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GLOBAL SOLUTIONS FOR A NEW CLASS OF ABSTRACT NEUTRAL DIFFERENTIAL EQUATIONS

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ABSTRACT. We study the existence of global solutions for a class of abstract neutral differential equations recently introduced in the literature. An application involving a partial neutral differential equation is presented.

1. INTRODUCTION

In this paper we continue our developments in [13] on the existence and qualitative properties of solutions of a class of abstract neutral equations of the form

$$u'(t) = Au(t) + f(t, u_t, u'_t), \qquad t \in [0, \infty), \tag{1.1}$$

$$u_0 = \varphi \in \mathcal{B}, \tag{1.2}$$

where $A : D(A) \subset X \to X$ is the infinitesimal generator of an analytic semigroup of bounded linear operator $(T(t))_{t\geq 0}$ defined on a Banach space $(X, \|\cdot\|)$, the history u_t belongs to an abstract Banach space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ defined axiomatically, the symbol u'_t denotes the derivative at t of the function $s \to u_s$ and $f(\cdot)$ is a suitable function.

Motivated by mathematical aspects and different applications, in [13] we introduced the abstract neutral system (1.1)-(1.2) and studied the existence of local in time solutions. In this paper we consider the problem of the existence of global strict solutions (solutions defined on $[0, \infty)$). In particular, we establish some results on the existence of asymptotically almost periodic solutions.

As pointed out in [13], the study of the neutral problem (1.1)-(1.2) via semigroup methods and the fixed point technique is highly nontrivial. The reason is simple and evident: the temporal derivative of the solution appears in the integral equation used to define the concept of a mild solution of (1.1)-(1.2); see Definition 2.1. As a consequence, we need to work on spaces of continuously differentiable functions, which is a complex problem under the semigroup framework.

To the best of our knowledge, the paper [13] is the first and only work treating neutral problems described in the abstract form (1.1)-(1.2). The general literature on abstract differential equations of neutral type is extensive and varied. For relevant developments and the current state of the theory of abstract neutral equations, we refer the reader to [1, 3, 5–12, 18] and the references therein.

The abstract problem (1.1)-(1.2) arises, for example, in the theory of heat conduction in fading memory material. In the classical theory of heat conduction, it

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is assumed that the internal energy and the heat flux depend linearly on the temperature $u(\cdot)$ and on its gradient $\nabla u(\cdot)$. Under these conditions, the classical heat equation describes sufficiently well the evolution of the temperature in different types of materials. However, this description is not satisfactory in materials with fading memory. In the theory developed in [4,17], the internal energy and the heat flux are described as functionals of u and u_x. The next system (see [2,16]) has been frequently used to describe this phenomena:

$$\begin{aligned} \frac{d}{dt}\left[u(t,x)+\int_{-\infty}^{t} k_1(t-s)u(s,x)ds\right] &= c\triangle u(t,x)+\int_{-\infty}^{t} k_2(t-s)\triangle u(s,x)ds, \\ u(t,y) &= 0. \end{aligned}$$

In this system, $\Omega \subset \mathbb{R}^n$ is open, bounded and has smooth boundary, $(t,x) \in [0,\infty)\times\Omega$, $y \in \partial\Omega$, $u(t,x)$ represents the temperature in x at the time t, c is a physical constant and $k_i : \mathbb{R} \to \mathbb{R}$, $i = 1,2$, are the internal energy and the heat flux relaxation respectively. If $u(\cdot)$ is a smooth enough solution of this problem, then we obtain the integro-differential system

$$\begin{aligned} u'(t,x) &= c\triangle u(t,x)+\int_{-\infty}^{t} k_2(t-s)\triangle u(s,x)ds-\int_{-\infty}^{t} k_1(t-s)u'(s,x)ds, \\ u(t,y) &= 0, \qquad y\in\partial\Omega, \end{aligned}$$

for $(t,x) \in [0,\infty)\times\Omega$. Finally, if we assume $u(\cdot)$ is known on $(-\infty,0]$, then we can represent this system in the abstract form (1.1)-(1.2).

This paper has three sections. By using the ideas in [13], in the next section we discuss the existence of strict solutions for the problem (1.1)-(1.2). In particular, in Theorem 2.2 and Theorem 2.4 we establish the existence of asymptotically almost periodic solutions. In the last section, an application involving a partial neutral differential equation is presented.

We include now some notation and technicalities. Let $(Z,\|\cdot\|_Z)$ and $(W,\|\cdot\|_W)$ be Banach spaces. In this paper, $\mathcal{L}(Z,W)$ represents the space of bounded linear operators from Z into W endowed with the norm of operators denoted $\|\cdot\|_{\mathcal{L}(Z,W)}$, and we write $\mathcal{L}(Z)$ and $\|\cdot\|_{\mathcal{L}(Z)}$ when $Z = W$. We use the notation $Z \hookrightarrow W$ to indicate that Z is continuously included in W.

Let $I \subset \mathbb{R}$. Next, $C_b(I;Z)$ is the space formed by all the bounded continuous functions from I into Z endowed with the sup-norm denoted by $\|\cdot\|_{C_b(I;Z)}$, and $C_b^1(I;Z)$ is the space formed for all the functions $u \in C_b(I;Z)$ for which $u' \in C_b(I;Z)$ endowed with the norm $\| u \|_{C_b^1(I;Z)} = \| u \|_{C_b(I;Z)} + \| u' \|_{C_b(I;Z)}$. In addition, $C_b^{\mu}([0,\infty);Z)$ (with $\mu \in (0,1)$) is the space formed by all the functions $\xi \in C_b([0,\infty);Z)$ such that $[\,\xi\,]_{C_b^{\mu}([0,\infty);Z)} = \sup_{t,s\in[0,\infty),t\neq s}\frac{\|\xi(s)-\xi(t)\|_Z}{|t-s|^{\mu}}$ is finite, provided with the norm $\| \xi \|_{C_b^{\mu}([0,\infty);Z)} = \| \xi \|_{C_b([0,\infty);Z)} + [\,\xi\,]_{C_b^{\mu}([0,\infty);Z)}$. The notation $C_b^{1+\mu}([0,\infty);Z)$ is used for the space of all the functions $\xi \in C_b^{\mu}([0,\infty);Z)$ such that $\xi' \in C_b^{\mu}([0,\infty);Z)$ endowed with the norm $\| \xi \|_{C_b^{1+\mu}([0,\infty);Z)} = \| \xi \|_{C_b^{\mu}([0,\infty);Z)} + \| \xi' \|_{C_b^{\mu}([0,\infty);Z)}$.

In this paper, $A : D(A) \subset X \to X$ is the generator of a uniformly stable analytic semigroup of bounded linear operators $(T(t))_{t\geq 0}$ on X and γ, C_i, $i \in \mathbb{N}$, are positive constants such that $\| A^iT(t) \|_{\mathcal{L}(X)} \leq \frac{C_i e^{-\gamma t}}{t^i}$ for all $t > 0$ and each $i \in \mathbb{N}$. For $\beta > 0$, we represent by X_β the domain of the fractional power $(-A)^\beta$ of $-A$ endowed with the norm $\| x \|_\beta = \| (-A)^\beta x \|$, and we assume that C_β is a positive constant such that $\| AT(t) \|_{\mathcal{L}(X_\beta,X)} \leq \frac{C_\beta e^{-\gamma t}}{t^\beta}$ for all $t > 0$. The notation $D_A(\eta,\infty)$, $\eta \in (0,1)$, stands for the space

$$D_A(\eta,\infty) = \{x \in X : [x]_{\eta,\infty} = \sup_{t\in(0,1)} \| t^{1-\eta} AT(t)x \| < \infty\},$$

with the norm $\| x \|_{\eta,\infty} = [x]_{\eta,\infty} + \| x \|$, and for $k \in \mathbb{N} \cup \{0\}$ we assume that there is a constant $C_{k,\eta} > 0$ such that $\| A^k T(t) \|_{\mathcal{L}(D_A(\eta,\infty),X)} \leq \frac{C_{k,\eta} e^{-\gamma t}}{t^{1-\eta}}$ for all $t > 0$. For additional details on analytic semigroups and interpolation spaces we cite [15].

In this work, $(\mathcal{B}, \| \cdot \|_{\mathcal{B}})$ is a Banach space formed by functions defined from a connected interval $\{0\} \subset J \subset (-\infty, 0]$ into X, satisfying the following conditions.

(**A**) If $x : (J + \{\sigma\}) \cup [\sigma, \sigma + b) \to X$, $b > 0, \sigma \in \mathbb{R}$, is continuous on $[\sigma, \sigma + b)$ and $x_\sigma \in \mathcal{B}$, then for every $t \in [\sigma, \sigma + b)$ the following conditions hold:

- (i) the function $s \to x_s$ belongs to $C_b([\sigma, \sigma + b), \mathcal{B})$,
- (ii) $\| x(t) \| \leq H \| x_t \|_{\mathcal{B}}$,
- (iii) $\| x_t \|_{\mathcal{B}} \leq K(t - \sigma) \sup\{\| x(s) \| : \sigma \leq s \leq t\} + M(t - \sigma) \| x_\sigma \|_{\mathcal{B}}$, where $H > 0$ is a constant, and $K, M \in C_b([0,\infty); \mathbb{R}^+)$ and $H, K(\cdot), M(\cdot)$ are independent of $x(\cdot)$. Next, to simplify, we assume that $\mathcal{K} > 0$ is a constant such that $\max\{K(t), M(t)\} \leq \mathcal{K}$ for all $t \geq 0$.
- (iv) If $(\psi^n)_{n\in\mathbb{N}}$ is a sequence in $C_b(J, X) \cap \mathcal{B}$ and $\psi^n \to \psi$ uniformly on compact subsets of J, then $\psi \in \mathcal{B}$ and $\| \psi^n - \psi \|_{\mathcal{B}} \to 0$ as $n \to \infty$.

For $\beta > 0$, we represent by $\mathcal{B}_\beta$ the space $\mathcal{B}_\beta = \{(-A)^{-\beta}\psi : \psi \in \mathcal{B}\}$ endowed with the norm $\| \psi \|_{\mathcal{B}} = \| (-A)^\beta \psi \|_{\mathcal{B}}$. We note that $\mathcal{B}_\beta$ verifies axiom **A** with X_β in place of X.

We include now some comments concerning almost periodic functions. To begin, we note that a set $\mathcal{P} \subset \mathbb{R}$ is said to be relatively dense in $[0,\infty)$ if there is $l > 0$ such that $[a, a + l] \cap \mathcal{P} \neq \emptyset$ for all $a \in [0,\infty)$.

Definition 1.1. A function $g \in C_b(\mathbb{R}; Z)$ is said to be almost periodic if for all $\varepsilon > 0$ there exists a relatively dense subset of $\mathbb{R}$, denoted by $\mathcal{R}(\varepsilon, g)$, such that $\|g(t + \xi) - g(t)\|_Z < \varepsilon$ for each $t \in \mathbb{R}$ and all $\xi \in \mathcal{R}(\varepsilon, g)$.

Definition 1.2. A function $h \in C_b([0,\infty); Z)$ is called asymptotically almost periodic if there exist an almost periodic function $g(\cdot)$ and a function $w \in C_b([0,\infty); Z)$ with $\lim_{t\to\infty} w(t) = 0$ such that $h = g + w$.

Next, we use the notation $AAP(Z)$ for the space formed by all the Z-valued asymptotically almost periodic functions endowed with the norm $\| \cdot \|_{C_b([0,\infty);Z)}$. It is well known that $AAP(Z)$ is a Banach space. In this paper, we use the following characterization of asymptotically almost periodic functions.

Lemma 1.1 ([19, Theorem 5.5])**.** *A function $h \in C_b([0,\infty); Z)$ is asymptotically almost periodic if and only if for all $\varepsilon > 0$ there exist $L_\varepsilon > 0$ and a relatively dense subset of $[0,\infty)$, denoted by $\mathcal{H}(\varepsilon, h)$, such that $\|h(t + \xi) - h(t)\|_Z < \varepsilon$ for all $t \geq L_\varepsilon$ and each $\xi \in \mathcal{H}(\varepsilon, h)$.*

2. Existence of global solutions

In this section we study the existence of global solutions for the abstract neutral problem (1.1)-(1.2). In the remainder of this work we assume that there are $\beta \in (0,1]$ and functions $f_1 \in C_b([0,\infty); \mathcal{L}(\mathcal{B}_\beta, X))$, $f_2 \in C_b([0,\infty); \mathcal{L}(\mathcal{B}, X))$ such that $f(t, \psi_1, \psi_2) = f_1(t, \psi_1) + f_2(t, \psi_2)$ for all $t \geq 0$, $\psi_1 \in \mathcal{B}_\beta$ and $\psi_2 \in \mathcal{B}$. To prove our results, we consider the following conditions.

($\mathbf{H_1}$) There is $\alpha \in (0,1)$ such that the function f_1 belongs to $C_b^\alpha([0,\infty); \mathcal{L}(\mathcal{B}_\beta, X))$ and $f_2 \in C_b^\alpha([0,\infty); \mathcal{L}(\mathcal{B}, X))$.

($\mathbf{H_2}$) There is a Banach space $(Y, \| \cdot \|_Y) \hookrightarrow (X, \| \cdot \|)$ and functions $H \in L^1([0,\infty); \mathbb{R}^+)$, $L_f \in C_b([0,\infty); \mathbb{R}^+)$ such that $\| AT(s) \|_{\mathcal{L}(Y,X)} \leq H(s)$ for all $s > 0$, $f_1 \in C_b([0,\infty); \mathcal{L}(\mathcal{B}_\beta, Y))$, $f_2 \in C_b([0,\infty); \mathcal{L}(\mathcal{B}, Y))$ and

$$\| f(t,\psi_1,\zeta_1) - f(t,\psi_2,\zeta_2) \|_Y \leq L_f(t)(\| \psi_1 - \psi_2 \|_{\mathcal{B}_\beta} + \| \zeta_1 - \zeta_2 \|_{\mathcal{B}}),$$

for all $\psi_i \in \mathcal{B}_\beta$, $\zeta_i \in \mathcal{B}$, $i = 1, 2$, and every $t \geq 0$.

Next, $u \in C_b(J \cup [0,\infty); X_\beta)$, $v \in C_b(J \cup [0,\infty); X)$ with $u_0 \in \mathcal{B}_\beta$ and $v_0 \in \mathcal{B}$, we use the notation $\mathcal{P}_v$ and $f_{u,v}$ for the functions $\mathcal{P}_v : [0,\infty) \to \mathcal{B}$ and $f_{u,v} : [0,\infty) \to X$ given by $\mathcal{P}_v(t) = v_t$ and $f_{u,v}(t) = f(t, u_t, v_t)$. We introduce now the following concepts of the solution.

Definition 2.1. A function $u : J \cup [0,\infty) \to X$ is called a mild solution of the abstract problem (1.1)-(1.2) if $u_0 = \varphi$, $\mathcal{P}_u \in C_b^1([0,\infty); \mathcal{B})$ and

$$u(t) = T(t)\varphi(0) + \int_0^t T(t-s) f_{u,u'}(s) ds, \qquad \forall\, t \in [0,\infty).$$

Definition 2.2. A function $u : J \cup [0,\infty) \to X$ is said to be a strict solution of (1.1)-(1.2) if $\mathcal{P}_u \in C_b^1([0,\infty); \mathcal{B})$, $u \in C_b([0,\infty); X_1)$ and $u(\cdot)$ satisfies (1.1)-(1.2).

To continue, it is convenient to include some comments on the abstract system

$$u'(t) = Au(t) + \xi(t), \quad t \in I, \tag{2.1}$$
$$u(0) = x \in X, \tag{2.2}$$

where $I = [0,a]$ or $I = [0,\infty)$ and $\xi \in L^1(I; X)$. We note that the function $u : I \to X$ given by $u(t) = T(t)x + \int_0^t T(t-s)\xi(s) ds$ is called a mild solution of (2.1)-(2.2) on I and that a function $v \in C_b(I; X)$ is said to be a strict solution of (2.1)-(2.2) on I if $v \in C_b^1(I; X) \cap C_b(I; X_1)$ and $v(\cdot)$ satisfies (2.1)-(2.2).

The proof of Proposition 2.1 follows ideas in the proof of [15, Theorem 4.3.1]. The basic difference between Proposition 2.1 and [15, Theorem 4.3.1] is the fact that in our case, the constants $M_i, M_{\alpha,1}$ are independent of the number T in [15].

Proposition 2.1. *Assume* $\xi \in C_b^\alpha([0,\infty); X)$, $x \in X_1$ *and let* $u(\cdot)$ *be the mild solution of* (2.1)-(2.2). *If* $x \in X_1$ *and* $Ax + \xi(0) \in D_A(\alpha,\infty)$, *then* $u(\cdot)$ *is a strict solution of* (2.1)-(2.2) *on* $[0,\infty)$ *and*

$$u'(t) = AT(t)x + \int_0^t AT(t-s)(\xi(s) - \xi(t)) ds + T(t)\xi(t), \quad t \geq 0. \tag{2.3}$$

Moreover, there exists a constant $\Lambda > 0$ *(which is independent of* $\xi(\cdot)$*) such that*

$$\| u \|_{C_b^{1+\alpha}([0,\infty);X)} + \| Au \|_{C_b^\alpha([0,\infty);X)} \leq \Lambda(\| Ax \| + \| Ax + \xi(0) \|_{\alpha,\infty} + \| \xi \|_{C_b^\alpha([0,\infty);X)}). \tag{2.4}$$

Proof. From the proof of [15, Theorem 4.3.1] we know that $u_{|_{[0,T]}}$ is a strict solution of (2.1)-(2.2) on $[0,T]$ and the representation (2.3) is valid on $[0,T]$ for all $T > 0$. Thus, $u(\cdot)$ is a strict solution on $[0,\infty)$ and (2.3) is satisfied for all $t \geq 0$. Moreover, by noting that the constants $M_i, M_{1,\alpha}$ are independent of the interval $[0,T]$, from the proof of [15, Theorem 4.3.1] we obtain that

$$
\begin{aligned}
&[\| u \|]_{C_b^\alpha([0,\infty);X_1)} \\
&\leq \frac{M_{1,\alpha}}{\alpha} \| Ax + \xi(0) \|_{\alpha,\infty} + (\frac{2M_1}{\alpha} + 3M_0 + 1 + \frac{M_2}{\alpha(1-\alpha)})[\| \xi \|]_{C_b^\alpha([0,\infty);X)},
\end{aligned}
$$

and from (2.3) we see that

$$
\begin{aligned}
\| u \|_{C_b([0,\infty);X_1)} &\leq M_0 \| Ax \| + M_1(\frac{1}{\gamma} + \frac{1}{\alpha})[\| \xi \|]_{C_b^\alpha([0,\infty);X)} \\
&\quad + (M_0 + 1) \| \xi \|_{C_b([0,\infty);X)} .
\end{aligned}
$$

By noting that $u(\cdot)$ is a strict solution and A is invertible, we have that

$$
\begin{aligned}
\| u' \|_{C_b^\alpha([0,\infty);X)} &\leq \| u \|_{C_b^\alpha([0,\infty);X_1)} + \| \xi \|_{C_b^\alpha([0,\infty);X)}, \\
\| u \|_{C_b^\alpha([0,\infty);X)} &\leq \| (-A)^{-1} \| \| u \|_{C_b^\alpha([0,\infty);X_1)} .
\end{aligned}
$$

Collecting the above inequalities we obtain that the inequality (2.4) is satisfied with $\Lambda = (2 + \| (-A)^{-1} \|)(\frac{M_{1,\alpha}}{\alpha} + 4M_0 + M_1(\frac{3}{\alpha} + \frac{1}{\gamma}) + 3 + \frac{M_2}{\alpha(1-\alpha)})$. □

For completeness, from [13] we note the following lemmas.

Lemma 2.1. *If $u \in C_b^1(J \cup [0,b]; X)$, then $\mathcal{P}_u \in C_b^1([0,b]; \mathcal{B})$ and $\frac{d}{dt}\mathcal{P}_u(t) = \mathcal{P}_{\frac{du}{dt}}(t)$ for all $t \in [0,b]$.*

Lemma 2.2. *Assume the condition $\mathbf{H_2}$ is satisfied, $x \in X_1$ and $\xi \in C([0,b]; Y)$. Then, the mild solution $w(\cdot)$ of (2.1)-(2.2) with $x = 0$ is a strict solution and*

$$
w'(t) = T(t)Ax + \int_0^t AT(t-s)\xi(s)ds + \xi(t), \quad \forall t \in [0,b]. \tag{2.5}
$$

We can establish now our first result on the existence of a global solution. In the remainder of this paper, Λ is the constant in Proposition 2.1 and $y : J \cup [0,\infty) \to X$ is the function given by $y(t) = \varphi(t)$ for $t \in J$ and $y(t) = T(t)\varphi(0)$ for $t \geq 0$.

Theorem 2.1. *Assume that condition $\mathbf{H_1}$ is satisfied and $C_b(J; X) \hookrightarrow \mathcal{B}$. Suppose $\varphi \in C_b^{1+\alpha}(J; X) \cap C_b(J; X_\beta)$, $\varphi(0) \in X_1$, $\frac{d^-\varphi}{dt}(0) = A\varphi(0) + f(0,\varphi,\varphi')$, $\{A\varphi(0), f(0,\varphi,\varphi')\} \subset D_A(\alpha,\infty)$, $\mathcal{P}_y \in C_b^{1+\alpha}([0,\infty); \mathcal{B})$ and*

$$
\mathfrak{L} = \mathcal{K}(1 + \| (-A)^{\beta-1} \|)\Lambda(\| f_1 \|_{C_b^\alpha([0,\infty);\mathcal{L}(\mathcal{B}_\beta,X))} + \| f_2 \|_{C_b^\alpha([0,\infty);\mathcal{L}(\mathcal{B},X))}) < 1.
$$

Then there exists a unique strict solution $u \in C_b^{1+\alpha}(J \cup \mathbb{R}^+; X) \cap C_b(J \cup \mathbb{R}^+; X_\beta)$ of the problem (1.1)-(1.2).

Proof. On the space

$$
\begin{aligned}
\mathfrak{S} = \{u \in C_b^{1+\alpha}(J \cup \mathbb{R}^+; X) \cap C_b(J \cup \mathbb{R}^+; X_\beta) : \\
\mathcal{P}_u \in C_b^{1+\alpha}([0,\infty); \mathcal{B}) \cap C_b([0,\infty); \mathcal{B}_\beta)\}
\end{aligned} \tag{2.6}
$$

endowed with the metric $\Phi(u,v) = \| \mathcal{P}_u - \mathcal{P}_v \|_{C_b^{1+\alpha}([0,\infty);\mathcal{B})} + \| \mathcal{P}_u - \mathcal{P}_v \|_{C_b([0,\infty);\mathcal{B}_\beta)}$, we define the map $\Gamma : \mathfrak{S} \to \mathfrak{S}$ by $(\Gamma u)_0 = \varphi$ and

$$
\Gamma u(t) = T(t)\varphi(0) + \int_0^t T(t-s)f_{u,u'}(s)ds, \quad t \geq 0. \tag{2.7}
$$

Let $u \in \mathfrak{S}$. Arguing as in the proof of [13, Theorem 2.1], but using Proposition 2.1 instead of [13, Proposition 2.1], it follows that $\mathcal{P}_{\Gamma u} \in C_b^{1+\alpha}([0,\infty);\mathcal{B}) \cap C_b([0,\infty);\mathcal{B}_\beta)$, $\frac{d}{dt}\mathcal{P}_{\Gamma u} = \mathcal{P}_{\frac{d}{dt}\Gamma u}$ and $u \in C_b^{1+\alpha}([0,\infty);X) \cap C_b^{\alpha}([0,\infty);X_1)$. Thus, Γ is a well defined function from $\mathfrak{S}$ into $\mathfrak{S}$.

Let $u, v \in \mathfrak{S}$. From Proposition 2.1 we infer that

$$\begin{aligned} \| \Gamma u - \Gamma v \|_{C_b([0,\infty);X_\beta)} &\leq \| (-A)^{\beta-1} \|_{\mathcal{L}(X)} \| A\Gamma u - A\Gamma v \|_{C_b([0,\infty);X)} \\ &\leq \| (-A)^{\beta-1} \|_{\mathcal{L}(X)} \Lambda \| f_{u,u'} - f_{v,v'} \|_{C_b^{\alpha}([0,\infty);X)} . \end{aligned} \tag{2.8}$$

To estimate $\| \frac{d}{dt}\mathcal{P}_{\Gamma u} - \frac{d}{dt}\mathcal{P}_{\Gamma v} \|_{C_b^{1+\alpha}([0,\infty);\mathcal{B})}$, we note that the function $\Gamma u - \Gamma v$ is a mild solution of problem (2.1)-(2.2) with $x = 0$ and $\xi = f_{u,u'} - f_{v,v'}$. Then, from Proposition 2.1 and (2.8) we get

$$\begin{aligned} &\| \mathcal{P}_{\Gamma u} - \mathcal{P}_{\Gamma v} \|_{C_b^{1+\alpha}([0,\infty);\mathcal{B})} + \| \mathcal{P}_{\Gamma u} - \mathcal{P}_{\Gamma v} \|_{C_b([0,\infty);\mathcal{B}_\beta)} \\ &\qquad \leq \mathcal{K} \| \Gamma u - \Gamma v \|_{C_b^{1+\alpha}([0,\infty);\mathcal{B})} + \mathcal{K} \| \Gamma u - \Gamma v \|_{C_b([0,\infty);X_\beta)} \\ &\qquad \leq \mathcal{K}(1+ \| (-A)^{\beta-1} \|_{\mathcal{L}(X)})\Lambda \| f_{u,u'} - f_{v,v'} \|_{C_b^{\alpha}([0,\infty);X)}, \end{aligned}$$

from which we obtain that $\Phi(\Gamma u, \Gamma v) \leq \mathfrak{L}\Phi(u,v)$. Thus, Γ is a contraction and there exists a unique mild solution $u \in \mathfrak{S}$ of (1.1)-(1.2). Finally, from Proposition 2.1 we infer that $u(\cdot)$ is a strict solution of (1.1)-(1.2). The proof is complete. □

To establish our next theorem on the existence of an asymptotically almost periodic strict solution, it is convenient to consider some lemmas first. In the next results, $(Z, \|\cdot\|_Z)$ is a Banach space.

Lemma 2.3. *Assume $(Z, \|\cdot\|_Z) \hookrightarrow (X, \|\cdot\|)$ and $u \in C_b(J \cup [0,\infty); Z)$. If $u_{|_{[0,\infty)}} \in AAP(Z)$ and $M(t) \to \infty$ as $t \to \infty$, then $\mathcal{P}_u \in AAP(\mathcal{B})$.*

Proof. To prove the assertion we use Lemma 1.1. Let i_c be the inclusion map from Z into X. For $\varepsilon > 0$ given, there exist $L_\varepsilon > 0$ and a relatively dense subset $\mathcal{H}(\varepsilon, u)$ of $[0,\infty)$ such that $\| u(t+\xi) - u(t) \|_Z \leq \varepsilon$ and $M(t) \leq \varepsilon$ for all $\xi \in \mathcal{H}(\varepsilon, u)$ and each $t \geq L_\varepsilon$. For $t \geq 2L_\varepsilon$ and $\xi \in \mathcal{H}(\varepsilon, u)$ we get

$$\begin{aligned} &\| \mathcal{P}_u(t+\xi) - \mathcal{P}_u(t) \|_{\mathcal{B}} \\ &\quad \leq M(t - L_\varepsilon) \| u_{L_\varepsilon} \|_{\mathcal{B}} + \mathcal{K} \| i_c \|_{\mathcal{L}(Z,X)} \sup_{s \geq L_\varepsilon} \| u(s+\xi) - u(s) \|_Z \\ &\quad \leq \varepsilon \left(M(L_\varepsilon) \| u_0 \|_{\mathcal{B}} + \mathcal{K} \| i_c \|_{\mathcal{L}(Z,X)} \| u \|_{C_b([0,\infty);Z)} \right) + \mathcal{K} \| i_c \|_{\mathcal{L}(Z,X)} \varepsilon \\ &\quad \leq \varepsilon\mathcal{K} \left(\| u_0 \|_{\mathcal{B}} + \| i_c \|_{\mathcal{L}(Z,X)} \| u \|_{C_b([0,\infty);Z)} + \| i_c \|_{\mathcal{L}(Z,X)} \right), \end{aligned}$$

which implies that $\mathcal{P}_u \in AAP(\mathcal{B})$. □

Lemma 2.4. *Assume $\mathcal{Q} \in C_b([0,\infty); \mathcal{L}(Z,X)) \cap L^1([0,\infty); \mathcal{L}(Z,X))$, $v \in AAP(Z)$ and let $u : [0,\infty) \to X$ be given by $u(t) = \int_0^t \mathcal{Q}(t-s)v(s)ds$. Then $u \in AAP(X)$.*

Proof. From the estimate

$$\|u(t)\| \leq \int_0^t \|\mathcal{Q}(t-s) \|_{\mathcal{L}(Z,X)} \| v(s)\|_Z ds \leq \| \mathcal{Q} \|_{L^1([0,\infty),\mathcal{L}(Z,X))} \|v\|_{C_b([0,\infty);Z)},$$

it is easy to infer that $v \in C_b([0,\infty); X)$.

Let $\varepsilon > 0$ be given. Since $v \in AAP(Z)$, there is a relatively dense subset $\mathcal{H}(\varepsilon, v)$ of $[0,\infty)$ and $L_\varepsilon > 0$ such that $\| v(s+\xi) - v(s) \|_Z \leq \varepsilon$ and $\| \mathcal{Q} \|_{L^1([L_\varepsilon,\infty),\mathcal{L}(Z,X))} \leq \varepsilon$ for all $s \geq L_\varepsilon$ and each $\xi \in \mathcal{H}(\varepsilon, v)$. Then, for $t \geq 2L_\varepsilon$ and $\xi \in \mathcal{H}(\varepsilon, v)$ we have that

$$\begin{aligned}
&\|u(t+\xi) - u(t)\| \\
\leq\ & \int_0^{\xi} \|\mathcal{Q}(t+\xi-s)\|_{\mathcal{L}(Z,X)} \| v(s)\|_Z ds \\
& + \int_0^{L_\varepsilon} \|\mathcal{Q}(t-s)\|_{\mathcal{L}(Z,X)} \| v(s+\xi) - v(s)\|_Z ds \\
& + \int_{L_\varepsilon}^{t} \|\mathcal{Q}(t-s)\|_{\mathcal{L}(Z,X)} \| v(s+\xi) - v(s)\|_Z ds \\
\leq\ & \|v\|_{C_b([0,\infty);Z)} \int_t^{t+\xi} \| \mathcal{Q}(s) \|_{\mathcal{L}(Z,X)}\, ds \\
& +2\|v\|_{C_b([0,\infty);Z)} \int_{t-L_\varepsilon}^{t} \| \mathcal{Q}(s) \|_{\mathcal{L}(Z,X)}\, ds + \varepsilon \int_0^{t-L_\varepsilon} \| \mathcal{Q}(s) \|_{\mathcal{L}(Z,X)}\, ds \\
\leq\ & 3\|v\|_{C_b([0,\infty);Z)} \| \mathcal{Q} \|_{L^1([L_\varepsilon,\infty),\mathcal{L}(Z,X))} + \varepsilon \| \mathcal{Q} \|_{L^1([0,\infty);\mathcal{L}(Z,X))} \\
\leq\ & \varepsilon(3\|v\|_{C_b([0,\infty);Z)} + \| \mathcal{Q} \|_{L^1([0,\infty),\mathcal{L}(Z,X))}),
\end{aligned}$$

which permits us to conclude that $u \in AAP(X)$. □

The next result concerns the existence of asymptotically almost periodic solutions.

Theorem 2.2. *Assume the assumptions in Theorem* 2.1 *are fulfilled,* $M(t) \to 0$ *as* $t \to \infty$, $f_1 \in AAP(\mathcal{L}(\mathcal{B}_\beta, X))$ *and* $f_2 \in AAP(\mathcal{L}(\mathcal{B}, X))$. *Then there exists a unique strict solution* $u \in AAP(X) \cap C_b^{1+\alpha}(J \cup \mathbb{R}^+; X) \cap C_b(J \cup \mathbb{R}^+; X_\beta)$ *of* (1.1)-(1.2).

Proof. Let $\mathfrak{S}$ and Γ be as in the proof of Theorem 2.1 and $\mathfrak{S}_{AAP(X)}$ be the subspace of $\mathfrak{S}$ defined by $\mathfrak{S}_{AAP(X)} = \{u \in \mathfrak{S} : u_{|[0,\infty)} \in AAP(X)\}$. From the proof of Theorem 2.1 we know that Γ is a contraction on $\mathfrak{S}$. Thus, to prove the assertion it is sufficient to show that $\Gamma(\mathfrak{S}_{AAP(X)}) \subset \mathfrak{S}_{AAP(X)}$.

Let $u \in \mathfrak{S}_{AAP(X)}$. Since u' is uniformly continuous, from [19, Theorem 5.2] and Lemma 2.3 we infer that $\mathcal{P}_{u'} \in AAP(\mathcal{B})$, and from the theory of asymptotically almost periodic functions, it follows that the functions $f_1 \circ \mathcal{P}_u$ and $f_2 \circ \mathcal{P}_{u'}$ belong to $AAP(X)$. Finally, by using Lemma 2.4 with $\mathcal{Q}(s) = T(s)$ and $Z = X$, we infer that $\Gamma u \in AAP(X)$ and $\Gamma(\mathfrak{S}_{AAP(X)}) \subset \mathfrak{S}_{AAP(X)}$. This completes the proof. □

In the next theorem, we study the existence of a solution via condition $\mathbf{H_2}$.

Theorem 2.3. *Assume that condition* $\mathbf{H_2}$ *is satisfied,* $C_b(J;X) \hookrightarrow \mathcal{B}$, $\varphi \in C_b^1(J;X)$, $\varphi(0) \in X_1$ *and* $\frac{d^-\varphi}{dt}(0) = A\varphi(0) + f(0,\varphi,\varphi')$. *Suppose there are* $\psi_1 \in \mathcal{B}_{\mathcal{B}}$ *and* $\psi_2 \in \mathcal{B}$ *such that* $f(\cdot, \psi_1, \psi_2) \in C_b([0,\infty); Y)$ *and*

$$\begin{aligned}
\Theta \ =\ & \mathcal{K} \| L_f \|_{C_b([0,\infty);\mathbb{R}^+)} (\| (-A)^{\beta-1} \| + 1) \| H \|_{L^1([0,\infty),\mathbb{R}^+)} \\
& + \mathcal{K} \| L_f \|_{C_b([0,\infty);\mathbb{R}^+)} \| i_c \|_{\mathcal{L}(Y,X)} (\frac{M_0}{\gamma} + 1) < 1,
\end{aligned}$$

where i_c *denotes the inclusion map from* Y *into* X. *Then there exists a unique strict solution* $u \in C_b^1(J \cup \mathbb{R}^+; X) \cap C_b(J \cup \mathbb{R}^+; X_\beta)$ *of* (1.1)-(1.2).

Proof. Let $\mathfrak{F}$ be the space

$$\begin{aligned}\mathfrak{F} = \{u \in C_b^1(J \cup [0,\infty); X) \cap C_b(J \cup [0,\infty); X_\beta) :\\ \mathcal{P}_u \in C_b([0,\infty); \mathcal{B}_\beta), \mathcal{P}_u \in C_b^1([0,\infty); \mathcal{B})\},\end{aligned} \tag{2.9}$$

endowed with the metric $\Phi(u,v) = \parallel \mathcal{P}_u - \mathcal{P}_v \parallel_{C_b^1([0,\infty);\mathcal{B})} + \parallel \mathcal{P}_u - \mathcal{P}_v \parallel_{C_b([0,\infty);\mathcal{B}_\beta)}$ and $\Gamma : \mathfrak{F} \to \mathfrak{F}$ defined as in the proof of Theorem 2.1. Next, we show that Γ is a contraction on $\mathfrak{F}$. To begin, we prove that Γ is a function with values in $\mathfrak{F}$.

Let $u \in \mathfrak{F}$. By noting that $f_{u,u'} \in C_b([0,\infty); Y)$, from condition $\mathbf{H_2}$ it is easy to see that

$$\begin{aligned}\parallel (-A)^\beta \Gamma u(t) \parallel &\leq M_0 e^{-\gamma t} \parallel (-A)^\beta \varphi(0) \parallel \\ &\quad + \parallel (-A)^{\beta-1} \parallel_{\mathcal{L}(X)} \int_0^t \parallel AT(t-s) \parallel_{\mathcal{L}(Y,X)} \parallel f_{u,u'}(s) \parallel_Y \, ds \\ &\leq M_0 e^{-\gamma t} \parallel (-A)^\beta \varphi(0) \parallel \\ &\quad + \parallel (-A)^{\beta-1} \parallel_{\mathcal{L}(X)} \parallel f_{u,u'} \parallel_{C_b([0,\infty);Y)} \parallel H \parallel_{L^1([0,\infty),\mathbb{R}^+)},\end{aligned}$$

which implies that $\Gamma u \in C_b(J \cup [0,\infty); X_\beta)$ and $\mathcal{P}_{\Gamma u} \in C_b(J \cup [0,\infty); \mathcal{B}_\beta)$. Proceeding as above, from Lemma 2.2 we have that

$$\begin{aligned}\parallel \frac{d}{dt}\Gamma u(t) \parallel &\leq e^{-\gamma t} \parallel A\varphi(0) \parallel + \parallel f_{u,u'} \parallel_{C_b([0,\infty);Y)} \parallel H \parallel_{L^1([0,\infty),\mathbb{R}^+)} \\ &\quad + \parallel f_{u,u'} \parallel_{C_b([0,\infty);Y)},\end{aligned}$$

which shows that $\frac{d}{dt}\Gamma u \in C_b(J \cup [0,\infty); X)$. Moreover, from Lemma 2.1, the compatibility condition $\frac{d^-\varphi}{dt}(0) = A\varphi(0) + f(0,\varphi,\varphi')$ and the fact that

$$\frac{d}{dt}\Gamma u(t) = AT(t)\varphi(0) + \int_0^t AT(t-s) f_{u,u'}(s)ds + f(t,u_t,u_t'), \ \forall t \geq 0, \tag{2.10}$$

we infer that $\mathcal{P}_{\Gamma u} \in C_b^1([0,\infty); \mathcal{B})$ and $\frac{d}{dt}\mathcal{P}_{\Gamma u} = \mathcal{P}_{\frac{d}{dt}\Gamma u}$, which completes the proof that Γ is a well defined function from $\mathfrak{F}$ into itself.

On the other hand, for $u, v \in \mathfrak{F}$ and $t \geq 0$ we get

$$\begin{aligned}\parallel \Gamma u(t) - \Gamma v(t) \parallel_\beta &= \int_0^t \parallel (-A)^\beta T(t-s)(f_{u,u'}(s) - f(s,v_s,v_s')) \parallel ds \\ &\leq \parallel (-A)^{\beta-1} \parallel \int_0^t \parallel AT(t-s) \parallel_{\mathcal{L}(Y,X)} L_f(s)\Phi(u,v)ds,\end{aligned}$$

and hence

$$\begin{aligned}&\parallel \Gamma u - \Gamma v \parallel_{C_b([0,\infty);X_\beta)} \\ &\quad\leq \parallel (-A)^{\beta-1} \parallel \parallel L_f \parallel_{C_b([0,\infty);\mathbb{R}^+)} \parallel H \parallel_{L^1([0,\infty);\mathbb{R}^+)} \Phi(u,v).\end{aligned} \tag{2.11}$$

Similarly, from the inequality

$$\parallel \Gamma u(t) - \Gamma v(t) \parallel \leq M_0 \int_0^t e^{-\gamma(t-s)} \parallel i_{Y,X} \parallel_{\mathcal{L}(Y,X)} \parallel f_{u,u'}(s) - f(s,v_s,v_s') \parallel_Y \, ds$$

we obtain

$$\parallel \Gamma u - \Gamma v \parallel_{C_b([0,\infty);X)} \leq \frac{M_0}{\gamma} \parallel i_{Y,X} \parallel_{\mathcal{L}(Y,X)} \parallel L_f \parallel_{C_b([0,\infty);\mathbb{R}^+)} \Phi(u,v). \tag{2.12}$$

In addition, by using formulae (2.5) it is easy to see that

$$\begin{aligned} &\| \frac{d}{dt}\Gamma u - \frac{d}{dt}\Gamma v \|_{C_b([0,\infty);X)} \\ (2.13) \qquad &\leq \quad \| L_f \|_{C_b([0,\infty);\mathbb{R}^+)} \left(\| H \|_{L^1([0,\infty);\mathbb{R}^+)} + \| i_{Y,X} \|_{\mathcal{L}(Y,X)}\right) \Phi(u,v). \end{aligned}$$

Finally, from inequalities (2.11)-(2.13) we obtain that $\Phi(\Gamma u, \Gamma v) \leq \Theta\Phi(u,v)$, which shows that Γ is a contraction on $\mathfrak{F}$ and there exits a unique mild solution $u \in \mathfrak{F}$ of (1.1)-(1.2). The fact that $u(\cdot)$ is a strict solution of (1.1)-(1.2) is a consequence of Lemma 2.2. The proof is complete. □

The proof of the next proposition follows reasoning similar to that in the proof of Theorem 2.2. We only note that to prove this result, we apply Lemma 2.4 with $\mathcal{Q}(s) = AT(s)$ and $Z = Y$ and we use Theorem 2.3 in place of Theorem 2.1.

Theorem 2.4. *Suppose that the assumptions in Theorem* 2.3 *are satisfied. If* $f_1 \in AAP(\mathcal{L}(\mathcal{B}_\beta, Y))$ *and* $f_2 \in AAP(\mathcal{L}(\mathcal{B}, Y))$*, then there exists a unique strict solution* $u \in AAP(X)$ *of the problem* (1.1)-(1.2).

3. Application

Next, we use the abstract results in Section 2 to study the existence of a global solution for the partial neutral differential problem

$$\begin{aligned} u'(t,\xi) &= \triangle u(t,\xi) + \int_{-\infty}^{t} K_1(t,t-s)\Delta u(s,\xi)ds \\ &\quad - \int_{-\infty}^{t} K_2(t,t-s)u'(s,\xi)ds, \qquad (t,\xi) \in [0,\infty)\times[0,\pi], \\ (3.1) \qquad & u(t,0) = u(t,\pi) = 0, \qquad t \in [0,\infty), \\ (3.2) \qquad & u(s,\xi) = \varphi(s,\xi) \qquad \xi \in [0,\pi],\ s \leq 0, \end{aligned}$$

where $\varphi(\cdot)$ is a function defined from $(-\infty,0]\times[0,\pi]$ into $\mathbb{R}$.

To study the above system, we consider the space $X = L^2([0,\pi])$ and the operator $A : D(A) \subset X \to X$ given by $Ax = x''$ on $D(A) = \{x \in X : x'' \in X,\ x(0) = x(\pi) = 0\}$. It is well known that A is the generator of an analytic semigroup $(T(t))_{t\geq 0}$ on X, A has discrete spectrum with eigenvalues $-n^2$, $n \in \mathbb{N}$, and associated normalized eigenvectors $z_n(\xi) = \left(\frac{2}{\pi}\right)^{1/2}\sin(n\xi)$. In addition, $\| T(t) \| \leq e^{-t}$, $\| AT(t) \| \leq e^{-\frac{t}{2}}t^{-1}$ and $\| A^2T(t) \| \leq 4e^{-t}t^{-2}$ for all $t > 0$. To simplify, next we use that $\| T(t) \| \leq e^{-\frac{t}{2}}$, $\| AT(t) \| \leq e^{-\frac{t}{2}}t^{-1}$ and $\| A^2T(t) \| \leq 4e^{-\frac{t}{2}}t^{-2}$ for $t > 0$.

As a phase space we consider the space $\mathcal{B} = C_r \times L^p(\rho, X)$. Let $r \geq 0, 1 \leq p < \infty$ and $\rho : (-\infty,-r] \to \mathbb{R}$ be a nonnegative measurable function which satisfies the conditions (g-5)-(g-7) in the terminology of [14]. The space $C_r \times L^p(\rho, X)$ is formed by all classes of functions $\psi : (-\infty, 0] \to X$ such that $\psi_{|_{[-r,0]}} \in C([-r,0],X)$, $\psi(\cdot)$ is Lebesgue-measurable and $\rho^{\frac{1}{p}}\psi \in L^p((-\infty,-r],X)$. The norm in $C_r \times L^p(\rho,\mathcal{D})$ is given by $\| \psi \|_{\mathcal{B}} = \| \psi \|_{C([-r,0];X)} + \| \rho^{\frac{1}{p}}\psi \|_{L^p((-\infty,-r],X)}$. From [14], we know that $\mathcal{B}$ satisfies the conditions in Section 2 and $\mathcal{B}$ is a uniform fading memory space. In this case, we have that $M(t) \to 0$ as $t \to \infty$ and $K(\cdot)$ is bounded; see [14, p. 190].

Next we assume $K_i \in C_b([0,\infty)\times[0,\infty);\mathbb{R})$ and there are $\alpha \in (0,1)$ and functions $L_{K_i} \in C_b([0,\infty);\mathbb{R}^+)$ such that $| K_i(t,\tau) - K_i(s,\tau) | \leq L_{K_i}(\tau) \mid t-s \mid^\alpha$ for all $t,s \in [0,\infty)$, $\tau \geq 0$, and $\theta_i = \sup_{t\geq 0}(\int_{-\infty}^{0} \frac{K_i^2(t,-\tau)}{\rho(\tau)}d\tau)^{\frac{1}{2}} + (\int_{-\infty}^{0} \frac{L_{K_i}^2(-\tau)}{\rho(\tau)}d\tau)^{\frac{1}{2}} < \infty$

for $i = 1, 2$. Under these conditions, the functions $f_1 : [0,\infty) \times \mathcal{B}_1 \to X$, $f_2 : [0,\infty) \times \mathcal{B} \to X$ given by

$$f_1(t,\psi)(\xi) = \int_{-\infty}^{0} K_1(t,-\tau)\Delta\psi(\tau,\xi)d\tau \text{ and } f_2(t,\psi)(\xi) = \int_{-\infty}^{0} K_2(t,-\tau)\psi(\tau,\xi)d\tau$$

are well defined. Moreover, $f_1 \in C_b^{\alpha}([0,\infty);\mathcal{L}(\mathcal{B}_1,X))$, $f_2 \in C_b^{\alpha}([0,\infty);\mathcal{L}(\mathcal{B},X))$, $\| f_1 \|_{C_b^{\alpha}([0,\infty);\mathcal{L}(\mathcal{B}_1,X))} \leq \theta_1$ and $\| f_2 \|_{C_b^{\alpha}([0,\infty);\mathcal{L}(\mathcal{B},X))} \leq \theta_2$.

Next, we say that $u \in C_b^1(J \cup \mathbb{R}^+; X) \cap C_b(J \cup \mathbb{R}^+; X_1)$ is a strict solution of (3.1)-(3.2) if $u(\cdot)$ is a strict solution of the associated problem (1.1)-(1.2).

Proposition 3.1. *Assume* $\varphi \in C_b^{1+\alpha}((-\infty,0];X)$, $\varphi(0,\cdot) \in X_1$, $\frac{d^-\varphi}{dt}(0,\cdot) = A\varphi(0,\cdot) + f_1(0,\varphi) + f_2(0,\varphi')$, $\{A\varphi(0,\cdot), f_1(0,\varphi) + f_2(0,\varphi')\} \subset D_A(\alpha,\infty)$ *and* $\mathcal{P}_y \in C_b^{1+\alpha}([0,\infty);\mathcal{B})$. *If* $\mathcal{K}(2+ \| (-A)^{-1} \|)(9 + \frac{3}{\alpha} + \frac{4}{\alpha(1-\alpha)} + \frac{M_{1,\alpha}}{\alpha})(\theta_1+\theta_2) < 1$, *then there exists a unique strict solution* $u \in C_b^1(J \cup \mathbb{R}^+; X) \cap C_b(J \cup \mathbb{R}^+; X_1)$ *of* (3.1)-(3.2). *If, in addition, there are* $a_i, k_i \in C_b([0,\infty);\mathbb{R})$ *such that* $K_i(t,s) = a_i(t)k_i(s)$, $a_i \in AAP(\mathbb{R})$ *and* $\eta_i = \| a_i \|_{C_b([0,\infty);\mathbb{R})} (\int_{-\infty}^{0} \frac{k_i^2(-\tau)}{\rho(\tau)d\tau})^{\frac{1}{2}} < \infty$, $i = 1, 2$, *then* $u \in AAP(X)$.

Proof. The existence of a strict solution $u \in C_b^1(J \cup \mathbb{R}^+; X) \cap C_b(J \cup \mathbb{R}^+; X_1)$ follows from Theorem 2.1 (we note that the number Λ in Proposition 2.1 appears explicitly in the last paragraph of the proof of Proposition 2.1). If $K_i(t,s) = a_i(t)k_i(s)$, then $f_1 \in AAP(\mathcal{L}(\mathcal{B}_1, X))$ and $f_2 \in AAP(\mathcal{L}(\mathcal{B}, X))$, which implies via Theorem 2.2 that $u \in AAP(X)$. ☐

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