A NOTE ON EXTREME POINTS OF $C^\infty$-SMOOTH BALLS IN POLYHEDRAL SPACES

A. J. GUIRAO, V. MONTESINOS, AND V. ZIZLER

(Communicated by Thomas Schlumprecht)

Abstract. Morris (1983) proved that every separable Banach space $X$ that contains an isomorphic copy of $c_0$ has an equivalent strictly convex norm such that all points of its unit sphere $S_X$ are unpreserved extreme, i.e., they are no longer extreme points of $B_X^{**}$. We use a result of Hájek (1995) to prove that any separable infinite-dimensional polyhedral Banach space has an equivalent $C^\infty$-smooth and strictly convex norm with the same property as in Morris’ result. We additionally show that no point on the sphere of a $C^2$-smooth equivalent norm on a polyhedral infinite-dimensional space can be strongly extreme, i.e., there is no point $x$ on the sphere for which a sequence $(h_n)$ in $X$ with $\|h_n\| \not\to 0$ exists such that $\|x \pm h_n\| \to 1$.

1. Introduction

It is known that in non-superreflexive spaces, there exist no equivalent $C^2$-smooth norms that would be at the same time locally uniformly rotund (cf. e.g. [FHIMZ, Exercise 9.16]). We show in this note that yet, in separable polyhedral spaces (all of which are non-superreflexive), there exist $C^\infty$-smooth norms with various degrees of rotundity weaker than local uniform rotundity.

If $(X, \|\cdot\|)$ is a normed space, its closed unit ball (its unit sphere) will be denoted alternatively by $B_X$, $B_{\|\cdot\|}$, or even $B_{(X,\|\cdot\|)}$ (respectively $S_X$, $S_{\|\cdot\|}$, or $S_{(X,\|\cdot\|)}$), according to the circumstances. If $x \in X$ and $\delta > 0$, we put $B_X(x; \delta)$, $B_{\|\cdot\|}(x; \delta)$, or even $B_{(X,\|\cdot\|)}(x; \delta)$, for $x + \delta B_X$. The norm on $X$, its dual norm on $X^*$, and its bidual norm on $X^{**}$, are denoted by the same notation. For standard notation, results, and undefined terms we refer, e.g., to [FHIMZ].

Extreme points of $B_X$ that are not extreme of $B_X^{**}$ are called unpreserved. On the other side, points in $S_X$ that are extreme points of $B_X^{**}$ are called preserved extreme points (see Figure I). Obviously, every preserved extreme point of $B_X$ is itself an extreme point of $B_X$.

The preserved extreme points coincide with the $w$-strongly extreme points of $B_X$ (see [GLT92] and the references therein). A point $x \in S_X$ is called ($w$-) strongly extreme of $B_X$ if given two sequences $\{y_n\}$ and $\{z_n\}$ in $B_X$ such that $(y_n + z_n) \to 2x$,
Figure 1. In (i), all points in $S_X$ are preserved extreme; none in (ii)

then $y_n \to x$ (respectively, $y_n \overset{w}{\to} x$). A norm $\| \cdot \|$ such that all points in $S_{\| \cdot \|}$ are strongly extreme is said to be midpoint locally uniformly rotund (for this notion, see, e.g., [LPT09] and the references therein).

Solving a question by Phelps, Katznelson (see the reference in [Mo83]) proved that the closed unit ball of the disk algebra contains unpreserved extreme points.

Let $x \in S_X$. The point $x$ is said to be strongly exposed (by a functional $f \in S_X^*$) if $f(x) = 1$ and $\text{diam}(S(f, \delta)) \to 0$ as $\delta \downarrow 0$, where $S(f, \delta) := \{x \in B_X : f(x) > 1 - \delta\}$ is a section of $B_X$ determined by $f$. The point $x$ is said to be denting if for every $\varepsilon > 0$ it is contained in a section of $B_X$ having diameter less than $\varepsilon$. It is easy to show that strongly exposed $\Rightarrow$ denting $\Rightarrow$ strongly extreme $\Rightarrow$ w-strongly extreme (= preserved extreme) $\Rightarrow$ extreme, and that if $X$ is locally uniformly rotund, then every point in $S_X$ is strongly exposed. For an example showing how big the gap between being strongly or w-strongly extreme is, see Theorem 2.4. It is simple to show that strongly exposed $\Rightarrow$ denting $\Rightarrow$ strongly extreme $\Rightarrow$ extreme, and that if $X$ is locally uniformly rotund, then every point in $S_X$ is strongly exposed. For an example showing how big the gap between being strongly or w-strongly extreme is, see Theorem 2.4. It is simple to show that a denting point of $S_X^{**}$ must belong to $X$, hence the example in Remark 2.5 hints also at the difference between being strongly extreme and denting.

Morris proved in [Mo83] the following result.

(M1) Any separable Banach space $X$ containing an isomorphic copy of $c_0$ can be renormed in such a way that all points of $S_X$ are unpreserved extreme points. (Observe that the new norm is then strictly convex.)

The space $c_0$ has the property that the set $\text{Ext}(B_X^*)$ of extreme points of the closed dual unit ball is countable. The set $\text{Ext}(B_X^*)$ is an example of a James boundary, i.e., a subset of $B_{X^*}$ where each element $x \in X$ attains its supremum on $B_{X^*}$. A Banach space with a countable James boundary has a separable dual space (this follows, e.g., from the fact that a countable James boundary is strong, i.e., its closed convex hull is the closed dual unit ball ([Ro81]; see also [Go87]).

A Banach space $X$ is called polyhedral if the ball of every finite-dimensional subspace (equivalently every two-dimensional subspace; see [K59]) of $X$ has only a finite number of extreme points. Every polyhedral separable space has a countable James boundary ([Fo80]; see also [Ve00]).

An example of polyhedral space is $c_0$ in its canonical norm ([K60]; see also [GM72] and [Go01]). The argument in [Go01] is so nice that we cannot help but to reproduce it here. It relies on the fact that the $\| \cdot \|_\infty$-norm on $c_0$ depends locally on a finite number of coordinates (see the precise definition of this term below). Let $E$ be a finite-dimensional subspace of $c_0$. For each $x \in S_E$ there exists $\varepsilon(x) > 0$ and a finite subset $F(x)$ of $X^*$ such that $\|y\|_\infty = \sup\{|\langle y, x^* \rangle| : x^* \in F(x)\}$ for all
y ∈ B_E(x; ε(x)). Since S_E is compact, there are x_1, . . . , x_n in S_E such that
$$S_E \subset \bigcup_{i=1}^{n} B_E(x_i, \varepsilon(x_i)).$$

Put F := \bigcup_{i=1}^{n} F(x_i). Then F is a finite subset of X* such that
$$\|x\|_\infty = \sup\{|\langle x, x^* \rangle| : x^* \in F\}$$
for all x ∈ E, hence E is isometric to a subspace of (R^{|F|}, \|·\|_\infty), a polyhedral space.

On the other side, the space c in its canonical norm is not polyhedral. The following argument was kindly provided by L. Veselý (personal communication): Consider the points
$$P_n := \exp\{i(1 - 1/n)\pi/4\}$$
in the plane, for all n ∈ N (see Figure 2). Let a_n x + b_n y = 1 be the equation of the line through P_n and P_{n+1} for all n ∈ N, and a_0 x + b_0 y = 1 the equation of the line through P_0 := (−1, 0). Then a := (a_n)_{n≥0} and b := (b_n)_{n≥0} are elements in c, and their linear span L is isometric to a plane equipped with the norm whose closed unit ball is the set \text{conv} \{±P_1, ±P_2, . . . , ±P_∞\}.

![Figure 2. The construction to prove that c is not polyhedral](image-url)

There is no infinite-dimensional reflexive polyhedral space ([L64]). Actually, no infinite-dimensional C(K) space in its canonical norm is polyhedral—although such space has, if K is a countable compact topological space, obviously, a countable James boundary. As seen below (see (H)), every C(K) space with K a countable and compact topological space is isomorphic to a polyhedral space.

We will need the following result:

(Z) Banach spaces with a countable James boundary are c_0-saturated, i.e., each closed subspace contains an isomorphic copy of c_0 ([Fo77], [PWZ81]; see also [FHHMZ, Theorem 10.9]).

In this note we slightly modify Morris technique by means of a result of P. Hájek ([Ha95]; see also [FHHMZ, Theorem 10.12]) on normed spaces with a countable James boundary—a characterization quoted below as (H)—to add, under these circumstances, smoothness—in fact, C∞-smoothness—to the kind of renorming shown by Morris.

The norm \|·\| of a Banach space is said to depend locally on a finite number of coordinates if given any x_0 ∈ S_X there exists δ > 0, continuous linear functionals \{ψ_1, ψ_2, . . . , ψ_n\} ⊂ X*, and a continuous function f : R^n → R such that, for every x ∈ B(x_0; δ) we have \|x\| = f(ψ_1(x), ψ_1(x), . . . , ψ_1(x)). The result of Hájek [Ha95] (see also [FHHMZ, Theorem 10.12]) mentioned above, an improvement of results in
The following result appears in [Mo83], with a different argument, as an ingredient of the proof of (M1) above; it will also be used in the proof of our main result.

(M2) There exists an infinite-dimensional $w^*$-closed subspace $M_0$ of $\ell_\infty$ such that $M_0 \cap c_0 = \{0\}$.

To see this, first note that every separable Banach space is isometric to a subspace of $\ell_\infty$, thus in particular $\ell_\infty$ contains an isometric copy $Z$ of a given infinite-dimensional separable reflexive space. By a result of Rosenthal (see, e.g., [FHHMZ, Lemma 4.62]), $Z$ is $w^*$-closed. Observe that $Z \cap c_0$ must be finite-dimensional, as any infinite-dimensional subspace of $c_0$ contains a copy of $c_0$. Then, a finite-codimensional subspace $M_0$ of $Z$ is what we need to finish the proof.

2. The results

Theorem 2.1. Let $(X, ||\cdot||_0)$ be a Banach space having a countable James boundary. Then there exists an equivalent (strictly convex) norm $|||\cdot|||$ on $X$ that is $C_\infty$-smooth away from the origin and such that every point in $S_{|||\cdot|||}$ is an unpreserved extreme point of $B_{|||\cdot|||}$.

Proof. By (H) above, the space $X$ has an equivalent $C_\infty$-smooth norm $||\cdot||$ that depends locally on a finite number of coordinates. Moreover, it contains an isomorphic copy $Z$ of $c_0$ (see (Z) above). The space $Z^{**}$ can be canonically identified to a closed subspace of $X^{**}$. Let $M$ be a $w^*$-closed infinite-dimensional subspace of $Z^{**}$ such that $M \cap Z = \{0\}$; it exists thanks to (M2) above. It is clear, too, that $M \cap X = \{0\}$.

Let $N := M_\perp \subset X^*$ (the orthogonal is taken with respect to the duality $\langle X^{**}, X^* \rangle$). Find a sequence $\{\phi_n\}$ in $N$ such that $\text{span} \{\phi_n : n \in \mathbb{N}\} = N$ and $\sum_{n=1}^{\infty} ||\phi_n||^2 < +\infty$. Define a linear operator $T : X \to \ell_2$ by $Tx := (\langle x, \phi_n \rangle)_{n=1}^{\infty}$ for $x \in X$; then $T$ is clearly bounded and one-to-one, and the mapping $x \to ||Tx||_2$ from $X$ into $\mathbb{R}$ is certainly $C_\infty$-smooth away from the origin.

Define a norm $|||\cdot|||$ on $X$ by

$$|||x||| := ||x|| + ||Tx||_2 \quad \text{for all } x \in X.$$  

Clearly $|||\cdot|||$ is strictly convex (see e.g. [DGZ, Chapter II]) and $C_\infty$-smooth away from the origin. Let us show that every point $x_0$ in $S_{|||\cdot|||}$ is unpreserved extreme. Find $\delta > 0$ such that $||\cdot||$ depends on $B_{||\cdot||}(x_0; \delta)$ on finitely many coordinates $\{\psi_1, \psi_2, \ldots, \psi_n\}$, i.e., $||x|| = f(\psi_1(x), \psi_2(x), \ldots, \psi_n(x))$ for $x \in B_{||\cdot||}(x_0; \delta)$, where $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous function. Due to the fact that $M$ is infinite-dimensional, we can find $h^{**} \in M \cap \bigcap_{k=1}^{n} \ker \psi_k$ with $0 < ||h^{**}|| \leq \delta$. 

[ Fo77 ] and [ PWZ81 ], is the equivalence (i) to (iv) in the following. For the property (v) see [ FLP01, Proposition 6.19 ] and, e.g., [ Ve00 ].

(H) For a Banach space $X$, the following are equivalent: (i) $X$ has a countable James boundary. (ii) $X$ has a James boundary that can be covered by a countable number of $\|\cdot\|$-compact subsets of $X^*$. (iii) $X$ is separable and has an equivalent norm that depends locally on a finite number of coordinates. (iv) $X$ is separable and has an equivalent norm that is $C_\infty$-smooth away from the origin and depends locally on a finite number of coordinates. (v) $X$ is separable and isomorphic to a polyhedral Banach space.
Find a net \( \{h_i : i \in I, \leq \} \) in \( B_{\|\cdot\|}(0; \delta) \) that \( w^* \)-converges to \( h^{**} \). Observe that 
\[ \|x_0 + h_i\| = f(\psi_1(x_0 + h_i), \psi_2(x_0 + h_i), \ldots, \psi_n(x_0 + h_i)), \text{ for all } i \in I. \]
Note that \( \psi_k(x_0 + h_i) \to \psi_k(x_0 + h^{**}) \) for all \( k = 1, 2, \ldots, n \), and so, by \((2.2)\),
\[
\|x_0 + h_i\| = f(\psi_1(x_0 + h_i), \psi_2(x_0 + h_i), \ldots, \psi_n(x_0 + h_i)) \\
\to f(\psi_1(x_0 + h^{**}), \psi_2(x_0 + h^{**}), \ldots, \psi_n(x_0 + h^{**})) \\
= f(\psi_1(x_0), \psi_2(x_0), \ldots, \psi_n(x_0)) = \|x_0\|.
\]
Since
\[(2.4)\]
\[x_0 + h_i \overset{w^*}{\to} x_0 + h^{**},\]
we get from \((2.3)\) and \((2.4)\) that \( \|x_0 + h^{**}\| \leq \|x_0\| \). In the same way we get
\[ \|x_0 - h^{**}\| \leq \|x_0\| , \text{ so finally by a standard convexity argument, } \|x_0\| = \|x_0 + h^{**}\| = \|x_0 - h^{**}\|. \]
Regarding the norm \( \|\cdot\| \), we have then
\[ \|x_0 + h^{**}\| = \|x_0 + h^{**}\| + \|T(x_0 + h^{**})\|, \]
as it is easy to show, hence, since \( T(h^{**}) = 0 \),
\[(2.5)\]
\[\|x_0 + h^{**}\| = \|x_0\| + \|Tx_0\| = \|x_0\| = 1.\]
Analogously,
\[(2.6)\]
\[\|x_0 - h^{**}\| = \|x_0\| = 1.\]
Equations \((2.5)\) and \((2.6)\) together show that \( x_0 \) is an unpreserved extreme point of \( B_{\|\cdot\|} \).

The following result extends what formerly was known for \( C^2 \)-smooth LUR norms (see, e.g., [FHIMZ, Exercise 9.16]) and later for \( C^2 \)-smooth norms with a strongly exposed point on its unit sphere [FWZ83, Theorem 3.3].

**Theorem 2.2.** Let \( (X, \|\cdot\|) \) be an infinite-dimensional \( C^2 \)-smooth Banach space. If there exists a strongly extreme point of \( B_{\|\cdot\|} \), then \( X \) is superreflexive.

**Proof.** Assume that \( x \) is a strongly extreme point of \( B_X \). The \( C^2 \)-differentiability of \( \|\cdot\| \) implies that there exists \( \delta > 0 \) such that the first derivative of \( \|\cdot\| \) is uniformly continuous on a \( 2\delta \)-ball around \( x \). Let \( g \) be the supporting functional to the ball at \( x \). For \( h \in g^{-1}(0) \), let \( f(h) = \|x + h\| + \|x - h\| - 2 \). Then \( f(h) \geq 0 \), \( f(0) = 0 \) and \( \inf_{\|h\|=\delta} f > 0 \). Indeed, otherwise there exists a sequence \( \{h_n\}_{n=1}^{\infty} \) in \( g^{-1}(0) \) such that \( \|h_n\| = \delta \) for all \( n \in \mathbb{N} \), and \( f(h_n) \to 0 \), meaning that \( \|x + h_n\| \to 1 \) and \( \|x - h_n\| \to 1 \), as \( \|x + h_n\| \geq g(x + h_n) = g(x) = 1 \). Thus, by the definition of the strong extremality of \( x \), \( \|h_n\| \to 0 \), a contradiction. Hence, by standard methods we can construct a bump function (i.e. a function with bounded non-empty support) on \( g^{-1}(0) \) with uniformly continuous derivative, meaning that \( X \) is superreflexive (see, e.g., [FHIMZ, Theorem 9.19]).

**Corollary 2.3.** Let \( (X, \|\cdot\|) \) be an infinite-dimensional \( C^2 \)-smooth Banach space. Assume that \( X \) does contain an isomorphic copy of \( c_0 \) (in particular, assume that \( X \) is isomorphic to a polyhedral space). Then no point of \( S_{\|\cdot\|} \) is a strongly extreme point of \( B_{\|\cdot\|} \).
Proof. Otherwise, according to Theorem 2.2, the space $X$ would be superreflexive. This is impossible since $X$ contains an isomorphic copy of $c_0$. In case that $X$ is isomorphic to a polyhedral space, so it is every separable subspace of $X$, thus the containment of $c_0$ follows from (Z) and (H) above.

**Theorem 2.4.** Let $X$ be a separable infinite-dimensional polyhedral Banach space. Then there exists an equivalent norm $\| \cdot \|$ on $X$ such that every point in $S_{\| \cdot \|}$ is preserved extreme non-strongly extreme of $B_{\| \cdot \|}$.

Proof. Let $\| \cdot \|$ be an equivalent $C^2$-smooth norm on $X$ (such a norm always exists, see (H) above). Let $\{f_i : i \in \mathbb{N}\}$ be a countable norm-dense subset of $B_{\|X^*\|}$ (recall that $X$ is Asplund). Then the equivalent norm $\| \cdot \|$ on $X$ defined by $\|x\|^2 := \|x\|^2 + \sum \frac{1}{2} f_i^2(x)$ for all $x \in X$, is weakly uniformly rotund, i.e., whenever $x_n, y_n$ are in $S_{\| \cdot \|}$ and $\|x_n + y_n\| \to 2$, then $x_n - y_n \to 0$ in the weak topology of $X$. This means that, in particular, the bidual norm of $\| \cdot \|$ is rotund (indeed, assume that $2x^* = y^* + z^*$ for some $x^* \in S_{\|X^*\|}$), where $y^*$ and $z^*$ are both in $B_{\|X^*\|}$ and $y^* \neq z^*$. Since $X^*$ is separable, there exist sequences $\{y_n\}$ and $\{z_n\}$ in $B_{\|X\|}$ such that $y_n \to y^*$ and $z_n \to z^*$ in the $w^*$-topology. This leads immediately to a contradiction. Moreover, the norm $\| \cdot \|$ on $X$ is clearly $C^2$-smooth. Thus all points in $S_{\| \cdot \|}$ are preserved extreme points and yet, no point there is a strongly extreme point of $B_{\| \cdot \|}$ by Corollary 2.3 (indeed, $X$ is not superreflexive, as it contains an isomorphic copy of $c_0$).

**Remark 2.5.** (1) Note that, in the setting of Theorem 2.4, no point in $S_{\| \cdot \|}$ is a point where the norm and weak topologies coincide, as otherwise, by a result in [LLTSS], such a point would be a strongly extreme point of $B_{\| \cdot \|}$.

(2) The James space $J$ can be renormed by a norm the second bidual norm of which has the property that all its points on its sphere are strongly extreme points (MOTV01; see also [LPT09]). None of the points in $S_{\|X\|}$ can be denting. Recall that a space is reflexive if its dual space admits an equivalent Fréchet differentiable dual norm ([FHHMZ, Corollary 7.26]).

(3) The space $\ell_\infty$ cannot be renormed so that all points on the sphere would be preserved extreme points ({HMS}).

(4) Hájek (Ha98) showed that, if $\Gamma$ is uncountable, then there exists no $C^2$-smooth and strictly convex norm on $c_0(\Gamma)$.

(5) We refer to, e.g., [HMZ12], for a survey on related topics.

**References**


Instituto de Matemática Pura y Aplicada. Universitat Politècnica de València, C/Vera, s/n, 46020 Valencia, Spain

E-mail address: anguisa2@mat.upv.es

Instituto de Matemática Pura y Aplicada. Universitat Politècnica de València, C/Vera, s/n, 46020 Valencia, Spain

E-mail address: vmontesinos@mat.upv.es

Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Alberta, Canada

E-mail address: vasekzizler@gmail.com