisfying the hypothesis of the previous lemma and having $\nu$-sequences $\{\beta_i - ki\}$ and $\{\gamma_i - ki\}$ respectively. Also note that in the commutative case this lemma is most frequently used with the sets $S_i$ being cosets of the maximal ideal, but in the noncommutative case the lemma may not apply since such sets need not be closed under conjugation.

Proof. First note that the hypotheses imply that if $s, s' \in S_1$, then $\nu(s - s') \geq k$ since, choosing $s''$ to be any element of $S_2$, we have $\nu(s - s') - \nu(s'' - s' - s) = \min(\nu(s - s''), \nu(s' - s'')) = k$ and similarly if $s, s' \in S_2$. Thus if $\{s_0, \ldots, s_n\} \subseteq S_1$, then for any $s \in S_1$, we have

$$\nu(f(s_0, \ldots, s_n)(s)) = \nu((s^f(s_0, \ldots, s_{n-1})(s) - s^f(s_0, \ldots, s_{n-1})(s_n))$$

$$\geq k + \nu(f(s_0, \ldots, s_{n-1})(s))$$

and so, by induction, that $\nu(f(s_0, \ldots, s_n)(s)) \geq k(n + 1)$.

Suppose that $\{s_i : i = 0, 1, 2, \ldots\}$ is a $\nu$-ordering of $S_1 \cup S_2$ and that $s_n \in S_1$. Reorder the set $\{s_i : i = 0, 1, \ldots, n - 1\}$ so that $s_i \in S_1$ if $i < \ell$ and $s_i \in S_2$ if $i \geq \ell$. The minimal polynomial of the set $\{s_i : i = 0, 1, \ldots, n - 1\}$ is independent of the ordering of the elements. Hence, by Lemma 2.74, we have that $s_n$ is the element minimizing the $\nu$-valuation over $S_1 \cup S_2$ of

$$f(s_0, \ldots, s_{n-1})(x) = g(x)f(s_1, \ldots, s_{\ell-1})(x)$$

with

$$g(x) = (x - s^f(s_0, \ldots, s_{n-2})(s_{n-1})) \ldots (x - s^f(s_0, \ldots, s_{\ell-1})(s_{\ell-1})).$$

If $s \in S_1$, then

$$\nu(g(s)) = \sum_{i=\ell}^{n-1} \nu(s^f(s_0, \ldots, s_{i-1})(s)) - s^f(s_0, \ldots, s_{i-1})(s_i)),$$

which is equal to $k(n - \ell)$ since $s^f(s_0, \ldots, s_{i-1})(s_i) \in S_1$ and $s^f(s_0, \ldots, s_{i-1})(s_i) \in S_2$. Thus

$$\nu(f(s_0, \ldots, s_{n-1})(s)) = k(n - \ell) + \nu(f(s_1, \ldots, s_{\ell-1})(s))$$

for $s \in S_1$. Since $s_n$ minimizes the left hand side over $S_1 \cup S_2$ it does so over $S_1$ alone and so minimizes $\nu(f(s_1, \ldots, s_{\ell-1})(s))$ over this set. Thus, by induction, the subsequence of elements of $\{s_i : i = 0, 1, 2, \ldots\}$ in $S_1$ forms a $\nu$-ordering of $S_1$. Symmetrically the same holds for $S_2$.

Conversely, suppose that $\{s_i : i = 0, 1, 2, \ldots\}$ and $\{s'_i : i = 0, 1, 2, \ldots\}$ are $\nu$-orderings of $S_1$ and $S_2$ respectively and that $\{s_i : i = 0, 1, 2, \ldots, \ell - 1\}$ and
The hypothesis of Lemma 5.2. Furthermore the may be described as

\[ \nu(s_0, \ldots, s_{\ell-1}, s'_0, \ldots, s'_{m-1})(s) \]

over \( S_1 \cup S_2 \). For elements of \( S_1 \) this valuation is \( km + \nu(s_0, \ldots, s_{\ell-1})(s) \), which is minimized by \( s_\ell \), while for \( s \in S_2 \) it is \( k\ell + \nu(s'_0, \ldots, s'_{m-1})(s) \), which is minimized by \( s'_m \). Thus the next term in a \( \nu \)-ordering of this set can be taken to be one or the other of these according to which of \( \nu(s_0, \ldots, s_{\ell-1})(s) - k\ell \) or \( \nu(s'_0, \ldots, s'_{m-1})(s)) - km \) is smaller. \( \square \)

6. Hurwitz quaternions

The Hurwitz quaternions are the maximal order in Hamilton’s quaternions and may be described as

\[ \mathbb{H} = \left\{ a + bi + cj + dk : (a, b, c, d) \subseteq \mathbb{Z} \quad \text{or} \quad (a, b, c, d) \subseteq \mathbb{Z} + \frac{1}{2} \right\} \]

or, letting \( \rho = (1 + i + j + k)/2 \),

\[ \mathbb{H} = \{ a\rho + bi + cj + dk : (a, b, c, d) \subseteq \mathbb{Z} \} . \]

We will use the notation \( \mathbb{H}\mathbb{Q} \) to denote the division algebra of quaternions with rational coefficients, \( N(z) = a^2 + b^2 + c^2 + d^2 \) and \( Tr(z) = 2a \) for the norm and trace of a quaternion \( z = a + bi + cj + dk \), \( \bar{z} = a - bi - cj - dk \) for its quaternionic conjugate, and we will let \( \pi = 1 + i \). The ideal (\( \pi \)) is the unique two sided prime ideal in \( \mathbb{H} \), and we will be concerned with the Ore localization with respect to this ideal of \( \mathbb{H} \), which we will denote \( \mathbb{H}_{(\pi)} \), and with polynomials taking values there when evaluated on \( \mathbb{H} \). It is the case that \( z \in (\pi) \) if and only if \( N(z) \) is even and so that \( \nu(z) = \nu_2(N(z)) \) defines a valuation on \( \mathbb{H}\mathbb{Q} \) such that \( \mathbb{H}_{(\pi)} = \{ z \in \mathbb{H}\mathbb{Q} : \nu(z) \geq 0 \} \).

Also, clearly \((\pi)^2 = (2)\). In order to compute the \( \nu \)-sequence of \( \mathbb{H} \) we use Lemma 5.2 and the following subsets of \( \mathbb{H} \):

**Definition 6.1.**

\[ S = \{ z \in \mathbb{H} : z \equiv \rho \pmod{\pi} \} \]
\[ T = \{ z \in \mathbb{H} : z \equiv 0 \pmod{\pi} \} \]
\[ T + 1 = \{ z \in \mathbb{H} : z \equiv 1 \pmod{\pi} \} \]
\[ T_1 = \{ z \in \mathbb{H} : z \equiv \pi, \rho\pi, \text{ or } \rho\pi \pmod{2} \} \]
\[ T_2 = \{ z \in \mathbb{H} : z \equiv 0 \pmod{2} \} \]
\[ = \{ z \in \mathbb{H} : \nu(z) > 1 \} . \]

Note that \( S \cup T \cup (T + 1) = \mathbb{H} \) and that \( T_1 \cup T_2 = T \) and that these sets satisfy the hypothesis of Lemma 5.2. Furthermore the \( \nu \)-sequences of \( T \) and \( T + 1 \) and of \( T_2 \) and \( \mathbb{H} \) are related by the following lemma:

**Lemma 6.2.**

(1) A polynomial \( f(x) \in \mathbb{H}\mathbb{Q}[x] \) is \( \mathbb{H}_{(\pi)} \) valued on \( T \) if and only if \( f(x-1) \) is \( \mathbb{H}_{(\pi)} \) valued on \( T + 1 \). The \( \nu \)-sequence of \( T \) and that of \( T + 1 \) are equal.
Lemma 6.3.

\[ S = \{ z \in \mathbb{H} : Tr(z) \equiv 1 \pmod{2} \text{ and } N(z) \equiv 1 \pmod{2} \}, \]

\[ T_1 = \{ z \in \mathbb{H} : Tr(z) \equiv 0 \pmod{2} \text{ and } N(z) \equiv 2 \pmod{4} \}. \]

Proof. Since 1 and 2 are in the center of \( \mathbb{H} \) it is the case that for \( z \in \mathbb{H} \) the value of the polynomial \( f(x-1) \) at \( z \) is equal to the value of \( f(x) \) at \( z-1 \) and the value of \( f(2x) \) at \( z \) is equal to the value of \( f(x) \) at \( 2z \). Since \( z \rightarrow z-1 \) and \( z \rightarrow 2z \) give bijections between \( T + 1 \) and \( T \) and between \( \mathbb{H} \) and \( T_2 \) respectively, it follows that \( f(x) \) is \( \mathbb{H}(\pi) \) valued on \( T \) if and only if \( f(x-1) \) is \( \mathbb{H}(\pi) \) valued on \( T + 1 \) and \( f(x) \) is \( \mathbb{H}(\pi) \) valued on \( T_2 \) if and only if \( f(2x) \) is \( \mathbb{H}(\pi) \) valued on \( \mathbb{H} \). The assertions concerning the \( \nu \)-sequences follow from the observation that the leading coefficients of \( f(x) \) and \( f(x-1) \) are equal and the leading coefficient of \( f(2x) \) is \( 2^{deg(f)} \) times that of \( f(x) \).

\( \square \)

Remark. That 1 and 2 are in the center of \( \mathbb{H} \) is essential to this result, and in particular the analogous statement to (ii) with 2 replaced by \( \pi \) is false. From \( \mathbb{H}(\pi) \) valued polynomial on \( \mathbb{H} \) of degree 3 must have coefficients in \( \mathbb{H}(\pi) \); hence the analogue of (ii) would assert that an \( \mathbb{H}(\pi) \) valued polynomial on \( \pi \mathbb{H} \) of degree 3 would have the property that each of its coefficients would be in \( \mathbb{H}(\pi) \) when multiplied by \( \pi^3 \). A direct calculation shows that the polynomial \( (x^3 - 2jx)/\pi^4 \) is an \( \mathbb{H}(\pi) \) valued polynomial on \( \pi \mathbb{H} \) however.

It follows that a recursive formula for the \( \nu \)-sequence of \( \mathbb{H} \) can be obtained if the \( \nu \)-sequences of \( S \) and \( T_1 \) are known. We now compute these.

\section*{Definition 6.4.}

(1) Let \( \phi = (\phi_1, \phi_2) : \mathbb{Z}_{\geq 0}^2 \rightarrow (1 + 2\mathbb{Z}_{\geq 0})^2 \) be defined by

\[ \phi(n) = (1 + 2 \sum_{i \geq 0} n_{2i}2^i, 1 + 2 \sum_{i \geq 0} n_{2i+1}2^i) \]

if \( n = \sum_{i \geq 0} n_i2^i \) is the expression of \( n \) in base 2, and let \( f_n(x) = \prod_{i=0}^{n-1}(x^2 - \phi_1(i)x + \phi_2(i)) \).
(2) Let \( \psi = (\psi_1, \psi_2) : \mathbb{Z}^{\geq 0} \to (2\mathbb{Z}^{\geq 0}) \times (2 + 4\mathbb{Z}) \) be defined by
\[
\psi(n) = (2 \sum_{i \geq 0} n_i 2^i, 2 + 4 \sum_{i \geq 0} n_{2i+1} 2^i)
\]
if \( n = \sum_{i \geq 0} n_i 2^i \) is the expression of \( n \) in base 2, and let \( g_n(x) = \prod_{i=0}^{n-1} (x^2 - \psi_1(i)x + \psi_2(i)) \).

**Lemma 6.5.**

1. If \( z \in S \), then \( \nu(f_n(z)) \geq 2n + 2 \sum_{k>0} [n/4^k] \) with equality if \( \text{Tr}(z) = \phi_1(n) \) and \( N(z) = \phi_2(n) \).
2. If \( z \in T_1 \), then \( \nu(g_n(z)) \geq 3n + \sum_{i>0} [n/2^i] \) with equality if \( \text{Tr}(z) = \psi_1(n) \) and \( N(z) = \psi_2(n) \).

**Proof.** Let \( z \in S \) and let \( 1 + 2 \sum_{k \geq 0} a_k 2^k \) be the expansion in base 2 of \( \text{Tr}(z) \) and let \( 1 + 2 \sum_{k \geq 0} b_k 2^k \) be the expansion in base 2 of \( N(z) \). Let \( m = \sum_{k \geq 0} a_k 2^{2k} + b_k 2^{2k+1} \) so that \( \phi(m) = (\text{Tr}(z), N(z)) \). If \( 0 \leq m < n \), then \( x^2 - \phi_1(m)x - \phi_2(m) \), which is the characteristic polynomial of \( z \), occurs as a quadratic factor of \( f_n(x) \), and so, since all of the coefficients of \( f_n(x) \) are in the centre of \( \mathbb{H} \), we have \( f_n(z) = 0 \). Thus we may assume that \( m \geq n \). For any \( 0 \leq k < n \) we have
\[
\begin{align*}
z^2 - \phi_1(k)z + \phi_2(k) &= z^2 - \phi_1(k)z + \phi_2(k) - (z^2 - \text{Tr}(z)z + N(z)) \\
&= (\phi_1(m) - \phi_1(k))z - (\phi_2(m) - \phi_2(k)),
\end{align*}
\]
and so
\[
\nu(z^2 - \phi_1(k)z + \phi_2(k)) = \nu((\phi_1(m) - \phi_1(k))z - (\phi_2(m) - \phi_2(k)))
\]
\[
= 2 \min(\nu_2(\phi_1(m) - \phi_1(k)), \nu_2(\phi_2(m) - \phi_2(k)))
\]
since \( \nu(az + b) = 0 \) if \( a \) and \( b \) are integers which are not both even. Thus
\[
\nu(f_n(z)) = 2 \sum_{k=0}^{n-1} \min(\nu_2(\phi_1(m) - \phi_1(k)), \nu_2(\phi_2(m) - \phi_2(k))).
\]
To evaluate this sum, note that if \( \sum k_i 2^i \) and \( \sum m_i 2^i \) are the base 2 expansions of \( k \) and \( m \), then \( \nu_2(m - k) \) is the least \( i \) for which \( k_i \neq m_i \), that \( \nu_2(\phi_1(m) - \phi_1(k)) \) is the least \( i \) for which \( k_{2i} \neq m_{2i} \) plus 1 and that \( \nu_2(\phi_2(m) - \phi_2(k)) \) is the least \( i \) for which \( k_{2i+1} \neq m_{2i+1} \) plus 1. Thus
\[
\min(\nu_2(\phi_1(m) - \phi_1(k)), \nu_2(\phi_2(m) - \phi_2(k))) = [(\nu_2(m - k))/2] + 1.
\]
We will use the notation \( \nu_4(m - k) = [(\nu_2(m - k))/2] \) since this is the largest power of 4 dividing \( k - m \). It is well known that for a prime \( p \) we have
\[
\nu_p(n!) = \sum_{i=1}^{n} \nu_p(i) = \sum_{i > 0} [n/p^i] = (n - \sum n_i)/(p - 1)
\]
if \( n = \sum n_i p^i \) is the expansion of \( n \) in base \( p \). The usual proof of this theorem by switching the order of summation in a double sum carries over to powers of primes also and shows that
\[
\sum_{i=1}^{n} \nu_4(i) = \sum_{i > 0} [n/4^i] = (n - \sum n_i)/3
\]
where $\sum n_i4^i$ is the expression of $n$ in base 4. Thus

$$\nu(f_n(z)) = 2 \sum_{k=0}^{n-1} 1 + \lfloor (\nu_2(m-k))/2 \rfloor$$

$$= 2n + 2 \left( \sum_{k=1}^{n} \left( \lfloor (\nu_2(k))/2 \rfloor \right) - \sum_{k=1}^{m-n} \left( \lfloor (\nu_2(k))/2 \rfloor \right) \right)$$

$$= 2n + 2 \left( \sum_{i>0} \left( m/4^i \right) - \sum_{i>0} \left( (m-n)/4^i \right) \right)$$

$$= 2n + 2((m - \sum m_i)/3 - ((m - n) - \sum (m - n)_i)/3).$$

Thus we wish to show that

$$(m - \sum m_i)/3 - ((m - n) - \sum (m - n)_i)/3 - (n - \sum n_i)/3$$

$$= (\sum (m - n)_i + \sum n_i - \sum m_i)/3 \geq 0.$$
and
\[ \nu(g_n(z)) = 3n + \sum_{k=0}^{n-1} \nu_2(m-k). \]

This sum may be evaluated as above to give
\[ \nu(g_n(z)) \geq 3n + \sum_{i>0} \lfloor n/2^i \rfloor \]
with equality if the addition of \( n \) and \( m-n \) in base 2 involves no carries, which is the case if \( n = m \). \( \square \)

With these polynomials we will construct the initial segment of \( \nu \)-orderings of \( S \) and \( T_1 \). It is first necessary, however, to check that we can pick elements of \( S \) and of \( T_1 \) with prescribed traces and norms modulo high powers of 2.

**Lemma 6.6.** For any \( N > 0 \) and any \( t, m \in (1+2\mathbb{Z}) \) there exists \( z \in \mathbb{H} \) such that \( \text{Tr}(z) \equiv t \pmod{2^N} \) and \( \text{N}(z) \equiv m \pmod{2^N} \). For any \( N > 0 \) and any \( t \in 2\mathbb{Z}, m \in (2+4\mathbb{Z}) \) there exists \( z \in \mathbb{H} \) such that \( \text{Tr}(z) \equiv t \pmod{2^N} \) and \( \text{N}(z) \equiv m \pmod{2^N} \).

**Proof.** Let \( U \) denote the set \( \{tx+x^2 \pmod{2^N} : x \in \mathbb{Z}\} \). From the formulas for the trace and norm of elements of \( \mathbb{H} \) it follows that the set of values of norms modulo \( 2^N \) of elements of \( \mathbb{H} \) with trace \( t \) is the set \( t^2 + U + U + U \). If \( t \) is odd, then completing the square shows that \( y \) is in \( U \) if and only if \( 4y + t^2 \) is a square modulo \( 2^{N+2} \). Since the odd squares modulo \( 2^{N+2} \) are those elements congruent to 1 modulo 8 it follows that \( U = 2\mathbb{Z}/(2^N) \) and so \( t^2 + U + U + U = 1 + 2\mathbb{Z}/(2^N) \) as required.

If \( t = 2v \) is even, then completing the square shows that the set \( U \) is equal to \( V - v^2 \) where \( V \) denotes the set of squares modulo \( 2^N \). Thus \( t^2 + U + U + U = v^2 + V + V + V \). The set of all squares modulo \( 2^N \) is given by
\[ V = \bigcup_{k \geq 0} (2^{2k} + 2^{2k+3}\mathbb{Z}) \pmod{2^N}; \]
hence we have
\[ V + V + V = \bigcup_{k \geq 0} ((2^{2k} + 2^{2k+3}\mathbb{Z}) \cup (2 \cdot 2^{2k} + 2^{2k+3}\mathbb{Z}) \cup (3 \cdot 2^{2k} + 2^{2k+3}\mathbb{Z}) \cup (2^{2k} + 2^{2k+1} + 2^{2k+3}\mathbb{Z}) \cup (2^{2k+1} + 2^{2k+1} + 2^{2k+3}\mathbb{Z})) \pmod{2^N}, \]
which we may write as
\[ \bigcup_{k \geq 0} ((1,2,3,5,6) \cdot 2^{2k} + 2^{2k+3}\mathbb{Z}) \pmod{2^N}. \]
Observing that
\[ \mathbb{Z}\backslash((1,2,3,5,6) + 2^3\mathbb{Z}) = (0,4,7) + 2^4\mathbb{Z} \]
and that, for each \( k > 0 \),
\[ (0,4) + 2^{2k+1}\mathbb{Z}\backslash((1,2,3,5,6) + 2^{2k+3}\mathbb{Z}) = (0,4,7) + 2^{2k+3}\mathbb{Z} \]
we have that \( V + V + V \) is the complement of the set
\[ \bigcup_{k \geq 0} (7 \cdot 2^{2k} + 2^{2k+3}\mathbb{Z}) \pmod{2^N}). \]
Since all elements of this set are congruent to either 0 or 3 modulo 4 and \(v^2\) is congruent to either 0 or 1 modulo 4, it follows that \(v^2 + V + V + V\) contains \(2 + 4\mathbb{Z}\) (mod \(2^N\)) as required.

**Proposition 6.7.** The \(\nu\)-sequence of \(S\) is given by
\[
\alpha_S(2n) = \alpha_S(2n + 1) = 2n + 2 \sum_{k>0} [n/4^k].
\]

*Proof.* For any \(N > 0\) we may, by the previous lemma, choose elements \(z_i \in \mathbb{H}\) with \((Tr(z_i), N(z_i)) \equiv (\phi_1(i), \phi_2(i))\) (mod \(2^N\)) and form the sequence \(\{z_0, z_1, z_2, z_3, \ldots\}\). From Lemma 6.8 both \(z_i\) and \(\tilde{z}_i\) are in \(S\) for each \(i\). Since \(z_i\) and \(\tilde{z}_i\) have the same trace and norm, the minimal polynomial of \(\{z_i, \tilde{z}_i\}\) is equal to the characteristic polynomial of \(z_i\) and has integer coefficients and is congruent modulo \(2^N\) to \(x^2 - \phi_1(i)x + \phi_2(i)\). Thus the minimal polynomial of \(\{z_i, \tilde{z}_i : i = 0, \ldots, n-1\}\) is congruent to \(f_n(x)\) modulo \(2^N\) and that of \(\{z_i, \tilde{z}_i : i = 0, \ldots, n-1\} \cup \{z_n\}\) is congruent to \(f_{n+1}(x)\). From the proof of Lemma 6.5 and the fact that \(\nu(z - \bar{z}) = 0\) for elements of \(S\) it follows that if \(N\) is large compared to \(n\), then \(\nu(f_n(z_n)) = \nu(f_n(z_n - \bar{z}_n)) = 2n + 2 \sum_{k>0} [n/4^k]\) is minimal over \(S\). Thus the sequence \(\{z_1, \tilde{z}_1, z_2, \tilde{z}_2, \ldots\}\) forms the beginning of a \(\nu\)-ordering of \(S\). \(\square\)

For the corresponding result about \(T_1\) we need the following analogue of the quaternionic conjugation for this set.

**Lemma 6.8.** The maps \(\chi_r : \mathbb{H} \rightarrow \mathbb{H}, r = 1, 2, 3\), defined by \(\chi_1(a + bi + cj + dk) = a + bi + dj + ck, \chi_2(a + bi + cj + dk) = a + di + cj + bk\) and \(\chi_3(a + bi + cj + dk) = a + ci + bj + dk\) preserve trace and norm and restrict to give bijections between the cosets \(\rho \pi + (\pi)^2\) and \(\bar{\rho} \pi + (\pi)^2\), between \(\pi + (\pi)^2\) and \(\rho \pi + (\pi)^2\) and between \(\pi + (\pi)^2\) and \(\bar{\rho} \pi + (\pi)^2\) respectively.

**Proposition 6.9.** The \(\nu\)-sequence of \(T_1\) is given by
\[
\alpha_{T_1}(2n) = 3n + \sum_{k>0} [n/2^k]
\]
and
\[
\alpha_{T_1}(2n + 1) = \alpha_{T_1}(2n) + 1.
\]

*Proof.* For any \(N > 0\) we may also choose elements \(w_i \in \mathbb{H}\) with \((Tr(w_i), N(w_i)) \equiv (\psi_1(i), \psi_2(i))\) (mod \(2^N\)) and take \(\tilde{w}_i = \chi_r(w_i)\) with \(r = 1, 2, 3\) according to whether \(w_i\) is an element of \(\rho \pi + (\pi)^2\), \(\rho \pi + (\pi)^2\) or \(\pi + (\pi)^2\). Note that \(w_i\) and \(\tilde{w}_i\) have the same trace and norm; hence the minimal polynomial of \(\{w_i, \tilde{w}_i\}\) is the characteristic polynomial of \(w_i\). Also \(\nu(w_i - \tilde{w}_i) = 1\) since these elements are in distinct cosets of \((\pi)^2\). Forming the sequence \(\{w_0, \tilde{w}_0, w_1, \tilde{w}_1, \ldots\}\) we obtain, for \(N\) large with respect to \(n\), the initial segment of a \(\nu\)-ordering of \(T_1\) by the same argument as for \(S\). \(\square\)

Let us denote the shuffle of two nondecreasing sequences \(\alpha\) and \(\beta\) into nondecreasing order by \(\alpha \wedge \beta\) and denote the sequence whose \(n\)-th term is \(kn\) by \((kn)\).

**Proposition 6.10.** The \(\nu\)-sequence of \(\mathbb{H}\) satisfies, and is uniquely determined by, the equation
\[
\alpha_{\mathbb{H}} = \alpha_S \wedge \left(\left((\alpha_{T_1} - (n)) \wedge (\alpha_{\mathbb{H}} + (n))\right) + (n)\right) \wedge \left(\left((\alpha_{T_1} - (n)) \wedge (\alpha_{\mathbb{H}} + (n))\right) + (n)\right)
\]
and the initial values \(\alpha_{\mathbb{H}}(n) = 0\) for \(n < 4\).
Proof. The values of $\alpha_H$ for $n \leq 4$ are computed in [5]. From Lemmas 5.2 and 6.2 we have the equations

\[
\begin{align*}
\alpha_H &= \alpha_S \wedge \alpha_T \wedge \alpha_{T+1} \\
\alpha_{T+1} &= \alpha_T \\
\alpha_T &= ((\alpha_{T_1} - (n)) \wedge (\alpha_{T_2} - (n))) + (n) \\
\alpha_{T_2} &= \alpha_H + (2n)
\end{align*}
\]

which combine to form the given equation. The sequences $\alpha_S$ and $\alpha_{T_1}$ are described above. Thus this equation equates $\alpha_H$ with the shuffle of copies of $\alpha_H$ and known sequences. Since this equates $\alpha_H(n)$ with a combination of $\alpha_H(i)$’s for $i < n$ and known quantities, it determines $\alpha_H(n)$ once the initial values are known. \hfill \Box

Since the operations in this equation (addition and shuffle) can be evaluated quickly this gives a fast algorithm for the computation of $\alpha_H$. The first 100 terms are

\[
\begin{align*}
0, 0, 0, 1, 1, 2, 2, 2, 4, 4, 4, 4, 5, 5, 5, 6, 6, 6, 8, 8, 10, 10, 10, 10, 11, 11, 12, 12, 12, 12, 14, 14, 14, 14, 15, 15, 16, 16, 17, 17, 19, 19, 20, 20, 21, 21, 22, 22, 22, 22, 24, 24, 24, 24, 24, 25, 25, 26, 26, 26, 26, 28, 28, 30, 30, 30, 30, 31, 31, 32, 32, 32, 32, 34, 34, 34, 35, 35, 36, 36, 36, 40, 40, 42, 42, 42, 42, 43, 43, 44, 44, 44, 44, 44, 46, 46, 46
\end{align*}
\]

and $\alpha_H(1000000) = 499990$.

This proposition also gives a recursive formula for a $\nu$-ordering for $H$. Using Propositions 6.7 and 6.9 the initial portions of $\nu$-orderings for the sets $S$ and $T_1$ may be constructed, and these, together with an initial segment of a $\nu$-ordering of $H$, can be combined using the shuffles determined by Proposition 6.10 to give a longer initial segment of a $\nu$-ordering of $H$. For example if the initial segment of a $\nu$-ordering of $H$ is taken to be $\rho, \bar{\rho}, 0, 1, \pi$

which can be deduced from the results of [5], and if

\[
\begin{align*}
\rho, \bar{\rho}, 3\rho + i + j, 3\bar{\rho} - i - j, \rho + i, \bar{\rho} - i, 3\rho + 2i, 3\bar{\rho} - 2i, 5\rho + 3i, 5\bar{\rho} - 3i \\
7\rho + i + j, 7\bar{\rho} - i - j, 5\rho + 2i + j + k, 5\bar{\rho} - 2i - j - k, 7\rho + 2i, 7\bar{\rho} - 2i
\end{align*}
\]

is used as the initial segment of a $\nu$-ordering of $S$ and

\[
\begin{align*}
i + j, i + k, 2\rho + i + j, 2\bar{\rho} + i + k, i + j + 2k, i + 2j + k, 2\rho + i + 3k, 2\rho + j + 3k
\end{align*}
\]

of $T_1$, then the resulting initial segment of a $\nu$-ordering of $H$ is

\[
\begin{align*}
\rho, \bar{\rho}, 0, 1, i + j, i + j + 1, 3\rho + i + j, 3\bar{\rho} - i - j, i + k, i + k + 1, \\
\rho + i, \bar{\rho} - i, 2, 3, 2\rho + i + j, 2\bar{\rho} + i + j + 1, 3\rho + 2i, 2\bar{\rho} - 2i, 2\rho + i + k, \\
\rho + i + k + 1, 2\rho, 2\rho + 1, 5\rho + 3i, 5\bar{\rho} - 3i, i + j + 2k, i + j + 2k + 1, \\
i + 2j + k, i + 2j + k + 1, 7\rho + i + j, 7\bar{\rho} - i - j, 2\rho, 2\bar{\rho} + 1, \\
5\rho + 2i + j + k, 5\bar{\rho} - 2i - j - k, 2\rho + i + 3k, 2\rho + i + 3k + 1, \\
2\rho + j + 3k, 2\rho + j + 3k + 1, 7\rho + 2i, 7\bar{\rho} - 2i, 2\pi, 2\pi + 1.
\end{align*}
\]

In general this process, starting with a segment of a $\nu$-ordering of $H$ of length $m$, will result in a segment of length $>8m$. 
Using the results in [7] we may also determine the asymptotic behavior of \( \alpha_{\mathbb{H}}(n)/n \).

**Proposition 6.11.** \( \lim_{n \to \infty} \alpha_{\mathbb{H}}(n)/n = 1/2 \).

**Proof.** From [7] if \( \alpha(n) \) and \( \beta(n) \) are superadditive sequences, then

\[
\lim_{n \to \infty} \frac{\alpha \land \beta(n)}{n} = \frac{1}{\lim_{n \to \infty} \frac{\alpha(n)}{n}} + \frac{1}{\lim_{n \to \infty} \frac{\beta(n)}{n}},
\]

hence

\[
\lim_{n \to \infty} \frac{\alpha_{\mathbb{H}}(n)}{n} = \frac{1}{\lim_{n \to \infty} \frac{\alpha_S(n)}{n}} + \frac{2}{\lim_{n \to \infty} \frac{\alpha_T(n)}{n}}.
\]

Direct calculation shows

\[
\lim_{n \to \infty} \frac{\alpha_S(n)}{n} = 4/3
\]

and

\[
\lim_{n \to \infty} \frac{\alpha_T(n)}{n} = 2
\]

and, by Proposition 6.1.10 and [7],

\[
\lim_{n \to \infty} \frac{\alpha_T(n)}{n} = 1/2 + 1.
\]

Combining these and simplifying yield the quadratic

\[
14\left( \lim_{n \to \infty} \frac{\alpha_{\mathbb{H}}(n)}{n} \right)^2 + 17\left( \lim_{n \to \infty} \frac{\alpha_{\mathbb{H}}(n)}{n} \right) - 12 = 0
\]

whose positive root is 1/2.

**References**


Department of Mathematics, Dalhousie University, Halifax, Nova Scotia, B3H 4R2, Canada

E-mail address: johnson@mathstat.dal.ca