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ON SYMMETRIC POWERS OF τ-RECURRENT SEQUENCES AND DEFORMATIONS OF EISENSTEIN SERIES

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Abstract. We prove the equality of several τ-recurrent sequences, which were first considered by Pellarin and which have close connections to Drinfeld vectorial modular forms. Our result has several consequences: an A-expansion for the l^{th} power ($1 \leq l \leq q$) of the deformation of the weight 2 Eisenstein series; relations between Drinfeld modular forms with A-expansions; and a new proof of relations between special values of Pellarin L-series.

1. Introduction and statement of results

Let $q = p^e$, with p a prime number and e a positive integer. Let A be the polynomial ring $\mathbb{F}_q[\theta]$, K its fraction field and A_+ the set of monic polynomials in A. Let $|\cdot|$ be the absolute value on K uniquely defined by $|a| = q^{\deg_\theta(a)}$ for $a \in A$, let K_∞ be the completion of K with respect to $|\cdot|$, and let $\mathbb{C}_\infty$ be the completion of a fixed algebraic closure of K_∞. Let $B_r \subset \mathbb{C}_\infty$ be the open disc of radius r centered at 0. The *Drinfeld upper half-plane* Ω is the set $\mathbb{C}_\infty \backslash K_\infty$ together with its rigid analytic structure as in [5, §1.6]. The group $\Gamma := \mathbf{GL}_2(A)$ acts on Ω by fractional linear transformations. Let ϕ_{Car} be the *Carlitz module* defined by $\phi_{\mathrm{Car}}(\theta) = \theta\tau^0 + \tau$ with τ the qth power Frobenius operator on $\mathbb{C}_\infty$. We fix $\tilde{\pi} \in \mathbb{C}_\infty$ so that the lattice corresponding to the Carlitz module is $\tilde{\pi}A$. The exponential function of $\tilde{\pi}A$ will be called *the Carlitz exponential* and will be denoted by $e_{\tilde{\pi}A}$.

Let t be a new variable independent of θ and consider the series

$$s_{\mathrm{Car}}(t) := \sum_{n=0}^{\infty} e_{\tilde{\pi}A}\left(\frac{\tilde{\pi}}{\theta^{n+1}}\right) t^n.$$

The series converges for $|t| < q$ (see [1, Proposition 2.3]).

The set of isomorphism classes of rank 2 Drinfeld A-modules over $\mathbb{C}_\infty$ corresponds to $\Gamma\backslash\Omega$ via the well-known equivalence of categories between Drinfeld modules and lattices ([18, 2.4]). For $z \in \Omega$, let $\Lambda_z := zA \oplus A$. The exponential function for Λ_z will be written as

$$e_{\Lambda_z}(\zeta) = \sum_{n=0}^{\infty} \alpha_n(z)\zeta^{q^n},$$

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where $\alpha_n : \Omega \to \mathbb{C}_\infty$ are functions given explicitly in [2, Theorem 3.1]. Following Pellarin [13] we consider

$$s_1(z,t) := \sum_{n=0}^{\infty} e_{\Lambda_z}\left(\frac{z}{\theta^{n+1}}\right) t^n, \qquad s_2(z,t) := \sum_{n=0}^{\infty} e_{\Lambda_z}\left(\frac{1}{\theta^{n+1}}\right) t^n.$$

Both s_1, s_2 converge for $(z,t) \in \Omega \times B_q$. For arithmetic consideration it is more convenient to work with normalizations of s_1 and s_2, namely

$$d_1(z,t) := \tilde{\pi} s_{\mathrm{Car}}^{-1}(t) s_1(z,t), \qquad d_2(z,t) := \tilde{\pi} s_{\mathrm{Car}}^{-1}(t) s_2(z,t).$$

The functions d_1, d_2 converge for any $z, t \in \mathbb{C}_\infty$ (see [11, Proposition 19]).

Let $\chi_t : A \to \mathbb{F}_q[t]$ be the ring homomorphism defined by $\chi_t(a) = a(t)$. If α, β are positive integers, then the *Pellarin L-function* is defined by

$$L(\chi_t^\alpha, \beta) := \sum_{a \in A_+} \chi_t(a)^\alpha a^{-\beta}. \tag{1.1}$$

Pellarin introduced $L(\chi_t^\alpha, \beta)$ in [12] as a deformation of the Carlitz zeta function and the more general Goss L-functions (see [7, Chapter 8]). In addition to Pellarin's original paper, the reader can find information about analytic continuation of Pellarin's L-function in [8] (note that we will only consider values of $L(\chi_t^\alpha, \beta)$ for positive integers α, β, i.e., values as in Equation (1.1)) and formulas for special values in [15]. In the course of the proof of our main result we will give a new proof of several relations between special values of Pellarin L-functions (see Corollary 3.4).

Let $\tau : \mathbb{C}_\infty((t)) \to \mathbb{C}_\infty((t))$ be the field automorphism that fixes t and acts as the Frobenius qth power operator on elements of $\mathbb{C}_\infty$. This agrees with the previous definition of τ on $\mathbb{C}_\infty$, so by abuse of notation we will use τ to denote both. If $f \in \mathbb{C}_\infty((t))$, then we will use the notation $f^{(i)}$ for $\tau^i f$.

For $l \in \mathbb{N}$ we define the sequence $\{\mathcal{G}_{l,k}\}_{k \in \mathbb{Z}}$ by

$$\mathcal{G}_{l,k} := \mathcal{G}_{l,k}(z,t) = \frac{1}{L(\chi_t^l, lq^k)} {\sum_{c,d \in A}}' \left(\frac{\chi_t(c) d_1 + \chi_t(d) d_2}{(cz+d)^{q^k}}\right)^l.$$

The primed sum $\sum'$ will be used throughout to denote a sum with the term where all summation indices are zero is omitted. We will prove (Proposition 2.4) that if $k \geq 0$ the series defining $\mathcal{G}_{l,k}$ is well-defined for all $(z,t) \in \Omega \times B_{q^{q^k}}$.

Pellarin explicitly computed $\{\mathcal{G}_{1,k}\}_{k \geq 0}$ in [12, Theorem 4]:

$$\mathcal{G}_{1,k} = -h^{q^k}(t - \theta^{q^k}) s_{\mathrm{Car}}^{(k)} \left(d_2^{(k+1)} d_1 - d_1^{(k+1)} d_2\right), \tag{1.2}$$

where h is the Drinfeld modular form of weight $q+1$ and type 1, which is defined by Equation (2.3) below. We give a different formula for $\mathcal{G}_{1,k}$ in (2.16).

The main result of the current paper is the computation of the sequence $\{\mathcal{G}_{l,k}\}_{k \geq 0}$ for l in the range $1 \leq l \leq q$:

Theorem 1.1. *Let $1 \leq l \leq q$ be fixed. For $k \geq 0$, we have*

$$\mathcal{G}_{l,k} = (-1)^{l+1} \mathcal{G}_{1,k}^l.$$

Prasenjit Bhowmik (work in preparation) has obtained similar results by a different method. Indeed, he computes Rankin brackets of certain families of Drinfeld modular forms and then applies a density argument ([14]).

The paper is organized as follows. In Section 2 we introduce the necessary background, in particular, vectorial Drinfeld modular forms, their deformations and τ-recurrent sequences. Subsection 2.3 gives a new method for the computation of the coefficients of d_2 based on the theory of shadowed partitions. In Section 3 we prove Theorem 1.1. Finally, Section 4 gives applications of Theorem 1.1 to deformations of Drinfeld modular forms. In particular, Theorem 4.4 gives an A-expansion with coefficients in $\mathbb{F}_q[t]$ for the lth power of the deformation of Gekeler's 'false Eisenstein series' (see (2.4)), $\mathbb{E}^l$, $1 \leq l \leq q$, extending the one given by Pellarin for $l = 1$, while Corollary 4.8 gives examples of Drinfeld modular forms that are eigenforms and can be expressed as products of eigenforms.

2. Deformations of vectorial modular forms

2.1. **Drinfeld modular forms and their generalizations.** For $z \in \Omega$ and $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$, we shall write $J_\gamma := cz + d$ and $L_\gamma := c/(cz+d)$.

The 'imaginary distance' $|z|_i$ of $z \in \Omega$ is defined by $|z|_i := \inf_{x \in K_\infty} |z - x|$. Let $u := u(z) = 1/e_{\tilde{\pi}A}(\tilde{\pi}z)$ be the normalized uniformizer at 'infinity'. We shall say that a rigid-analytic function $f : \Omega \to \mathbb{C}_\infty$ has a u-expansion if there exists $\delta_f > 0$ such that for $z \in \Omega$ with $|z|_i > \delta_f$ we have

$$f(z) = \sum_{n=n_0}^{\infty} a_n u^n,$$

for some $n_0 \in \mathbb{Z}$, $a_n \in \mathbb{C}_\infty$. Since $u \in \mathbb{C}_\infty$, τ acts on u as a qth power Frobenius. The function f is said to have an integral u-expansion if $n_0 \in \mathbb{Z}_{\geq 0}$. A rigid-analytic function f which satisfies $f(z + a) = f(z)$ for all $a \in A$ has a u-expansion, and this u-expansion determines f uniquely.

For $c \in A_+$, set $u_c := u(cz)$. We have (see [4, (6.2)] for instance)

$$u_c = u^{q^{\deg_\theta(c)}} + \text{higher order terms in } u. \tag{2.1}$$

Definition 2.1. A *Drinfeld modular form* of weight w, type m for Γ is a rigid-analytic function $f : \Omega \to \mathbb{C}_\infty$ such that

$$f(\gamma(z)) = J_\gamma^w \det(\gamma)^{-m} f(z),$$

and such that f has an integral u-expansion. The set of Drinfeld modular forms of weight w and type m is a finite-dimensional $\mathbb{C}_\infty$-vector space, which we denote by $M_{w,m}$.

Among the most important examples of Drinfeld modular forms are

$$g := 1 - (\theta^q - \theta) \sum_{c \in A_+} u_c^{q-1} = 1 - (\theta^q - \theta) u^{q-1} + \cdots \in M_{q-1,0}, \tag{2.2}$$

$$h := \sum_{c \in A_+} c^q u_c = u + \cdots \in M_{q+1,1}, \qquad \triangle := -h^{q-1} \in M_{q^2-1,0}. \tag{2.3}$$

The forms g and h generate the space of Drinfeld modular forms of any weight and type. An important rigid-analytic function, which is not a Drinfeld modular form but is closely connected with the theory, is Gekeler's 'false Eisenstein series':

$$E := \sum_{c \in A_+} c u_c.$$

The reader can find more about the properties of $g, h, \triangle$ and E in one of the standard references [4], [6], [5].

Bosser and Pellarin [1] introduced the concept of *almost-A-quasi-modular forms*, which encompasses Drinfeld modular forms as well as d_2 and the function

$$\mathbb{E} := -hd_2^{(1)}. \tag{2.4}$$

We do not recall the general definition of almost-A-quasi-modular forms ([1, Definition 2.9]) here.

The function d_1 does not fall into the framework of almost-A-quasi-modular forms, but the following facts show that it has to be studied together with d_2 when considering modular properties.

Proposition 2.2.

(1) *The functions d_1, d_2 satisfy*

$$J_\gamma d_1(\gamma(z)) = \chi_t(a)d_1 + \chi_t(b)d_2, \qquad J_\gamma d_2(\gamma(z)) = \chi_t(c)d_1 + \chi_t(d)d_2. \tag{2.5}$$

(2) *The function d_2 has a u-expansion with $\mathbb{F}_q[\theta, t]$ coefficients:*

$$d_2 = 1 + (\theta - t)u^{q-1} + (\theta - t)u^{(q-1)(q^2-q+1)} + \cdots, \tag{2.6}$$

while d_1 does not have a u-expansion.

(3) *The functions d_1 and d_2 form a basis for the solution space of the τ-difference equation*

$$X^{(2)} = \frac{1}{\triangle(t - \theta^q)}\left(X - gX^{(1)}\right). \tag{2.7}$$

Proof. Results (2.5), (2.6), are Lemmas 6, 8 in [13], respectively, while (2.7) is [13, (24)]. □

Keeping with Pellarin's notation, let $\mathbb{T}_{<r}$ be the Tate algebra of formal power series $\sum_{n\geq 0} c_n t^n \in \mathbb{C}_\infty[[t]]$ that converge for $|t| < r$. If $r = \infty$, then we write $\mathbb{T}_\infty$ instead of $\mathbb{T}_{<\infty}$. Let $\mathcal{R}_{<r}$ be the ring that consists of formal series $\sum_{n\geq 0} f_n t^n$ such that: (1) for all n, f_n is a rigid-analytic function $\Omega \to \mathbb{C}_\infty$; (2) for all $z \in \Omega$, $\sum_{n\geq 0} f_n(z)t^n$ is an element of $\mathbb{T}_{<r}$. The fraction field of $\mathcal{R} := \mathcal{R}_{<1}$ will be denoted by $\mathcal{L}$, while the fraction field of

$$\mathcal{R}_\infty = \bigcap_{r>0} \mathcal{R}_{<r}$$

will be denoted by $\mathcal{L}_\infty$. In this notation, $d_1, d_2, \mathbb{E} \in \mathcal{R}_\infty$.

2.2. **Vectorial modular forms and τ-recurrent sequences.** Let ρ be a representation

$$\rho : \Gamma \to \mathbf{GL}_s(\mathbb{F}_q((t))).$$

The following definition is due to Pellarin.

Definition 2.3. A *deformation of a vectorial modular form* of weight w, dimension s, type m and radius $r > 0$ associated with the representation ρ is a vector $\mathcal{F}$ with entries in $\mathcal{R}_{<r}$ such that

$$\mathcal{F}(\gamma(z)) = J_\gamma^w \det(\gamma)^{-m} \rho(\gamma)\mathcal{F}(z), \qquad \forall \gamma \in \Gamma.$$

The set of such vectors is denoted by $\mathcal{M}^s_{w,m}(\rho, r)$.

Consider the representation $\rho_{t,1}$ defined by

$$\rho_{t,1} = \begin{bmatrix} \chi_t(a) & \chi_t(b) \\ \chi_t(c) & \chi_t(d) \end{bmatrix},$$

and its lth symmetric power $\rho_{t,l} := \mathrm{Sym}^l(\rho_{t,1})$. By definition $\rho_{t,l} = \mathrm{Sym}^l(\rho_{t,1})$ can be realized on the vector space of homogeneous polynomials of degree l via

$$X^i Y^{l-i} \mapsto (\chi_t(a)X + \chi_t(b)Y)^i (\chi_t(c)X + \chi_t(d)Y)^{l-i}.$$

Let

$$\Phi_l := {}^{\mathtt{TR}}(d_1^l, d_1^{l-1}d_2, \ldots, d_1 d_2^{l-1}, d_2^l),$$

where $\mathtt{TR}$ is the usual transpose. Property (2.5) implies $\Phi_l \in \mathcal{M}^{l+1}_{-1,0}(\rho_{t,l}, \infty)$.

We let $\mathcal{E}_l$ be defined to be the transpose of the row vector

$$\frac{1}{L(\chi_t^l, l)} \sum_{c,d\in A}{}' \left(\frac{\chi_t(c)^l}{(cz+d)^l}, \frac{\binom{l}{1}\chi_t(c)^{l-1}\chi_t(d)}{(cz+d)^l}, \ldots, \frac{\binom{l}{l-1}\chi_t(c)\chi_t(d)^{l-1}}{(cz+d)^l}, \frac{\chi_t(d)^l}{(cz+d)^l} \right).$$

An equivalent (but more 'modular') definition[1] and other properties of $\mathcal{E}_l$ can be found in [11, Section 3.3.2]. Part 1 from [11, Proposition 21] shows that for $\gamma \in \Gamma$, we have

$$\mathcal{E}_l(\gamma(z)) = J_\gamma^l \left({}^{\mathtt{TR}}(\rho_{t,l}^{-1})(\gamma) \right) \mathcal{E}_l(z). \tag{2.8}$$

Proposition 2.4. *The series defining $\tau^k \mathcal{E}_l$, $k \geq 0$, converges for $(z,t) \in \Omega \times B_{q^{q^k}}$.*

Proof. Note that if $f(t)$ converges for $|t| < r$, then $\tau f(t)$ converges for $|t| < r^q$; thus it suffices to prove the case $k = 0$. Write $|t| = q^\epsilon$, with $\epsilon < 1$. For such t, the series

$$L(\chi_t^l, l) = \sum_{c \in A_+} \frac{\chi_t(c)^l}{c^l}$$

converges, since

$$\lim_{|c|\to\infty} \left|\chi_t(c)^l c^{-l}\right| = \lim_{|c|\to\infty} |c|^{(\epsilon-1)l} = 0.$$

Assume that $|z| \geq 1$. By property (2.8), we see that, for $a \in A$, $\mathcal{E}_l(z+a) = M\mathcal{E}_l(z)$, where M is a matrix with coefficients in $\mathbb{C}_\infty[t]$ that do not depend on z. Therefore the convergence of the series defining $\mathcal{E}_l$ is not affected by transformations of the form $z \mapsto z + a$, $a \in A$. Since K_∞ is locally compact we know that $|z|_i = |z - x_0|$ for some $x_0 \in K_\infty$. By applying transformations of the form $z \mapsto z + a$, $a \in A$, we can assume that the absolute value of z is strictly larger than the absolute value of x_0, hence $|z|_i = |z - x_0| = \max\{|z|, |x_0|\} = |z|$. We can therefore assume without loss of generality that $|z| = |z|_i$. For such z, we have $|cz + d| = \max\{|cz|, |d|\}$ for $c, d \in A$.

We need to show that the series

$$\sum_{c,d\in A}{}' \frac{\chi_t(c)^i \chi_t(d)^{l-i}}{(cz+d)^l} \tag{2.9}$$

converges for any i such that $0 \leq i \leq l$.

[1] The reader should be aware that in his original preprint Pellarin omits the binomial coefficients from the definition of $\mathcal{E}_l$, but we have confirmed with Pellarin that the binomial coefficients in the definition of $\mathcal{E}_l$ need to be present.

If $|cz| < |d|$, then

$$\left|\frac{\chi_t(c)^i\chi_t(d)^{l-i}}{(cz+d)^l}\right| = \frac{|c|^{i\epsilon}|d|^{(l-i)\epsilon}}{|d|^l} < \frac{1}{|z|^{\epsilon i}}|d|^{(\epsilon-1)l}.$$

Since $|cz| < |d|$ implies that $|c| \to \infty \Rightarrow |d| \to \infty$, the last quantity tends to 0 as $|c| \to \infty$ or $|d| \to \infty$. A similar argument shows that when $|cz| > |d|$ if $|c| \to \infty$ or $|d| \to \infty$, we have

$$\left|\frac{\chi_t(c)^i\chi_t(d)^{l-i}}{(cz+d)^l}\right| \to 0.$$

We conclude that for $|z| \geq 1$, $|t| = q^\epsilon$, $\epsilon < 1$, the series (2.9) converges. Since every $z \in \Omega$ is equivalent under the action of Γ to an element in $\mathfrak{F} = \{z \in \Omega : |z|_i = |z| \geq 1\}$ and $\mathcal{E}_l$ satisfies (2.8) under the action of Γ (i.e., the action of Γ permutes the components of $\mathcal{E}_l$), the result for $|z| \geq 1$ implies that for all $z \in \Omega$. ☐

Property (2.8) and the previous proposition show that $\mathcal{E}_l \in \mathcal{M}^{l+1}_{l,0}({}^{\mathtt{TR}}\rho^{-1}_{t,l}, q)$. By definition

$$\mathcal{G}_{l,k} = (\tau^k\mathcal{E}_l)\cdot\Phi_l, \tag{2.10}$$

where the dot denotes the usual inner product of vectors. The modular properties of $\mathcal{E}_l$ and Φ_l imply that for $k \geq 0$,

$$\mathcal{G}_{l,k} \in \mathcal{M}^1_{lq^k-l,0}(\mathbf{1}, q^{q^k}),$$

where $\mathbf{1}$ is the trivial representation.

Next we recall the theory of τ-recurrent sequences as described in [11, Section 2]. Let $\mathcal{K}$ be a field together with an infinite order automorphism τ. The fixed field of τ will be denoted by $\mathcal{K}^\tau$. Let $L = A_0\tau^0 + \cdots + A_s\tau^s \in \mathcal{K}[\tau]$ be a τ-linear operator such that $A_0 \neq 0, A_s \neq 0$ (s is the *order* of L). Given a sequence $\mathcal{G} = \{\mathcal{G}_k\}_{k\in\mathbb{Z}}$ with elements in $\mathcal{K}$, we write $L(\mathcal{G})$ for the sequence

$$\{A_0\tau^0\mathcal{G}_k + \cdots + A_s\tau^s\mathcal{G}_{k-s}\}_{k\in\mathbb{Z}}.$$

The sequence $\mathcal{G}$ is a *τ-recurrent sequence* for L if $L(\mathcal{G}) \equiv 0$. The space of all τ-recurrent sequences for L is a finite-dimensional $\mathcal{K}$-vector space, which we denote by $V(L)$. The space of solutions to the associated *τ-difference equation*

$$A_0\tau^0X + \cdots + A_s\tau^sX = 0 \tag{2.11}$$

is denoted by $V^\tau(L)$. Any solution x to (2.11) gives an element of $V(L)$ by simply taking the constant sequence $\{x\}_{k\in\mathbb{Z}}$. Pellarin shows (see [11, Propositions 10, 11]) that if $x_1, \ldots, x_s$ are $\mathcal{K}^\tau$-linearly independent elements of $\mathcal{K}$, then there exists an explicit procedure for computing a τ-linear operator L of order s (unique if we assume the normalization $A_s = 1$; otherwise L is unique up to left multiplication) such that $\{x_i : 1 \leq i \leq s\}$ is a basis of $V^\tau(L)$. In addition, for any vector $\mathcal{E} = (e_1, \ldots, e_s) \in \mathcal{K}^s$ the sequence $\mathcal{G}$, defined by

$$\mathcal{G}_k = (\tau^k\mathcal{E})\cdot(x_1, \ldots, x_s),$$

belongs to $V(L)$, and any $\mathcal{G} \in V(L)$ is of this form for a unique $\mathcal{E} \in \mathcal{K}^s$.

Applying this to $\mathcal{K}=\mathcal{L}$, $(x_1,\dots,x_s)=(d_1^l,d_1^{l-1}d_2,\dots,d_2^l)$ we have

Proposition 2.5. *The sequence $\{\mathcal{G}_{l,k}\}_{k\in\mathbb{Z}}$, defined by*

$$\mathcal{G}_{l,k}:=(\tau^k\mathcal{E}_l)\cdot\Phi_l,\qquad k\in\mathbb{Z},$$

is a τ-recurrent sequence that satisfies the unique normalized τ-difference equation L_l satisfied by $d_1^l,d_1^{l-1}d_2,\dots,d_1d_2^{l-1},d_2^l$.

Note that the linear independence of $\{d_1^l,d_1^{l-1}d_2,\dots,d_1d_2^{l-1},d_2^l\}$ over $\mathcal{K}^\tau=\mathbb{F}_q(t)$ is also part of the result (see [11, Lemmas 14 and 20]).

Examples. The sequence $\{\mathcal{G}_{1,k}\}_{k\in\mathbb{Z}}$ is a τ-recurrent sequence for

$$L_1:=\tau^0-g\tau^1-\triangle(t-\theta^q)\tau^2$$

and $\{\mathcal{G}_{2,k}\}_{k\in\mathbb{Z}}$ is a τ-recurrent sequence for

$$\begin{aligned}L_2:=\tau^0&-g^{1-q}(g^{1+q}+\triangle(t-\theta^q))\tau^1\\&-\triangle(t-\theta^q)(g^{1+q}+\triangle(t-\theta^q))\tau^2+g^{1-q}\triangle^{1+2q}(t-\theta^q)(t-\theta^{q^2})^2\tau^3.\end{aligned}\tag{2.12}$$

In [12, Theorem 4] Pellarin determines $\{\mathcal{G}_{1,k}\}_{k\in\mathbb{Z}}$ completely by computing its first two non-negative terms: $\mathcal{G}_{1,0}=-1$ and $\mathcal{G}_{1,1}=-g$. Our main theorem (Theorem 1.1) shows that $\mathcal{G}_{l,k}=(-1)^{l+1}\mathcal{G}_{1,k}^l$.

It follows from the following proposition that $\mathcal{G}_{1,k}^l$ is part of a basis for $V(L_l)$.

Proposition 2.6. *Let $\{(\mathcal{G}_k),(\mathcal{H}_k)\}$ be a basis of $V(L_1)$. Then for any $l\geq 1$,*

$$\{(\mathcal{G}_k^i\mathcal{H}_k^{l-i}):0\leq i\leq l\}$$

forms a basis of $V(L_l)$, where L_l is the unique normalized τ-difference equation satisfied by $d_1^l,d_1^{l-1}d_2,\dots,d_1d_2^{l-1},d_2^l$.

Proof. According to [11, Proposition 11], there exist $a,b,c,d\in\mathbb{C}_\infty[t]$ such that

$$\mathcal{G}_k=a^{(k)}d_1+b^{(k)}d_2,\qquad \mathcal{H}_k=c^{(k)}d_1+d^{(k)}d_2.\tag{2.13}$$

We thus get

$$\mathcal{G}_k^i\mathcal{H}_k^{l-i}=\sum_{m=0}^{l}C_{mi}^{(k)}d_1^md_2^{l-m},\tag{2.14}$$

where

$$C_{mi}=\sum_{j=0}^{i}\binom{i}{j}\binom{l-i}{m-j}a^jb^{i-j}c^{m-j}d^{l-m-(i-j)}.$$

Thus, again by the same proposition from [11], we see that (2.14) is equivalent to $(\mathcal{G}_k^i\mathcal{H}_k^{l-i})\in V(L_l)$. The linear independence of $(\mathcal{G}_k)$ and $(\mathcal{H}_k)$ is equivalent to $ad-bc\neq 0$. However, we have the equality of matrices

$$(C_{mi})_{0\leq i,m\leq l}=\mathrm{Sym}^l\begin{pmatrix}a&b\\c&d\end{pmatrix},$$

which yields

$$\det(C_{mi})=(ad-bc)^{\frac{l^2+l}{2}}\neq 0,$$

again implying the linear independence of our proposed basis. This proves the result as $V(L_l)$ has dimension at most $l+1$. ∎

2.3. **Using shadowed partitions to approximate d_2.** In this subsection, we give a new formula (2.16) for the sequence $\mathcal{G}_{1,k}$, $k \geq 1$. This formula can be used to give a method for computing the u-expansion of d_2 which, in contrast with the original computation of Pellarin, is not recursive in the coefficients of d_2. This is done via *shadowed partitions* as in [3]. If $S \subset \mathbb{Z}$ and $j \in \mathbb{Z}$, then let $S + j := \{i + j : i \in S\}$. Let $r, n \in \mathbb{N}$. We define the *order r index-shadowed partition of n* by

$$P_r(n) := \big\{(S_1, S_2, \ldots, S_r) : S_i \subset \{0, 1, \ldots, n-1\},$$
$$\text{and } \{S_i + j : 1 \leq i \leq r, 0 \leq j \leq i-1\} \text{ form a partition of } \{0, 1, \ldots, n-1\}\big\}.$$

Theorem 2.7. *For $k \geq 1$, we have*

$$d_2 - \sum_{(S_1,S_2)\in P_2(k)} \prod_{j\in S_1} \prod_{i\in S_2} g^{q^j}(t-\theta^{q^{i+1}})\triangle^{q^i} \in u^{q^{k-1}(q-1)}\mathbb{F}_q[\theta,t][[u]]. \tag{2.15}$$

Proof. According to Lemma 3.3 from [3], for $k \geq 1$ we have

$$\mathcal{G}_{1,k} = -\sum_{(S_1,S_2)\in P_2(k)} \prod_{j\in S_1} \prod_{i\in S_2} g^{q^j}(t-\theta^{q^{i+1}})\triangle^{q^i}. \tag{2.16}$$

Using this we see that $\mathcal{G}_{1,1} = -g$, $\mathcal{G}_{1,2} = -g^{q+1} - (t-\theta^q)\triangle$. By comparing the u-expansions of d_2 (2.6) with the u-expansions of $\mathcal{G}_{1,1}$ and $\mathcal{G}_{1,2}$, we see that

$$d_2 + \mathcal{G}_{1,1} \in u^{q-1}\mathbb{F}_q[\theta,t][[u]] \quad \text{and} \quad d_2 + \mathcal{G}_{1,2} \in u^{q(q-1)}\mathbb{F}_q[\theta,t][[u]].$$

Since $\mathcal{G}_{1,k}$ is a τ-recurrent sequence for L_1 and $L_1(d_2) = 0$, it follows by induction that $d_2 + \mathcal{G}_{1,k} \in u^{q^{k-1}(q-1)}\mathbb{F}_q[\theta,t][[u]]$. □

3. The proof of Theorem 1.1

We start the proof of Theorem 1.1 with several lemmas.

Lemma 3.1.

$$1 + \sum_{u\in\mathbb{F}_q} \frac{X+u}{Y+u} = \frac{Y^q - X}{Y^q - Y}.$$

Proof. It is well-known that $\prod_{u\in\mathbb{F}_q}(Y-u) = Y^q - Y$. By logarithmic differentiation

$$\sum_{u\in\mathbb{F}_q} \frac{1}{Y+u} = \frac{-1}{Y^q - Y}.$$

If $1 \leq l \leq q-1$, then

$$\sum_{u\in\mathbb{F}_q} \frac{Y^l}{Y-u} - \sum_{u\in\mathbb{F}_q} \frac{u^l}{Y-u} = \sum_{u\in\mathbb{F}_q} \sum_{j=0}^{l-1} Y^j u^{l-1-j} = \sum_{j=0}^{l-1} Y^j \sum_{u\in\mathbb{F}_q} u^{l-1-j} = 0.$$

Hence, for $1 \leq l \leq q-1$,

$$\sum_{u\in\mathbb{F}_q} \frac{u^l}{Y+u} = (-1)^l \sum_{u\in\mathbb{F}_q} \frac{u^l}{Y-u} = (-1)^l \sum_{u\in\mathbb{F}_q} \frac{Y^l}{Y+u} = \frac{(-1)^{l+1}Y^l}{Y^q - Y}.$$

Now the result is a simple computation:

$$\sum_{u\in\mathbb{F}_q} \frac{X+u}{Y+u} = \sum_{u\in\mathbb{F}_q} \frac{X}{Y+u} + \sum_{u\in\mathbb{F}_q} \frac{u}{Y+u} = -\frac{X}{Y^q - Y} + \frac{Y}{Y^q - Y} = \frac{Y^q - X}{Y^q - Y} - 1. \ \square$$

Next, we recall Lucas' Theorem [10] which states that for prime p, positive e, and $0 \le n_0, n_1, \dots, n_m, i_0, i_1, \dots, i_m \le p^e - 1$ we have

$$\binom{n_0 + n_1 p^e + \cdots + n_m p^{em}}{i_0 + i_1 p^e + \cdots + i_m p^{em}} \equiv \binom{n_0}{i_0}\binom{n_1}{i_1}\cdots\binom{n_m}{i_m} \pmod p.$$

We will use Lucas' Theorem in the proof of the next lemma.

Lemma 3.2. *For* $1 \le l \le q$,

$$1 + \sum_{u \in \mathbb{F}_q} \left(\frac{X+u}{Y+u}\right)^l = \left(1 + \sum_{u \in \mathbb{F}_q} \left(\frac{X+u}{Y+u}\right)\right)^l.$$

Proof. According to Lemma 3.1, we have

$$\left(1 + \sum_{u \in \mathbb{F}_q} \left(\frac{X+u}{Y+u}\right)\right)^l = \left(\frac{Y^q - X}{Y^q - Y}\right)^l.$$

Consider

$$P(X) := 1 + \sum_{u \in \mathbb{F}_q} \left(\frac{X+u}{Y+u}\right)^l$$

as a polynomial in X. If $X = Y^q$, then

$$\begin{aligned} P(Y^q) &= 1 + \sum_{u \in \mathbb{F}_q} \left(\frac{Y^q+u}{Y+u}\right)^l = 1 + \sum_{u \in \mathbb{F}_q} (Y+u)^{l(q-1)} \\ &= 1 + \sum_{u \in \mathbb{F}_q} \sum_{i=0}^{l(q-1)} \binom{l(q-1)}{i} Y^{l(q-1)-i} u^i = 1 - 1 - \sum_{i=1}^{l-1} \binom{l(q-1)}{i(q-1)} Y^{(l-i)(q-1)} \\ &= 1 - 1 = 0. \end{aligned}$$

We have used that for $1 \le i \le l-1$, $\binom{l(q-1)}{i(q-1)} \equiv 0$ in $\mathbb{F}_q$, which follows by Lucas' Theorem since in $\mathbb{F}_q$ we have

$$\binom{l(q-1)}{i(q-1)} = \binom{(l-1)q + (q-l)}{(i-1)q + (q-i)} = \binom{l-1}{i-1}\binom{q-l}{q-i} = 0$$

as $q - l < q - i$. Thus $X = Y^q$ is a root of $P(X)$.

Next, we show that this root repeats l times. To that end, we compute for any $1 \le j \le l-1$

$$\begin{aligned} \frac{d^j P}{dX^j}(Y^q) &= l(l-1)\cdots(l-j+1) \sum_{u \in \mathbb{F}_q} (Y+u)^{(l-j)q-l} \\ &= l(l-1)\cdots(l-j+1) \sum_{u \in \mathbb{F}_q} \sum_{i=0}^{(l-j)q-l} \binom{(l-j)q-l}{i} Y^{(l-j)q-l-i} u^i \\ &= -l(l-1)\cdots(l-j+1) \sum_{i=1}^{l-\lceil \frac{jq}{q-1} \rceil} \binom{(l-j)q-l}{i(q-1)} Y^{(l-j)q-l-i(q-1)} = 0, \end{aligned}$$

since all the binomial coefficients vanish by Lucas' Theorem.

This shows that $P(X)$ and $\left(\frac{Y^q-X}{Y^q-Y}\right)^l$ are equal up to a constant. Substituting $X=Y$ shows that the constant is 1 and finishes the proof. □

Lemma 3.3. *Let F be a field that contains $\mathbb{F}_q$. Let $V_1,\dots,V_n$ be arbitrary elements of F and $W_1,\dots,W_n$ be a set of $\mathbb{F}_q$-linearly independent elements of F. For $1\le l\le q$,*

$$\sideset{}{'}\sum_{u_1,\dots,u_n\in\mathbb{F}_q}\left(\frac{u_1V_1+\cdots+u_nV_n}{u_1W_1+\cdots+u_nW_n}\right)^l=(-1)^{l+1}\left(\sideset{}{'}\sum_{u_1,\dots,u_n\in\mathbb{F}_q}\frac{u_1V_1+\cdots+u_nV_n}{u_1W_1+\cdots+u_nW_n}\right)^l.$$

Proof. We will use induction on n. If $n=1$, then we have

$$\sideset{}{'}\sum_{u\in\mathbb{F}_q}\left(\frac{uV_1}{uW_1}\right)^l=\left(\frac{V_1}{W_1}\right)^l\sideset{}{'}\sum_{u\in\mathbb{F}_q}1=-\left(\frac{V_1}{W_1}\right)^l,$$

while

$$(-1)^{l+1}\left(\sideset{}{'}\sum_{u\in\mathbb{F}_q}\frac{uV_1}{uW_1}\right)^l=(-1)^{l+1}\left(\frac{V_1}{W_1}\right)^l\left(\sideset{}{'}\sum_{u\in\mathbb{F}_q}1\right)^l=-\left(\frac{V_1}{W_1}\right)^l.$$

Assume that the result holds for $n-1$. Until the end of the proof, let $u_1,\dots,u_n$ be elements of $\mathbb{F}_q$.

First, assume that $V_n\neq 0$. Set $V_i':=\frac{V_i}{V_n}, W_i':=\frac{W_i}{W_n}$. We have

$$\begin{aligned}
&\sideset{}{'}\sum_{u_1,\dots,u_n}\left(\frac{u_1V_1+\cdots+u_nV_n}{u_1W_1+\cdots+u_nW_n}\right)^l\\
&=\sideset{}{'}\sum_{u_1,\dots,u_{n-1}}\sum_{u_n}\left(\frac{u_1V_1+\cdots+u_nV_n}{u_1W_1+\cdots+u_nW_n}\right)^l+\sideset{}{'}\sum_{u_n}\left(\frac{u_nV_n}{u_nW_n}\right)^l\\
&=\left(\frac{V_n}{W_n}\right)^l\left(-1+\sideset{}{'}\sum_{u_1,\dots,u_{n-1}}\sum_{u_n}\left(\frac{u_1V_1'+\cdots+u_n}{u_1W_1'+\cdots+u_n}\right)^l\right)\\
&=\left(\frac{V_n}{W_n}\right)^l\sideset{}{'}\sum_{u_1,\dots,u_{n-1}}\left(\sum_{u_n}\left(\frac{u_1V_1'+\cdots+u_n}{u_1W_1'+\cdots+u_n}\right)^l+1\right)\\
&=\left(\frac{V_n}{W_n}\right)^l\sideset{}{'}\sum_{u_1,\dots,u_{n-1}}\left(\sum_{u_n}\left(\frac{u_1V_1'+\cdots+u_n}{u_1W_1'+\cdots+u_n}\right)+1\right)^l && \text{by Lemma 3.2}\\
&=\left(\frac{V_n}{W_n}\right)^l\sideset{}{'}\sum_{u_1,\dots,u_{n-1}}\left(\frac{u_1V_1''+\cdots+u_{n-1}V_{n-1}''}{u_1W_1''+\cdots+u_{n-1}W_{n-1}''}\right)^l && \text{by Lemma 3.1,}
\end{aligned}$$

where $V_i''=(W_i')^q-V_i', W_i''=(W_i')^q-W_i'$. Note that the linear independence over $\mathbb{F}_q$ of W_i'' for $1\le i\le n-1$ is equivalent to the linear independence over $\mathbb{F}_q$ of W_j for $1\le j\le n$. By the induction hypothesis the last expression is equal to

$$(-1)^{l+1}\left(\frac{V_n}{W_n}\right)^l\left(\sideset{}{'}\sum_{u_1,\dots,u_{n-1}}\frac{u_1V_1''+\cdots+u_{n-1}V_{n-1}''}{u_1W_1''+\cdots+u_{n-1}W_{n-1}''}\right)^l.$$

On the other hand,

$$\left(\sum_{u_1,\dots,u_n}' \frac{u_1V_1+\cdots+u_nV_n}{u_1W_1+\cdots+u_nW_n}\right)^l = \left(\frac{V_n}{W_n}\right)^l\left(\sum_{u_1,\dots,u_n}' \frac{u_1V_1'+\cdots+u_n}{u_1W_1'+\cdots+u_n}\right)^l$$
$$= \left(\frac{V_n}{W_n}\right)^l\left(\sum_{u_1,\dots,u_{n-1}}' \frac{u_1V_1''+\cdots+u_{n-1}V_{n-1}''}{u_1W_1''+\cdots+u_{n-1}W_{n-1}''}\right)^l,$$

where the last equality follows from Lemma 3.1.

Finally, assume that $V_n=0$. We compute

$$\sum_{u_1,\dots,u_n}' \left(\frac{u_1V_1+\cdots+u_nV_n}{u_1W_1+\cdots+u_nW_n}\right)^l$$
$$= \sum_{u_1,\dots,u_{n-1}}' \sum_{u_n} \left(\frac{u_1V_1+\cdots+u_{n-1}V_{n-1}}{u_1W_1+\cdots+u_{n-1}W_{n-1}+u_nW_n}\right)^l$$
$$= W_n^{-l} \sum_{u_1,\dots,u_{n-1}}' \sum_{u_n} \left(\frac{u_1V_1+\cdots+u_{n-1}V_{n-1}}{u_1W_1'+\cdots+u_{n-1}W_{n-1}'+u_n}\right)^l$$
$$= W_n^{-l} \sum_{u_1,\dots,u_{n-1}}' \left(\sum_{u_n} \frac{u_1V_1+\cdots+u_{n-1}V_{n-1}}{u_1W_1'+\cdots+u_{n-1}W_{n-1}'+u_n}\right)^l$$
$$= (-1)^l W_n^{-l} \sum_{u_1,\dots,u_{n-1}}' \left(\frac{u_1V_1+\cdots+u_{n-1}V_{n-1}}{u_1W_1''+\cdots+u_{n-1}W_{n-1}''}\right)^l.$$

The second-to-last equality is the equality of Goss polynomials $\sum_{u\in\mathbb{F}_q}\left(\frac{1}{Y+u}\right)^l = \left(\sum_{u\in\mathbb{F}_q}\frac{1}{Y+u}\right)^l$. On the other hand,

$$(-1)^{l+1}\left(\sum_{u_1,\dots,u_n}' \frac{u_1V_1+\cdots+u_{n-1}V_{n-1}}{u_1W_1+\cdots+u_{n-1}W_{n-1}+u_nW_n}\right)^l$$
$$= (-1)^{l+1}W_n^{-l}\left(\sum_{u_1,\dots,u_{n-1}}' \sum_{u_n} \frac{u_1V_1+\cdots+u_{n-1}V_{n-1}}{u_1W_1'+\cdots+u_{n-1}W_{n-1}'+u_n}\right)^l$$
$$= (-1)^{2l+1}W_n^{-l}\left(\sum_{u_1,\dots,u_{n-1}}' \frac{u_1V_1+\cdots+u_{n-1}V_{n-1}}{u_1W_1''+\cdots+u_{n-1}W_{n-1}''}\right)^l.$$

Thus the result for n follows from the result for $n-1$, completing the proof. □

Lemma 3.3 gives a new proof of the following relations between some values of Pellarin L-functions, which are special cases of Theorem 1.3 in [15].

Corollary 3.4. *Let $1\le l\le q$. We have*

$$L(\chi_t^l, l) = L(\chi_t, 1)^l.$$

Proof. Let $A(n) = \{a = \sum_{i=0}^{n-1} a_i\theta^i \colon a_i \in \mathbb{F}_q\}$. Applying Lemma 3.3 with $W_i = \theta^{i-1}$ and $V_i = t^{i-1}$ we see that

$$\sum_{a\in A(n)}{}' \frac{\chi_t(a)^l}{a^l} = \left(\sum_{a\in A(n)}{}' \frac{\chi_t(a)}{a}\right)^l.$$

But we have

$$-L(\chi_t^l, l) = \sum_{a\in A}{}' \frac{\chi_t(a)^l}{a^l} = \lim_{n\to\infty} \sum_{a\in A(n)}{}' \frac{\chi_t(a)^l}{a^l},$$

and a short calculation gives the result. □

We are ready to complete the proof of Theorem 1.1:

Proof. As in the proof of Corollary 3.4, let $A(n) = \{a \in A : \deg(a) < n\}$. Applying Lemma 3.3 with

$$V_i = \begin{cases} d_1 t^{i-1} & \text{if } 1 \le i \le n, \\ d_2 t^{i-n-1} & \text{if } n+1 \le i \le 2n \end{cases}$$

and

$$W_i = \begin{cases} z\theta^{i-1} & \text{if } 1 \le i \le n, \\ \theta^{i-n-1} & \text{if } n+1 \le i \le 2n, \end{cases}$$

we have

$$\sum_{c,d\in A(n)}{}' \left(\frac{\chi_t(c)d_1 + \chi_t(d)d_2}{(cz+d)^{q^k}}\right)^l = (-1)^{l+1} \left(\sum_{c,d\in A(n)}{}' \frac{\chi_t(c)d_1 + \chi_t(d)d_2}{(cz+d)^{q^k}}\right)^l. \tag{3.1}$$

But

$$\mathcal{G}_{l,k} = \frac{1}{\tau^k L(\chi_t^l, l)} \lim_{n\to\infty} \sum_{c,d\in A(n)}{}' \left(\frac{\chi_t(c)d_1 + \chi_t(d)d_2}{(cz+d)^{q^k}}\right)^l.$$

By Corollary 3.4 we have $L(\chi_t^l, l) = L(\chi_t, 1)^l$; combining this with (3.1), we see that

$$\mathcal{G}_{l,k} = (-1)^{l+1}\mathcal{G}_{1,k}^l.$$

□

4. Consequences of Theorem 1.1

Theorem 4.1. *Let $1 \le l \le q$, $0 \le j \le l$. For $(z,t) \in \Omega \times B_q$ we have*

$$\sum_{c\in A_+}\sum_{d\in A} \frac{\chi_t(c)^{l-j}\chi_t(d)^j}{(cz+d)^l} = \left(\sum_{c\in A_+}\sum_{d\in A} \frac{\chi_t(c)}{cz+d}\right)^{l-j} \left(\sum_{c\in A_+}\sum_{d\in A} \frac{\chi_t(d)}{cz+d}\right)^j.$$

Proof. By Theorem 1.1, if $1 \le l \le q$, then

$$\mathcal{G}_{l,k} = (-1)^{l+1}\mathcal{G}_{1,k}^l, \qquad \forall k \in \mathbb{Z}.$$

Since $\mathcal{G}_{l,k} = (\tau^k\mathcal{E}) \cdot \Phi_l$ for a unique vector $\mathcal{E}$ (see [11, Proposition 10]), it follows that

$$\sum_{c,d\in A}{}' \frac{\chi_t(c)^{l-j}\chi_t(d)^j}{(cz+d)^l} = (-1)^{l+1} \left(\sum_{c,d\in A}{}' \frac{\chi_t(c)}{cz+d}\right)^{l-j} \left(\sum_{c,d\in A}{}' \frac{\chi_t(d)}{cz+d}\right)^j.$$

The proof is complete by observing that for $0 \leq j \leq l$,

$$\sum_{c,d\in A}{}' \frac{\chi_t(c)^{l-j}\chi_t(d)^j}{(cz+d)^l} = -\sum_{c\in A_+}\sum_{d\in A} \frac{\chi_t(c)^{l-j}\chi_t(d)^j}{(cz+d)^l}.$$

□

Remark 4.2. Since $\mathcal{G}_{l,k}$ is completely determined by $\mathcal{E}_l$ one sees that Theorem 4.1 is in fact equivalent to Theorem 1.1.

The result of Theorem 4.1 when $j = 0$ actually holds for $|t| < q^q$ as long as z is in the neighborhood of 'infinity', $\Omega_1 = \{z \in \Omega : |z|_i > 1\}$.

Corollary 4.3. *Let $1 \leq l \leq q$. For $(z,t) \in \Omega_1 \times B_{q^q}$, we have*

$$\sum_{c\in A_+}\sum_{d\in A} \frac{\chi_t(c)^l}{(cz+d)^l} = \left(\sum_{c\in A_+}\sum_{d\in A} \frac{\chi_t(c)}{(cz+d)}\right)^l.$$

Proof. According to [4, (5.5)] and (2.1), for $z \in \Omega_1$,

$$|\tilde{\pi}u(cz)| = \left|\sum_{d\in A} \frac{1}{cz+d}\right| \leq q^{-|c|}.$$

If, in addition, $|t| < q^q$, then $|\chi_t(c)| < q^{|c|}$ and therefore the series

$$\tilde{\pi}\sum_{c\in A_+} \chi_t(c)u(cz) = \sum_{c\in A_+}\sum_{d\in A} \frac{\chi_t(c)}{(cz+d)}$$

converges. The same estimates and properties of Goss polynomials show that for $(z,t) \in \Omega_1 \times B_{q^q}$ the series

$$\tilde{\pi}^l \sum_{c\in A_+} \chi_t(c)^l u(cz)^l = \sum_{c\in A_+}\sum_{d\in A} \frac{\chi_t(c)^l}{(cz+d)^l}$$

converges. By Theorem 4.1 we know that for $|t| < q$, $1 \leq l \leq q$,

$$\sum_{c\in A_+}\sum_{d\in A} \frac{\chi_t(c)^l}{(cz+d)^l} = \left(\sum_{c\in A_+}\sum_{d\in A} \frac{\chi_t(c)}{cz+d}\right)^l,$$

and so by analytic continuation this equality extends to $|t| < q^q$ provided that $z \in \Omega_1$. □

Recall that we have defined $\mathbb{E}$ as $hd_2^{(1)}$. Corollary 5 from [11] shows that for $|t| < q^q$ we have the following series expansion:

$$\mathbb{E} = \sum_{c\in A_+} \chi_t(c)u_c,$$

where $u_c : \Omega \to \mathbb{C}_\infty$ is the function $u_c := e_{\tilde{\pi}A}(\tilde{\pi}cz)^{-1}$.

As a special case of Theorem 4.1 we obtain the following generalization, which was first conjectured in [16, Remark 3.7].

Theorem 4.4. *Let $1 \leq l \leq q$. For $(z,t) \in \Omega_1 \times B_{q^q}$, we have*

$$\mathbb{E}^l = \sum_{c\in A_+} \chi_t(c)^l u_c^l.$$

Proof. This follows immediately from Corollary 4.3. □

Remark 4.5. The range $1 \leq l \leq q$ is natural because of properties of Goss polynomials, since in this range the lth Goss polynomial is just X^l (see [4, (3.4)]). It is not difficult to find counterexamples to possible extensions of Theorem 4.4 if we go beyond $l = q$.

Remark 4.6. Theorem 4.4 also provides examples of deformations of Drinfeld modular forms with A-expansions. That is, expansions of the form

$$\sum_{c\in A_+} a_c(t)G_n(u_c),$$

where $a_c(t) \in A[t] = \mathbb{F}_q[\theta, t]$, and G_n is the nth Goss polynomial of the lattice $\tilde{\pi}A$ as defined in [6, Proposition 2.17]. It is natural to wonder if there are more examples of these, just as in the case of Drinfeld modular forms (see [16, Theorem 1.3]).

Computations with SAGE [17] suggest that this is indeed the case. For example, our computations suggest that for

$$f_s = \sum_{c\in A_+} c^{1+s(q-1)}u_c \in M_{2+s(q-1),1},$$

we have

$$\mathbf{f}_s := f_s d_2 = \sum_{c\in A_+} \chi_t(c)c^{s(q-1)}u_c,$$

for $s = 1, \ldots, q, q+2, q+3, \ldots, q^2$. We hope to return to this topic in future work.

Next, we turn to applications of Theorem 4.4 to Drinfeld modular forms with A-expansions (see [9], [16]). We assume throughout that $1 \leq l \leq q$. For $\nu \in \mathbb{N}$, define

$$f_{l,\nu} := \sum_{c\in A_+} c^{lq^\nu}u_c^l \in M_{lq^\nu+l,l}.$$

We will use Theorem 4.4 to give a recursive formula for $f_{l,\nu}$.

Theorem 4.7. *We have $f_{l,1} = h^l$, $f_{l,2} = h^l g^{lq}$ and the recursive formula for $\nu \geq 2$,*

$$f_{l,\nu} = \left(\frac{g^q}{h^{q-1}} f_{2,\nu-1}^q - \frac{[\nu-2]^{q^2}}{h^{q-1}} f_{2,\nu-2}^{q^2}\right)^l.$$

Proof. Let $\texttt{Frob}$ be the qth power Frobenius map acting on $\mathbb{C}_\infty((t))$. Then

$$\left((\texttt{Frob}\circ\tau^{-1})^\nu\mathbb{E}^l\right)_{|t=\theta} = f_{l,\nu}.$$

Pellarin has shown ([13, Proposition 9]) that

$$\mathbb{E} = \frac{g^q}{h^{q-1}}\tau\mathbb{E} - \frac{(u-\theta^{q^2})}{h^{q-1}}\tau^2\mathbb{E}.$$

Therefore

$$\mathbb{E}^l = \left(\frac{g^q}{h^{q-1}}\tau\mathbb{E} - \frac{(t-\theta^{q^2})}{h^{q-1}}\tau^2\mathbb{E}\right)^l.$$

Applying $(\texttt{Frob}\circ\tau^{-1})^\nu$ to both sides and plugging in $t = \theta$ finishes the proof. □

Corollary 4.8. *For $\nu \in \mathbb{N}$, we have the eigenproduct identity of Drinfeld modular forms $f_{1,\nu}^l = f_{l,\nu}$.*

Proof. Indeed, both $f_{1,\nu}$ and $f_{l,\nu}$ are eigenforms according to [16, Theorem 2.3]. □

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