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FOULKES CHARACTERS FOR COMPLEX REFLECTION GROUPS

ALEXANDER R. MILLER

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Abstract. We investigate Foulkes characters for a wide class of reflection groups which contains all finite Coxeter groups. In addition to new results, our general approach unifies, explains, and extends previously known (type A) results due to Foulkes, Kerber–Thürlings, Diaconis–Fulman, and Isaacs.

Introduction

Foulkes discovered a marvelous set of characters for the symmetric group by summing Specht modules of certain ribbon shapes according to height. These characters have many remarkable properties and have been the subject of many investigations, including a recent one [3] by Diaconis and Fulman which established some new formulas, a conjecture of Isaacs, and a connection with Eulerian idempotents.

We widen our consideration to complex reflection groups and find ourselves equipped from the start with a simple formula for *(generalized) Foulkes characters* which explains and extends these properties. In particular, it gives a factorization of the Foulkes character table which explains Diaconis and Fulman's formula for the determinant, their link to Eulerian idempotents, and their formula for the inverse. We present a natural extension of a conjecture of Isaacs and then use properties of Foulkes characters which resemble those of supercharacters to establish the result. We also discover a remarkable refinement of Diaconis and Fulman's determinantal formula by considering Smith normal forms.

Classic type A Foulkes characters have connections with adding random numbers, shuffling cards, the Veronese embedding, and combinatorial Hopf algebras; see [3, 9]. Our formula brings *Orlik–Solomon coexponents* from [14] the cohomology theory of [12] complements into the picture with the geometry of the Milnor fiber complex [10], and it gives rise to a curious classification at the end of the paper.

The paper is structured as follows. Section 1 introduces Foulkes characters for *Shephard* and *Coxeter groups*. Key properties are quickly gathered, including our main formula. In Section 2, properties of type A Foulkes characters are explained and extended from the symmetric group to the infinite family of wreath products. In Section 3, Isaacs' type A conjecture is sharpened for the Coxeter–Shephard–Koster family. Diaconis and Fulman's type A determinantal formula is also extended here. Lastly, we determine exactly when the Foulkes characters form a basis for the space of class functions $\chi(g)$ that depend only on the dimension of the fixed space of g.

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1. Foulkes characters for complex reflection groups

Let V be an ℓ-dimensional $\mathbb{C}$ vector space, and let W be a finite group of the form

$$(\star) \qquad \langle r_1, r_2, \ldots, r_\ell \mid r_i^{p_i} = 1, \quad r_i r_j r_i \cdots = r_j r_i r_j \cdots \quad \forall\, i < j \rangle$$

with $m_{ij} \geq 2$ factors on both sides of the second relation, $p_i \geq 2$, and $p_i = p_j$ whenever m_{ij} is odd.[1] *Finite Coxeter groups* are the ones where every p_i is 2, and each has a canonical faithful representation $W \subset \mathrm{GL}(V)$ as a (complex) reflection group, by which we mean that the r_i's act on V as reflections in the sense that the fixed spaces $\ker(1 - r_i)$ are hyperplanes. This familiar picture for Coxeter groups extends [4] to all W presented above, and we agree to identify the abstract group with its faithful representation as a reflection group. Henceforth, assume that this representation is irreducible. When W is not a Coxeter group, it is known as a *Shephard group* (the symmetry group of an object called a *regular complex polytope*). Write $R = \{r_1, r_2, \ldots, r_\ell\}$ and call ℓ the *rank* of W.

For each Shephard and Coxeter group W there exists [10] a simplicial complex Δ called the *Milnor fiber complex* (which is a strong deformation retract of a Milnor fiber) whose faces are indexed by [11] the cosets wW_I ($I \subseteq R$) of standard parabolic subgroups $W_I = \langle I \rangle$ ordered by inclusion. In the case of Coxeter groups, this simplicial complex is the *Coxeter complex*, which is the intersection of the real sphere $\mathbb{S}^{\ell-1}$ with the polyhedral pieces cut out in $\mathbb{R}^\ell$ by the reflecting hyperplanes of a real form of the group. In general, each *type-selected* subcomplex Δ_S ($S \subseteq R$) is homotopy equivalent to a bouquet of spheres, and the $\mathbb{C}W$-module on the top homology group is called a *ribbon representation.* Its character ρ^S has the following description as an alternating sum of characters induced from principal characters of parabolic subgroups:

$$\rho^S = \sum\nolimits_I (-1)^{|S \setminus I|}\, \mathbb{1}\uparrow_{W_{R \setminus I}}$$

where the sum ranges over all subsets I of S; see [7] for details and history.

In the special case when W is the symmetric group, a ribbon representation is a Specht module of a certain skew ribbon shape [10] and Foulkes' construction for the symmetric group translates to summing the ρ^S according to cardinality $|S|$.

Main definition. For a Shephard or Coxeter group with generators R as in ($\star$), and for any integer s with $0 \leq s \leq \ell$, the *Foulkes character* ϕ^s is the sum of all ribbon characters ρ^S for subsets $S \subseteq R$ with $|S| = s$.

The main tools of this paper are Section 1.1, and an explicit formula in Section 1.2 for Foulkes characters, which gives a factorization of the (resp. *reduced*) *Foulkes character table* in Section 1.3. Our main results will follow from this factorization. In particular, it elucidates the type A theory, which previously rested on ad hoc proofs by induction.

Our formula for $\phi^s(g)$ will depend on the fixed space of g, and we will be particularly interested in the case when it depends only on the *dimension* of the fixed space. When this happens the Foulkes character table reduces to a remarkable square matrix called the *reduced Foulkes character table.*

[1] If m_{ij} is odd, then the two reflections r_i, r_j must in fact be conjugate because in this case the braid relation says that $(r_i r_j)^{(m_{ij}-1)/2} r_i = r_j (r_i r_j)^{(m_{ij}-1)/2}$.

1.1. **A decomposition of the regular character.** Let ϵ^i denote the character of the ith exterior power of the irreducible reflection representation V, so that ϵ^i is irreducible for $0 \le i \le \ell$ by a theorem of Steinberg; see [2, Chapter V, Exercise 3]. The following properties are corollaries of [7, Theorem 12.1 and Theorem 12.2] (after the Coxeter case [17]), two finer results for individual ribbon representations.

- The sum $\phi^0 + \phi^1 + \ldots + \phi^\ell$ is the character of the regular representation.
- The ϵ^i-isotypic component of the regular character is a constituent of ϕ^i.

In this way, Foulkes characters bear resemblance to *supercharacters*; cf. [3, p. 429].

1.2. **A formula for Foulkes characters.**

Theorem 1. *Let $g \in W$ and consider its fixed space $X = \{v \in V \mid gv = v\}$. Then*

$$\phi^s(g) = \sum\nolimits_i (-1)^{s-i} \binom{\ell - i}{s - i} f_{i-1}(\Delta \cap X) \tag{1}$$

where $f_{i-1}(\Delta \cap X)$ denotes the number of $(i-1)$-dimensional faces of $\Delta \cap X$.

Proof. Write ϕ^s in terms of the ρ's, and the ρ's in terms of cosets, so that

$$\begin{aligned}
\phi^s(g) &= \sum_{|S|=s} \sum_{I \subseteq S} (-1)^{|S \setminus I|} \, \mathbb{1}\uparrow_{W_{R \setminus I}}(g) \\
&= \sum_i \sum_{|I|=i} (-1)^{s-i} \binom{\ell - i}{s - i} \, \mathbb{1}\uparrow_{W_{R \setminus I}}(g) \\
&= \sum_i (-1)^{s-i} \binom{\ell - i}{s - i} \sum_{|I|=i} \mathbb{1}\uparrow_{W_{R \setminus I}}(g).
\end{aligned}$$

The inner sum is the number of $(i-1)$-faces of Δ that are stabilized by g. Since Δ is a *balanced simplicial complex* (see [7]), it follows that a face is stabilized by g if and only if it is fixed pointwise by g, or in other words, is a face of $\Delta \cap X$. □

1.2.1. *A formula for calculating the face numbers $f_{i-1}(\Delta \cap X)$.* Let L be the collection of all intersections of reflecting hyperplanes of W ordered by reverse inclusion, and write μ for its usual Möbius function. For each $X \in L$ define a polynomial $B_X(t) \in \mathbb{Z}[t]$ by $B_X(t) = (-1)^{\dim X} \sum_{Y \ge X} \mu(X, Y)(-t)^{\dim Y}$. Also associated with W is a set of numbers called *exponents*, the smallest of which is denoted by m_1; we will review these numbers in Section 3.2 below. Orlik [10] (after Orlik–Solomon in the Coxeter case) showed that

$$f_{i-1}(\Delta \cap X) = \sum\nolimits_Y B_Y(m_1) \tag{2}$$

where Y varies over all i-dimensional subspaces that lie above X in L. Note that $f_{i-1}(\Delta \cap X)$ and $B_X(t)$ are determined by the restriction

$$L^X = \{Y \in L \mid Y \ge X\}.$$

1.3. **The Foulkes character table.** Define the *Foulkes character table* of W to be the matrix $\Phi = [\phi^i(g_j)]_{ij}$ with rows indexed by $\phi^0, \phi^1, \ldots, \phi^\ell$ and columns indexed by (conjugacy) class representatives $g_0, g_1, \ldots, g_c$, ordered in such a way that respects fixed space codimension; in particular, $g_0 = 1$ fixes the whole space and g_c fixes only the origin. The Foulkes character table factors according to Theorem 1.

Corollary 1.1. *The Foulkes character table has the following factorization:*

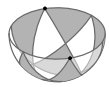
$$\Phi_{ik} = \left[(-1)^{i-j}\binom{\ell-j}{i-j}\right]_{ij} \times \left[f_{j-1}(\Delta \cap X_k)\right]_{jk} \tag{3}$$

where $0 \le i, j \le \ell$ *and* X_k *is the (pointwise) fixed space of representative* g_k.

Example 1.1. The Foulkes character table of the symmetric group $\mathfrak{S}_4$ is

	(1)(2)(3)(4)	(12)(3)(4)	(12)(34)	(123)(4)	(1234)
ϕ^0	1	1	1	1	1
ϕ^1	11	3	-1	-1	-3
ϕ^2	11	-3	-1	-1	3
ϕ^3	1	-1	1	1	-1

The (pointwise) fixed subcomplexes $\Delta \cap X$ and their cell counts are as follows:

					
f_{-1}	1	1	1	1	1
f_0	14	6	2	2	
f_1	36	6			
f_2	24				

In this way, equation (3) predicts this factorization of the Foulkes character table:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 11 & 3 & -1 & -1 & -3 \\ 11 & -3 & -1 & -1 & 3 \\ 1 & -1 & 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} \binom{3}{0} & & & \\ -\binom{3}{1} & \binom{2}{0} & & \\ \binom{3}{2} & -\binom{2}{1} & \binom{1}{0} & \\ -\binom{3}{3} & \binom{2}{2} & -\binom{1}{1} & \binom{0}{0} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 14 & 6 & 2 & 2 & \\ 36 & 6 & & & \\ 24 & & & & \end{bmatrix}$$

1.3.1. *The reduced Foulkes character table.* Example 1.1 shows that some groups have characters $\phi^i(g)$ which depend only on the dimension of the fixed space of g. When this happens we consider the *reduced Foulkes character table* $[\phi^i(g_j)]$, whose columns are indexed by a subset of class representatives $g_0, g_1, \ldots, g_\ell$ subject to the condition that the fixed space of g_i has codimension i, so that $g_0 = 1$ fixes the whole space and g_ℓ fixes only the origin. The reduced table stores the same information as the full table and is obtained by deleting redundant columns. In particular, it also factors according to (3); this will be the key to understanding the determinant and inverse formulas of Diaconis and Fulman in the next section.

Example 1.2. The reduced Foulkes character table of the symmetric group $\mathfrak{S}_4$ is

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 11 & 3 & -1 & -3 \\ 11 & -3 & -1 & 3 \\ 1 & -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} \binom{3}{0} & & & \\ -\binom{3}{1} & \binom{2}{0} & & \\ \binom{3}{2} & -\binom{2}{1} & \binom{1}{0} & \\ -\binom{3}{3} & \binom{2}{2} & -\binom{1}{1} & \binom{0}{0} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 14 & 6 & 2 & \\ 36 & 6 & & \\ 24 & & & \end{bmatrix}$$

The first column gives the values $\phi^i(g)$ for any permutation g with exactly four cycles (namely the identity), the second column gives the values for any permutation with exactly three cycles, and so on. It is obtained from the full table in Example 1.1 by deleting one of the two columns indexed by an element with exactly two cycles.

2. Foulkes characters for wreath products

Fix $W = \mathbb{Z}_r \wr \mathfrak{S}_n$, so that $\ell = n$ if $r > 1$, and $\ell = n-1$ in the case when $W = \mathfrak{S}_n$. Write L_p for the intersection lattice of $\mathbb{Z}_r \wr \mathfrak{S}_p$, so that $L_n = L$.

2.1. **Dimension dependence and basis result.** It is well known that in this case $L^X \simeq L_{n-k}$ for $X \in L$ of codimension k; see [14].

Theorem 2. *$\phi^s(g)$ depends only on the dimension of the fixed space of g.*

Proof. Let X be the fixed space of g. The dimension of X determines L^X, and hence the cell counts of (2). Now (1) implies the result. □

Theorem 3. *The Foulkes characters $\phi^0, \phi^1, \ldots, \phi^\ell$ form a $\mathbb{Q}$ basis for the space of class functions χ that depend only on the dimension of the fixed space. Moreover,*

$$\chi = \sum_i \frac{\langle \chi, \epsilon^i \rangle}{\dim \epsilon^i} \phi^i \tag{4}$$

where ϵ^i is the character of the ith exterior power of V (so $\dim \epsilon^i = \binom{\ell}{i}$).

Proof. Recall from Section 1.1 that ϵ^i is irreducible for $0 \le i \le \ell$, and that it is a constituent of ϕ^j if and only if $i = j$. Hence the Foulkes characters are linearly independent, and the basis claim follows from Theorem 2. In turn, the second claim follows from the fact that ϵ^i appears in ϕ^i with multiplicity equal to its dimension. □

2.2. **Explicit formulas and a second factorization.** Another consequence is the following explicit formula for the cell counts appearing in Theorem 1, which in turn gives an explicit formula for the character values $\phi^s(g)$.

Proposition 4. *Let $X \in L$ and let k denote the codimension of X. Then*

$$f_{i-1}(\Delta \cap X) = \sum_j (-1)^{i-j} \binom{n-\ell+i}{i-j} (rj+1)^{n-k}. \tag{5}$$

In particular, $f_{i-1}(\Delta \cap X) = f_{i-1}(\Delta_{n-k})$ where Δ_{n-k} is the complex for $\mathbb{Z}_r \wr \mathfrak{S}_{n-k}$.

Proof. For $Y \in L$ of dimension i, one has $L^Y \simeq L_{n-\ell+i}$, and so (2) tells us that $B_Y(m_1)$ is the number of top cells in the Milnor fiber complex of $\mathbb{Z}_r \wr \mathfrak{S}_{n-\ell+i}$. These top cells are indexed by the elements of the group, and so (2) further implies that

$$f_{i-1}(\Delta \cap X) = \#\{i\text{-dimensional } Y \in L^X\} \times r^{n-\ell+i}(n-\ell+i)!.$$

The first factor is a *Whitney number* of the *Dowling lattice* $L^X \simeq L_{n-k}$, and it has a well-known expression (see [14]) which can be written as [1, Eq. 6]

$$\frac{1}{r^i i!} \sum_j (-1)^{i-j} \binom{i}{i-j} (rj+1)^{\ell-k}.$$

Equation (5) follows if $r > 1$, when one has $n = \ell$. In the case when $r = 1$, so that $n = \ell + 1$, equation (5) follows from the identity $(i+1)\binom{i}{i-j} = (j+1)\binom{i+1}{i-j}$. □

Proposition 4 gives the following formula for the character values $\phi^s(g)$.

Theorem 5. *Let $g \in W$ and let k be the codimension of the fixed space of g. Then*

$$\phi^s(g) = \sum_j (-1)^{s-j} \binom{n+1}{s-j} (rj+1)^{n-k}. \tag{6}$$

Proof. Let X denote the fixed space of g, so that by (1) and (5) one has

$$\phi^s(g) = \sum\nolimits_i (-1)^{s-i}\binom{\ell-i}{s-i}\sum\nolimits_j (-1)^{i-j}\binom{n-\ell+i}{i-j}(rj+1)^{n-k}.$$

Now switch the order of summation to get

$$\sum\nolimits_j (-1)^{s-j}(rj+1)^{n-k}\sum\nolimits_i \binom{\ell-i}{\ell-s}\binom{n-(\ell-i)}{n-\ell+j}$$

and note that the inside sum is equal to $\binom{n+1}{s-j}$, since $n \geq \ell$. □

The second part of Proposition 4 sharpens our main factorization (3), while Theorem 5 gives a new, second factorization that involves a special Vandermonde matrix. We collect these special factorizations for $\mathbb{Z}_r \wr \mathfrak{S}_n$ in the following corollary.

Corollary 5.1. *Maintain the notation of this section. In particular, $W = \mathbb{Z}_r \wr \mathfrak{S}_n$. Then the reduced Foulkes character table Φ has the following factorizations:*

(F1) $$\Phi = \left[(-1)^{i-j}\tbinom{\ell-j}{i-j}\right]_{ij} \times \left[f_{j-1}(\Delta_{n-k})\right]_{jk} \qquad (0 \leq i,j,k \leq \ell).$$

(F2) $$\Phi = \left[(-1)^{i-j}\tbinom{n+1}{i-j}\right]_{ij} \times \left[(rj+1)^{n-k}\right]_{jk} \qquad (0 \leq i,j,k \leq \ell).$$

The second factor of (F2) *is a special instance of a Vandermonde matrix.* □

Example 5.1. In the case of Weyl group $A_2 = \mathbb{Z}_1 \wr \mathfrak{S}_3$ one has

$$\Phi = \begin{bmatrix} \binom{2}{0} & & \\ -\binom{2}{1} & \binom{1}{0} & \\ \binom{2}{2} & -\binom{1}{1} & \binom{0}{0} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 6 & 2 & \\ 6 & & \end{bmatrix}$$
$$= \begin{bmatrix} \binom{4}{0} & & \\ -\binom{4}{1} & \binom{4}{0} & \\ \binom{4}{2} & -\binom{4}{1} & \binom{4}{0} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2^3 & 2^2 & 2 \\ 3^3 & 3^2 & 3 \end{bmatrix}.$$

Example 5.2. In the case of Weyl group $B_3 = \mathbb{Z}_2 \wr \mathfrak{S}_3$ one has

$$\Phi = \begin{bmatrix} \binom{3}{0} & & & \\ -\binom{3}{1} & \binom{2}{0} & & \\ \binom{3}{2} & -\binom{2}{1} & \binom{1}{0} & \\ -\binom{3}{3} & \binom{2}{2} & -\binom{1}{1} & \binom{0}{0} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 26 & 8 & 2 & \\ 72 & 8 & & \\ 48 & & & \end{bmatrix}$$
$$= \begin{bmatrix} \binom{4}{0} & & & \\ -\binom{4}{1} & \binom{4}{0} & & \\ \binom{4}{2} & -\binom{4}{1} & \binom{4}{0} & \\ -\binom{4}{3} & \binom{4}{2} & -\binom{4}{1} & \binom{4}{0} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3^3 & 3^2 & 3 & 1 \\ 5^3 & 5^2 & 5 & 1 \\ 7^3 & 7^2 & 7 & 1 \end{bmatrix}.$$

2.3. **The determinant of the Foulkes character table.** In the case of the symmetric group, Isaacs conjectured (see [3, p. 429]) that the determinant of Φ is divisible by $n!$. Diaconis and Fulman confirmed this by in fact showing that the determinant is equal to $n!(n-1)!\cdots 2!$. They argued by induction. Our first factorization (F1) elucidates and extends this formula for the determinant: the left triangular factor of (F1) has 1's along its diagonal, while along the right diagonal we find the group cardinalities $f_{\mathrm{top}}(\Delta_i) = |\mathbb{Z}_r \wr \mathfrak{S}_i|$. Alternatively, one may use our second factorization (F2) and the well-known Vandermonde determinant formula.

Theorem 6. $\det \Phi = r^{n(n+1)/2}\, n!(n-1)!\cdots 2!$. □

2.4. **The branching rule and a recursion for Foulkes characters.** The next theorem gives a recursion for the Foulkes character tables and follows from either one of our factorizations (F1) or (F2). Write $\Phi^{(p)}$ for the (reduced) Foulkes character table of $\mathbb{Z}_r \wr \mathfrak{S}_p$, so that $\Phi^{(n)} = \Phi$.

Theorem 7. $\Phi^{(n)}_{i,k} = \Phi^{(n-1)}_{i,k-1} - \Phi^{(n-1)}_{i-1,k-1}$ *for* $k > 0$.

Proof. Factor $\Phi^{(n-1)}$ according to (F2), so that the row subtraction takes place in the left binomial factor, while the right, Vandermonde factor is unchanged. Now the claim follows from the identity $\binom{n+1}{i-j} = \binom{n}{i-j} + \binom{n}{i-1-j}$. □

The following branching rule says that the Foulkes characters ϕ^s satisfy the recurrence [18, Lemma 16] for Steingrímsson's colored Eulerian numbers $E(n,r,s)$, i.e., $E(n,r,s) = (r(n+1) - (rs+1))E(n-1,r,s-1) + (rs+1)E(n-1,r,s)$.

Theorem 8. $\phi^s\big\downarrow_{\mathbb{Z}_r \wr \mathfrak{S}_{n-1}} = (r(n+1) - (rs+1))\phi^{s-1} + (rs+1)\phi^s$.

Proof. The formula translates to

$$\Phi^{(n)}_{s,k} = (r(n+1) - (rs+1))\Phi^{(n-1)}_{s-1,k} + (rs+1)\Phi^{(n-1)}_{s,k}, \quad k < \ell.$$

Equation (6) tells us that the right side is equal to

$$\begin{aligned}(r(n+1) - (rs+1)) &\sum_{j=0}^{s-1} (-1)^{s-1-j}(rj+1)^{n-1-k}\binom{n}{s-j-1} \\ &+ (rs+1)\sum_{j=0}^{s} (-1)^{s-j}(rj+1)^{n-1-k}\binom{n}{s-j}.\end{aligned} \tag{7}$$

Consider the terms indexed by j and observe that

$$-r(n+1)\binom{n}{s-j-1} + (rs+1)\left(\binom{n}{s-j-1} + \binom{n}{s-j}\right) = (rj+1)\binom{n+1}{s-j}.$$

It follows that (7) is equal to $\sum_j (-1)^{s-j}(rj+1)^{n-k}\binom{n+1}{s-j}$, which is precisely our expression (6) for $\Phi^{(n)}_{sk}$. □

Corollary 8.1. $\phi^s(1)$ *is the Eulerian number* $E(n,r,s)$. □

2.5. **The inverse of the Foulkes character table and Eulerian idempotents.** For the symmetric group, Diaconis and Fulman showed [3, Thm. 3.1, Cor. 3.2] that the rows of the inverse of the Foulkes character table are evaluations of Eulerian idempotents. They accomplished this by first verifying an explicit formula for the inverse, then comparing coefficients in order to connect with Eulerian idempotents.

We take the opposite approach and give a simple reason for why Eulerian idempotents appear, then get the formula for the inverse for free. The key is our second factorization (F2). There are three bases in [6] Loday's classic treatment of Eulerian idempotents, and what Diaconis and Fulman showed is that the transpose of Φ is the transition matrix between two of them, known as the c's and e's. Our factorization elucidates this by passing through the third, known as the λ basis.

Eulerian idempotents $e_0, e_1, \ldots, e_\ell$ were introduced by Reutenauer and extended to $\mathbb{Z}_r \wr \mathfrak{S}_n$ by Moynihan [8, p. 94], after Bergeron–Bergeron in the type B case.

They are defined according to the formula

$$\sum_i \binom{n+\frac{x-1}{r}-i}{n} c_i = \sum_k x^{n-k} e_{\ell-k} \tag{8}$$

where $i, k = 0, 1, \ldots, \ell$ and the c's are certain sums in the group algebra $\mathbb{Z}W$.

Define a third set of elements λ_j by evaluating at $x = rj + 1$:

$$\lambda_j = \sum_i \binom{n+j-i}{n} c_i \tag{9}$$

or equivalently

$$\lambda_j = \sum_k (rj+1)^{n-k} e_{\ell-k}. \tag{10}$$

The transition matrix from the λ_j's to the c_i's is upper triangular with ones along the diagonal, and its inverse is given by [6, Eq. 1.6.1] the formula

$$c_i = \sum_j (-1)^{i-j} \binom{n+1}{i-j} \lambda_j. \tag{11}$$

It follows from (11) and (10) that the transition matrix from the c_i's to the $e_{\ell-k}$'s is

$$\left[(rj+1)^{n-k}\right]_{kj} \times \left[(-1)^{i-j}\tbinom{n+1}{i-j}\right]_{ji} \tag{12}$$

which is precisely the transpose of Φ from (F2). Hence the following theorem.

Theorem 9. *The transpose of Φ is the transition matrix from the c_i's to the $e_{\ell-j}$'s. Equivalently, $\Phi^{-\mathsf{T}}$ is the transition matrix from the $e_{\ell-j}$'s to the c_i's.* □

Corollary 9.1. *Let $s(k,l)$ denote the Stirling numbers of the first kind. Then*

$$\Phi^{-1}_{ij} = \sum_{k,l} \frac{s(k,l)(-1)^{n-i-l}}{k!r^l} \binom{l}{n-i} \binom{n-j}{n-k}.$$

Proof. Theorem 9 and equation (8) tell us that the i, j entry of the inverse is equal to the coefficient of x^{n-i} in $\binom{n+(x-1)/r-j}{n}$. Now write

$$\begin{aligned}\binom{n+\frac{x-1}{r}-j}{n} &= \sum_k \binom{\frac{x-1}{r}}{k}\binom{n-j}{n-k} \\ &= \sum_k \binom{n-j}{n-k} \frac{1}{k!} \sum_l s(k,l) \left(\frac{x-1}{r}\right)^l\end{aligned}$$

and use the binomial theorem to get the desired formula; cf. [3, Thm. 3.1 proof]. □

3. General results for Shephard and Coxeter groups

In this section we sharpen and generalize Theorem 6, which gave a formula for the determinant of the reduced Foulkes character table for a wreath product. In general, however, the full table may not reduce to a square matrix; exactly when this happens is the subject of Section 3.3. Here we consider a finer invariant called the *Smith normal form*, which exists for any matrix with integer entries; cf. [5, Ch. III, §7].

Let M be an $(\ell+1) \times (n+1)$ matrix with integer entries. The $\mathbb{Z}$-module $\mathbb{Z}^{n+1}/(\mathbb{Z}$ row space of $M)$ decomposes as $\mathbb{Z}/s_0\mathbb{Z} \oplus \mathbb{Z}/s_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/s_n\mathbb{Z}$ for a unique sequence of nonnegative integers $s_0, s_1, \ldots, s_n$ subject to the condition that s_i divides s_{i+1} for all i. Recall that the Smith normal form of M encodes the same

information. An integral matrix $D = [d_{ij}]$ is said to be in *Smith form* if it is diagonal in the sense that $d_{ij} = 0$ whenever $i \neq j$, and its diagonal entries d_{ii} are nonnegative and satisfy the condition $d_{00} \mid d_{11} \mid \ldots \mid d_{\min(\ell,n)}$. The matrix M can be brought into Smith form by an appropriate change of basis, i.e., there exist matrices $P \in \mathrm{GL}_{\ell+1}(\mathbb{Z})$ and $Q \in \mathrm{GL}_{n+1}(\mathbb{Z})$ such that $PMQ = D$ is in Smith form. The resulting Smith form is unique and its diagonal entries are given by $d_{ii} = s_i$. These entries $s_0, s_1, \ldots, s_{\min(\ell,n)}$ are called the *Smith entries of* M. Note that if M is square, then $\det M = \pm s_0 s_1 \cdots s_\ell$.

3.1. **A sharp generalization of Isaacs' conjecture.** Isaacs conjectured that the determinant of the Foulkes character table for the symmetric group is always divisible by $n!$. A stronger conjecture would be that the last Smith entry s_ℓ is divisible by $n!$, or perhaps even equal to $n!$. This is in fact true.

Theorem 10. *Let W be an irreducible Coxeter or Shephard group. Then the last Smith entry of the Foulkes character table Φ is equal to $|W|$.*

Proof. Let $\mathrm{CL}_{\mathbb{Z}}$ denote the $\mathbb{Z}$-valued class functions on W. Recall that the Foulkes characters of W are linearly independent, and so the last Smith entry s_ℓ is the smallest positive integer s such that the following holds: whenever $\sum q_i \phi^i \in \mathrm{CL}_{\mathbb{Z}}$ for some $q_i \in \mathbb{Q}$, then one has that $sq_i \in \mathbb{Z}$. We show that s is the order of the group.

To see that $|W|$ divides s, recall that the sum of the Foulkes characters of W is equal to the character of the regular representation of W, or in other words, that

$$\frac{1}{|W|}(\phi^0 + \phi^1 + \ldots + \phi^\ell) = \delta_{\mathrm{id}}$$

for δ_K the class function that is 1 on the elements of class K, and 0 elsewhere.

To see that s divides $|W|$, suppose that $\sum q_i \phi^i \in \mathrm{CL}_{\mathbb{Z}}$ for some $q_i \in \mathbb{Q}$. Write

$$\sum\nolimits_i q_i \phi^i = \sum\nolimits_K z_K \delta_K \tag{13}$$

for integers z_K indexed by conjugacy classes K. To show that $|W| q_i \in \mathbb{Z}$ it suffices to show that $|W| q_i$ is an algebraic integer because $|W| q_i$ is a rational number. Recall from Section 1.1 the character ϵ^i of the ith exterior power of the reflection representation. Apply $\langle \epsilon^i, - \rangle$ to both sides of (13) to get

$$\epsilon^i(1) \times q_i = \sum\nolimits_K z_K \frac{|K|\, \epsilon^i(g_K)}{|W|}$$

or equivalently

$$|W| \times q_i = \sum\nolimits_K z_K \frac{|K|\, \epsilon^i(g_K)}{\epsilon^i(1)}$$

for $g_K \in K$. Since ϵ^i is an irreducible character of a finite group, a well-known result from character theory tells us that the terms on the right are algebraic integers. Hence $|W| q_i$ is an algebraic integer, as desired. □

3.2. **A sharp generalization of the Diaconis–Fulman formula.** Diaconis and Fulman confirmed Isaacs' conjecture by showing that in fact the determinant of the Foulkes character table for $\mathfrak{S}_n$ is equal to $n!(n-1)! \cdots 2!$ via induction; see [3, p. 429]. We give a remarkable generalization that holds whenever the determinant is available, that is, whenever the full table reduces to a square matrix.

Recall that the group acts on polynomial functions on V by $gp(v) = p(g^{-1}v)$, and that there exists a set of homogeneous polynomials $p_1, p_2, \ldots, p_\ell$ such that the subalgebra of W-invariant polynomials is given by $\mathbb{C}[p_1, p_2, \ldots, p_\ell]$. The *degrees* $d_i = \deg(p_i)$ are uniquely determined by the group, and we agree to number them so that $d_1 \leq d_2 \leq \ldots \leq d_\ell$. For example, when W is the symmetric group acting irreducibly on the subspace of $\mathbb{C}^n$ where $x_1 + x_2 + \ldots + x_n = 0$, then the elementary symmetric functions $e_2, e_3, \ldots, e_n$ form such a set of homogeneous polynomials, and so the degrees are $2, 3, \ldots, n$. In general, one has that $|W| = d_1 d_2 \cdots d_\ell$.

Theorem 11. *Let W be an irreducible Coxeter or Shephard group. Assume that each $\phi^i(g)$ depends only on the dimension of the fixed space of g. Then*

$$\det \Phi = d_1^1 d_2^2 \cdots d_\ell^\ell.$$

A much sharper result is the following.

Theorem 12. *Let W be an irreducible Coxeter or Shephard group. Assume that each $\phi^i(g)$ depends only on the dimension of the fixed space of g. Then the Smith entries $s_0, s_1, \ldots, s_\ell$ of the Foulkes character table are given by*

$$s_i = d_1 d_2 \cdots d_i.$$

The empty product that occurs when $i = 0$ is defined to be 1.

Theorem 11 and Theorem 12 will follow from Theorem 1 and the next proposition. The proof of the proposition relies on classic results of Orlik and Solomon [13–15], which were established using the Shephard–Todd classification [16] and nontrivial calculations showing that for every $X \in L$ of dimension p there exist positive integers $b_1^X \leq b_2^X \leq \ldots \leq b_p^X$ such that

$$B_X(t) = (t + b_1^X)(t + b_2^X) \cdots (t + b_p^X).$$

Call these positive integers *Orlik–Solomon coexponents*. When $X = V$ they are the usual coexponents $n_1, n_2, \ldots, n_\ell$ from the invariant theory of the group, which in the case of Shephard and Coxeter groups satisfy a remarkable duality involving the *exponents* $m_1, m_2, \ldots, m_\ell$ (defined as $m_i = d_i - 1$) which says that $m_1 + n_i = d_i$.

Proposition 13. *Let W be an irreducible Coxeter or Shephard group such that each $\phi^i(g)$ depends only on the dimension of the fixed space of g. Let $X \in L$ and write $p = \dim X$. Then the following hold:*

(i) *The cell counts $f_i(\Delta \cap X)$ depend only on p.*
(ii) $f_{p-1}(\Delta \cap X) = d_1 d_2 \cdots d_p$.

Proof. For (i), suppose otherwise and choose a p-dimensional $Y \in L$ such that $f_{s-1}(\Delta \cap X) \neq f_{s-1}(\Delta \cap Y)$ for some s. Fix s to be the smallest such number, and choose $g, h \in W$ with fixed spaces X, Y. Then (1) implies that $\phi^s(g) \neq \phi^s(h)$.

For (ii), recall that $f_{p-1}(\Delta \cap X) = B_X(m_1)$, and hence

$$f_{p-1}(\Delta \cap X) = (m_1 + b_1^X)(m_1 + b_2^X) \cdots (m_1 + b_p^X).$$

Orlik and Solomon observed that for each Shephard and Coxeter group there exists some p-dimensional $Y \in L$ such that $b_i^Y = n_i$ for $1 \leq i \leq p$. Since (i) tells us that $f_{p-1}(\Delta \cap X) = f_{p-1}(\Delta \cap Y)$, it follows that

$$f_{p-1}(\Delta \cap X) = (m_1 + n_1)(m_1 + n_2) \cdots (m_1 + n_p).$$

Now use the duality $m_1 + n_i = d_i$ to rewrite the product as $d_1 d_2 \cdots d_p$. $\square$

Proof of Theorem 11 *and Theorem* 12. Choose a sequence $X_0, X_1, \ldots, X_\ell \in L$ such that $\dim X_i = i$. Define $\Phi' = [f_{i-1}(\Delta \cap X_j)]_{ij}$ where $0 \le i, j \le \ell$. Note that the left factor of (3) is square and has determinant equal to 1. Hence Φ and Φ' have the same determinant and the same Smith normal form.

Theorem 11 follows from Proposition 13(ii), which says Φ' has diagonal entries

$$f_{i-1}(\Delta \cap X_i) = d_1 d_2 \cdots d_i.$$

These diagonal entries are the claimed Smith entries, and so for Theorem 12 it suffices to show that each diagonal entry divides the entries to its right in Φ', i.e., that $f_{i-1}(\Delta \cap X_i)$ divides $f_{i-1}(\Delta \cap X_j)$ whenever $i \le j$. But (2) tells us that

$$\begin{aligned} f_{i-1}(\Delta \cap X_j) &= \sum\nolimits_X f_{i-1}(\Delta \cap X) \\ &= \sum\nolimits_X f_{i-1}(\Delta \cap X_i), \end{aligned}$$

where the sums are over all i-dimensional subspaces $X \in L$ that lie above X_j and the second equality follows from Proposition 13(i). This completes the proof. □

3.3. **A curious classification.**

Theorem 14. *Let W be an irreducible Coxeter or Shephard group. Then the following are equivalent:*

(a) *The characters $\phi^i(g)$ depend only on the dimension of the fixed space of g.*
(b) *The characters $\phi^0, \phi^1, \ldots, \phi^\ell$ form a $\mathbb{Q}$ basis for the space of class functions χ that depend only on the dimension of the fixed space, and* (4) *holds.*
(c) *The reduced Foulkes character table Φ is square and* $\det \Phi = d_1^1 d_2^2 \cdots d_\ell^\ell$.
(d) *The Smith entries $s_0, s_1, \ldots, s_\ell$ of the table Φ are given by $s_i = d_1 d_2 \cdots d_i$.*
(e) *The isomorphism class of L^X depends only on the dimension of X.*
(f) *The cell counts $f_i(\Delta \cap X)$ depend only on the dimension of X.*
(g) *The numbers $B_X(m_1)$ depend only on the dimension of X.*
(h) *The Orlik–Solomon coexponents b_i^X depend only on the dimension of X.*
(i) *The coexponent sequence $n_1, n_2, \ldots, n_\ell$ is arithmetic.*
(j) *The degree sequence $d_1, d_2, \ldots, d_\ell$ is arithmetic.*
(k) *The group W is not F_4, H_4, E_6, E_7, E_8, or D_ℓ for $\ell \ge 4$.*

Proof. The proof of Theorem 3 gives the forward implication of (a)⇔(b); the converse is trivial. (a)⇔(c) follows from Theorem 11. (a)⇒(d) is Theorem 12. The proof of Theorem 2 gives the implication (e)⇒(a).

(i)⇔(j)⇔(k). These equivalences follow from [13, Table 2].

(a)⇔(f)⇒(g). The implication (a)⇒(f) is Proposition 13(i), and the converse follows from equation (1). The implication (f)⇒(g) follows from the special case $f_{\dim X - 1}(\Delta \cap X) = B_X(m_1)$ of equation (2).

(g)⇔(h)⇔(i). The exceptional cases follow from the tables of [14, 15]. When W is a dihedral group, the rank is 2, and properties (g)–(i) follow immediately. There are two other nonexceptional cases: $\mathbb{Z}_r \wr \mathfrak{S}_n$ and the Weyl group D_ℓ when $\ell \ge 4$. In the case when $W = \mathbb{Z}_r \wr \mathfrak{S}_n$, property (i) follows from [13, Table 2], while (g) and (h) follow from the fact that L^X, and hence $B_X(t)$, is determined by the dimension of X. When $W = D_\ell$, [13, Table 2] tells us that the coexponent sequence is not arithmetic, and we claim that (g) and (h) fail in this case as well. In usual coordinates [15], let X be the codimension-2 space where $x_1 = x_2$ and $x_3 = x_4$, and

let Y be the space where $x_1 = x_2 = x_3$. Then $B_Y(m_1) < B_X(m_1) = d_1 d_2 \cdots d_{\ell-2}$ by [15, Prop. 2.6], and the claim follows.

(d)⇒(g). Assume that (g) fails. We claim that $s_1 s_2 \cdots s_\ell < d_1^1 d_2^2 \cdots d_\ell^\ell$. As in the previous proof, it suffices to consider the matrix $\Phi' = [f_{i-1}(\Delta \cap Y)]_{i,Y}$, $Y \in L$. Recall Orlik and Solomon's observation that for each Shephard and Coxeter group, and for each integer p with $0 \leq p \leq \ell$, there exists a p-dimensional $Y \in L$ such that $b_i^Y = n_i$ for $1 \leq i \leq p$. From the tables of [14, 15] and the previous paragraph, it follows that there exists a sequence $X_0, X_1, \ldots, X_\ell \in L$ such that $\dim X_i = i$ and $B_{X_i}(m_1) \leq d_1 d_2 \cdots d_i$, with at least one strict inequality. Hence the determinant of the submatrix $[f_{i-1}(\Delta \cap X_j)]_{ij}$ is strictly less than $d_1^1 d_2^2 \cdots d_\ell^\ell$ by triangularity. The claim now follows from a well-known result about Smith forms which says that $s_0 s_1 \cdots s_{i-1}$ is the greatest common divisor of all $i \times i$ subdeterminants.

To end, it suffices to show that (k)⇒(e). Since (e) is a well-known property of the infinite family $\mathbb{Z}_r \wr \mathfrak{S}_n$, let us suppose that W is dihedral or exceptional of rank ℓ. The cases $\dim X = \ell, 1, 0$ are trivial, and when $\dim X = 2$, the structure of L^X is determined by $B_X(t) = (t + b_1^X)(t + b_2^X)$, which apparently depends only on the dimension of X by (k)⇒(h). Having dispensed with the case when $\ell \leq 3$, the Shephard–Todd classification tells us that the only remaining case is when $\ell = 4$ and W is the group known as G_{32}. We need only consider the case when X is a reflecting hyperplane, and this case follows from the fact that G_{32} acts transitively on its reflecting hyperplanes; indeed, each reflection of a reflection group is conjugate to a power of some generating reflection r_i, and in the case of G_{32}, all of the generating reflections are clearly conjugate. □

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School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455

Current address: Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, Illinois 61801

E-mail address: `arm@illinois.edu`