

## NON-LINEAR NOISE EXCITATION AND INTERMITTENCY UNDER HIGH DISORDER

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ABSTRACT. Consider the semilinear heat equation  $\partial_t u = \partial_x^2 u + \lambda \sigma(u)\xi$  on the interval  $[0, L]$  with Dirichlet zero-boundary condition and a nice non-random initial function, where the forcing  $\xi$  is space-time white noise and  $\lambda > 0$  denotes the level of the noise. We show that, when the solution is intermittent [that is, when  $\inf_z |\sigma(z)/z| > 0$ ], the expected  $L^2$ -energy of the solution grows at least as  $\exp\{c\lambda^2\}$  and at most as  $\exp\{c\lambda^4\}$  as  $\lambda \rightarrow \infty$ . In the case that the Dirichlet boundary condition is replaced by a Neumann boundary condition, we prove that the  $L^2$ -energy of the solution is in fact of sharp exponential order  $\exp\{c\lambda^4\}$ . We show also that, for a large family of one-dimensional randomly forced wave equations on  $\mathbf{R}$ , the energy of the solution grows as  $\exp\{c\lambda\}$  as  $\lambda \rightarrow \infty$ . Thus, we observe the surprising result that the stochastic wave equation is, quite typically, significantly less noise-excitable than its parabolic counterparts.

### 1. INTRODUCTION

Consider a parabolic stochastic PDE [SPDE] of the following type:

$$(1.1) \quad \begin{cases} \partial_t u_t(x) = \partial_x^2 u_t(x) + \lambda \sigma(u_t(x))\xi & \text{for } 0 < x < L \text{ and } t > 0, \\ u_t(0) = u_t(L) = 0 & \text{for } t > 0, \end{cases}$$

where  $L > 0$  is a positive constant,  $\sigma : \mathbf{R} \rightarrow \mathbf{R}$  is a Lipschitz-continuous function,  $\xi$  denotes space-time white noise, and the initial function  $u_0 : [0, L] \rightarrow \mathbf{R}$  is a non-random function in  $L^2([0, L])$  such that  $\|u_0\|_{L^2([0, L])} > 0$  and  $u_0(0) = u_0(L) = 0$ . For the sake of simplicity, we assume further that  $\sigma(0) = 0$ , though some of our work remains valid when  $\sigma(0) \neq 0$  as well.

There is a large body of literature that is devoted to the analysis of models that are closely related to (1.1). One of the chief findings of this literature is that when  $\sigma$  is exactly a linear function [2, 3, 6, 8, 10, 13, 16, 24]—for those related models— $E(|u_t(x)|^k) = \exp\{\gamma(k)(t + o(t))\}$  as  $t \rightarrow \infty$ , where  $k \mapsto \gamma(k)/k$  is *strictly* increasing for  $k \geq 2$ . This property is known as *moment intermittency* [“intermittency,” to be brief], and is known to hold for some fully non-linear variants of (1.1) as well [15].

Intermittency is a mathematically-rigorous way to describe the property that the solution  $u_t(x)$  tends to develop very tall peaks that are distributed over small islands

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[3, 6, 17, 23, 24, 26]; we can think of this latter distribution property as “physical intermittency.” For a sample of results that are related to this general area see [1, 3, 6–10, 13, 16, 19, 20, 22, 24, 26], together with their substantial bibliographies.

The main goal of this article is to make the case that physical intermittency is a natural manifestation of noise in roughly-linear systems of the type (1.1). The existing literature concentrates on one way in which the solution is exposed to a great deal of noise: As  $t \rightarrow \infty$ , the net amount of exposure to the noise grows, and hence the system behaves intermittently. Here, we will make precise the statement that intermittency can also be caused by direct introduction of large noise [ $\lambda \gg 1$ ], and intermittent behavior can be observed at any time  $t$ , and not just when  $t \gg 1$ . This notion of “non-linear noise excitation” seems to be wholly new, though it has roots in the literature of noise excitation in NMR spectroscopy [4]. Our goal is to present our large-noise view in a way that is as simple and non-technical as possible. Therefore, we consider only the energy of the solution to a relatively-simple SPDE such as (1.1), though (1.1) turns out to be complex enough to elude a detailed analysis [compare Theorems 1.1 and 1.2]. It is however possible to develop a much more sophisticated mathematical theory of non-linearly noise-excitable systems; that is done elsewhere [21].

It is well known for example that if  $u_0 \in L^\infty[0, L]$ , then the stochastic heat equation (1.1) has an a.s.-unique continuous solution that has the property that

$$(1.2) \quad \sup_{x \in [0, L]} \sup_{t \in [0, T]} \mathbf{E}(|u_t(x)|^k) < \infty \quad \text{for all } T > 0 \text{ and } k \in [2, \infty).$$

We will be interested in the effect of the level  $\lambda$  of the noise on the energy  $\mathcal{E}_t(\lambda)$  of the solution at time  $t$ ; the latter quantity is defined as <sup>1</sup>

$$(1.3) \quad \mathcal{E}_t(\lambda) := \sqrt{\mathbf{E}(\|u_t\|_{L^2}^2)} \quad (t > 0).$$

Throughout, we use the following notation:

$$(1.4) \quad \ell_\sigma := \inf_{z \in \mathbf{R} \setminus \{0\}} |\sigma(z)/z|, \quad L_\sigma := \sup_{z \in \mathbf{R} \setminus \{0\}} |\sigma(z)/z|.$$

Clearly,  $0 \leq \ell_\sigma \leq L_\sigma$ . Moreover,  $L_\sigma < \infty$  because  $\sigma$  is Lipschitz and  $\sigma(0) = 0$ .

The following two theorems contain quantitative descriptions of the non-linear noise excitability of (1.1). These are two of our main three findings.

**Theorem 1.1.** *For all  $t > 0$ ,*

$$(1.5) \quad \ell_\sigma^2 t / (2L) \leq \liminf_{\lambda \rightarrow \infty} \lambda^{-2} \log \mathcal{E}_t(\lambda), \quad \limsup_{\lambda \rightarrow \infty} \lambda^{-4} \log \mathcal{E}_t(\lambda) \leq L_\sigma^4 t / 4.$$

We are not sure how to bridge the gap between the  $\exp\{c\lambda^2\}$  lower bound and the  $\exp\{c\lambda^4\}$  upper bound for the energy in part because the heat semigroup for the Dirichlet Laplacian is not conservative. However, in a companion paper [21] we prove that  $\exp\{c\lambda^4\}$  is a typical lower bound for the energy of a large number of intermittent complex systems. In particular, the energy of the solution for the stochastic heat equation on  $[0, L]$  with a *periodic* boundary condition is shown to be of sharp exponential order  $\exp\{c\lambda^4\}$ . The following theorem says that we can get the same kind of result when we replace the Dirichlet boundary condition with the Neumann boundary condition, provided additionally that the initial profile remains bounded uniformly away from zero.

<sup>1</sup>This is, in fact, the  $L^2(\Omega)$ -norm of the energy of the solution, where  $(\Omega, \mathcal{F}, \mathbf{P})$  denotes the underlying probability space.

**Theorem 1.2.** *Suppose that we replace the Dirichlet boundary condition in (1.1) by a Neumann boundary condition. If, in addition,  $\inf_{x \in [0, L]} u_0(x) > 0$ , then for every  $t > 0$ ,*

$$(1.6) \quad \ell_\sigma^4 t / 16 \leq \liminf_{\lambda \uparrow \infty} \lambda^{-4} \log \mathcal{E}_t(\lambda) \leq \limsup_{\lambda \uparrow \infty} \lambda^{-4} \log \mathcal{E}_t(\lambda) \leq L_\sigma^4 t / 4.$$

Theorems 1.1 and 1.2 together show that, under fairly natural regularity conditions [that include  $\ell_\sigma > 0$ ], the energy behaves roughly as  $\exp\{c\lambda^4\}$ , which is a fastly-growing function of the level  $\lambda$  of the noise. In Section 4 we document the somewhat surprising fact that, by contrast, the stochastic wave equation on  $\mathbf{R}$  has typically an energy that grows merely as  $\exp\{c\lambda\}$ . In other words, the stochastic wave equation is typically substantially less noise excitable than the stochastic heat equation.

Throughout this paper we write  $L^2$  in place of the Hilbert space  $L^2[0, L]$ .

## 2. PROOF OF THE LOWER BOUNDS

In this section we establish the lower bounds for the energy  $\mathcal{E}_t(\lambda)$  in both Theorems 1.1 and 1.2.

**2.1. The Lower bound for Theorem 1.1.** Let us begin the proof by writing the solution to the stochastic heat equation (1.1) in integral form [or the *mild form*],

$$(2.1) \quad u_t(x) = (P_t u_0)(x) + \lambda \int_{(0, t) \times (0, L)} p_{t-s}(x, y) \sigma(u_s(y)) \xi(ds dy),$$

where  $\{P_t\}_{t \geq 0}$  denotes the semigroup of the Dirichlet Laplacian on  $[0, L]$  and  $\{p_t\}_{t > 0}$  denotes the corresponding heat kernel. That is, in particular,  $P_0 h \equiv h$  for every  $h \in L^\infty[0, L]$ , and  $(P_t h)(x) := \int_0^L p_t(x, y) h(y) dy$  for all  $t > 0$  and  $x \in [0, L]$ . One can expand the heat kernel  $p_t(x, y)$  in terms of the eigenfunctions of the Dirichlet Laplacian as follows:

$$(2.2) \quad p_t(x, y) := \sum_{n=1}^{\infty} \phi_n(x) \phi_n(y) e^{-\mu_n t}; \quad \text{where} \\ \mu_n := (n\pi/L)^2 \quad \text{and} \quad \phi_n(x) := (2/L)^{1/2} \sin(n\pi x/L),$$

for all  $n \geq 1$  and  $0 \leq x \leq L$ . By the maximum principle: (i)  $p_t(x, y) > 0$  for all  $t > 0$  and  $x, y \in (0, L)$ ; and (ii)  $\int_0^L p_t(x, y) dx < 1$  for all  $y \in [0, L]$  and  $t > 0$ .

First, let us assume that  $u_0 \in C([0, L])$ . Let  $u_t^{(0)}(x) := u_0(x)$  for all  $x \in [0, L]$  and  $t \geq 0$ , and then define iteratively, for all  $k \geq 0$ ,

$$(2.3) \quad u_t^{(k+1)}(x) = (P_t u_0)(x) + \lambda \int_{(0, t) \times (0, L)} p_{t-s}(x, y) \sigma(u_s^{(k)}(y)) \xi(ds dy).$$

By the theory of parabolic SPDEs, the sequence  $\{u^{(k)}\}_{k=1}^{\infty}$  is locally uniformly Cauchy in  $L^2(\Omega)$ , whence the limit  $u_t(x) := \lim_{k \rightarrow \infty} u_t^{(k)}(x)$  exists in  $L^2(\Omega)$  for all  $t > 0$  and  $x \in [0, L]$ . Moreover,  $u$  solves (2.1), and is unique among all mild solutions that satisfy (1.2). See [25, Chapter 3], and in particular [11]. When  $u_0 \in L^2[0, L]$ , the existence and uniqueness of a solution as in  $L^2[0, L]$  follows from the density argument since  $C([0, L])$  is dense in  $L^2[0, L]$  (see for example [21, Section 8]).

Now we implement a projection method that is motivated by a classical idea of Kaplan[18], and especially the more recent work of Bonder and Groisman [5]. Since  $p_t(x, y) = p_t(y, x)$ , the stochastic Fubini theorem [25, Theorem 2.6, p. 296] implies the following for all  $n \geq 1$  and  $t > 0$ : With probability one,

$$(2.4) \quad \begin{aligned} (u_t, \phi_n) &= (u_0, P_t \phi_n) + \lambda \int_{(0,t) \times (0,L)} (P_{t-s} \phi_n)(y) \sigma(u_s(y)) \xi(ds dy) \\ &= e^{-\mu_n t} (u_0, \phi_n) + \lambda \int_{(0,t) \times (0,L)} e^{-\mu_n(t-s)} \phi_n(y) \sigma(u_s(y)) \xi(ds dy). \end{aligned}$$

Consequently, the Walsh isometry [25, Theorem 2.5, p. 295] shows us that

$$(2.5) \quad \begin{aligned} \mathbb{E} [(u_t, \phi_n)^2] &= e^{-2\mu_n t} (u_0, \phi_n)^2 + \lambda^2 \int_0^t ds \int_0^L dy e^{-2\mu_n(t-s)} |\phi_n(y)|^2 \mathbb{E} (|\sigma(u_s(y))|^2) \\ &\geq e^{-2\mu_n t} (u_0, \phi_n)^2 + \lambda^2 \ell_\sigma^2 \int_0^t e^{-2\mu_n(t-s)} ds \int_0^L dy |\phi_n(y)|^2 \mathbb{E} (|u_s(y)|^2). \end{aligned}$$

We may apply Jensen’s inequality, together with the Fubini theorem, in order to see that  $\int_0^L |\phi_n(y)|^2 \mathbb{E} (|u_s(y)|^2) dy \geq L^{-1} \mathbb{E} [(u_s, \phi_n)^2]$ . Thus we see that, for every fixed  $n \geq 1$ , the function  $F(t) := e^{2\mu_n t} \mathbb{E} [(u_t, \phi_n)^2]$  ( $t > 0$ ) satisfies

$$(2.6) \quad F(t) \geq (u_0, \phi_n)^2 + \frac{\lambda^2 \ell_\sigma^2}{L} \int_0^t F(s) ds \quad \text{for all } t > 0.$$

By the Gronwall’s inequality,  $F(t) \geq (u_0, \phi_n)^2 \exp(\lambda^2 \ell_\sigma^2 t / L)$  for  $t > 0$ ; i.e.,

$$(2.7) \quad \mathbb{E} [(u_t, \phi_n)^2] \geq (u_0, \phi_n)^2 \exp ([L^{-1} \lambda^2 \ell_\sigma^2 - 2\mu_n] t) \quad (t > 0).$$

We sum both sides over all  $n \geq 1$  to see that  $|\mathcal{E}_t(\lambda)|^2 \geq \|P_t u_0\|_{L^2}^2 e^{\lambda^2 \ell_\sigma^2 t / L}$  for all  $\lambda, t > 0$ . Since  $\sum_{n=1}^\infty (u_0, \phi_n)^2 = \|u_0\|_{L^2}^2 > 0$ , by the non-triviality assumption on  $u_0$ , it follows that  $\|P_t u_0\|_{L^2}^2 = \sum_{n=1}^\infty (u_0, \phi_n)^2 e^{-2\mu_n t} > 0$  for all  $t \geq 0$  and the proof is complete. □

**2.2. The lower bound for Theorem 1.2.** We now consider (1.1) with Neumann boundary. We will assume without loss of generality that  $\ell_\sigma > 0$ ; otherwise, the energy lower bound of Theorem 1.2 has no content. The existence and uniqueness of a continuous solution to (1.1) with Neumann boundary is found in Ref. [25, Chapter 3]. That solution solves (2.1), where  $\{P_t\}_{t \geq 0}$  now denotes the semigroup of the Neumann Laplacian on  $[0, L]$  and  $\{p_t\}_{t > 0}$  denotes the corresponding heat kernel. By the method of images,

$$(2.8) \quad p_t(x, y) := \sum_{n=-\infty}^\infty [\Gamma_t(x - y - 2nL) + \Gamma_t(x + y - 2nL)],$$

where  $\Gamma_t(z) := (4\pi t)^{-1/2} \exp\{-z^2/(4t)\}$  for all  $t > 0$  and  $z \in \mathbf{R}$ .

Owing to (2.8),  $p$  is conservative; that is,  $\int_0^L p_t(x, y) dy = 1$  for all  $t > 0$ . Therefore, we apply the Walsh isometry for stochastic integrals to find that for all fixed  $\varepsilon, t > 0$  and  $x \in [0, L]$ ,

$$(2.9) \quad \begin{aligned} \mathbb{E} \left( |u_t(x)|^2 \right) &\geq \varepsilon^2 + \lambda^2 \int_0^t ds \int_0^L dy [p_{t-s}(x, y)]^2 \mathbb{E} \left( |\sigma(u_s(y))|^2 \right) \\ &\geq \varepsilon^2 + (\lambda \ell_\sigma)^2 \int_0^t ds \int_0^L dy [p_{t-s}(x, y)]^2 \mathbb{E} \left( |u_s(y)|^2 \right), \end{aligned}$$

where  $\varepsilon := \inf_{x \in [0, L]} u_0(x) > 0$ . By (2.8),  $\int_0^L [p_r(x, y)]^2 dx = p_{2r}(y, y) \geq \Gamma_{2r}(0) = (8\pi r)^{-1/2}$  for all  $r > 0$  and  $y \in (0, L)$ .<sup>2</sup> Define

$$(2.10) \quad \mathcal{S}(r) := r^{-1/2} \quad (r > 0),$$

and integrate both sides of (2.9) in order to obtain

$$|\mathcal{E}_t(\lambda)|^2 \geq \varepsilon^2 L + \frac{(\lambda \ell_\sigma)^2}{\sqrt{8\pi}} \int_0^t \mathcal{S}(t-s) |\mathcal{E}_s(\lambda)|^2 ds \geq \varepsilon^2 L \sum_{j=0}^\infty \left( \frac{(\lambda \ell_\sigma)^2}{\sqrt{8\pi}} \right)^j \mathcal{S}^{*(j)}(t),$$

where  $\mathcal{S}^{*(0)}(r) := 1$  and  $\mathcal{S}^{*(k+1)}(r) := (\mathcal{S} * \mathcal{S}^{*(k)})(r)$  for all reals  $r > 0$  and integers  $k \geq 0$ . A calculation with beta integrals shows that

$$(2.11) \quad \mathcal{S}^{*(j)}(t) = \frac{(t\pi)^{j/2}}{\Gamma((j+2)/2)},$$

and hence

$$(2.12) \quad |\mathcal{E}_t(\lambda)|^2 \geq \varepsilon^2 L \sum_{j=0}^\infty \left( (\lambda \ell_\sigma)^2 \sqrt{t/8} \right)^j \frac{1}{\Gamma(\frac{j}{2} + 1)} \geq \varepsilon^2 L \exp \left( \frac{(\lambda \ell_\sigma)^4 t}{8} \right),$$

as can be seen by summing along even  $j \geq 0$ . The lower bound follows. □

### 3. PROOF OF THE UPPER BOUNDS

**3.1. The upper bound for Theorem 1.1.** For the upper bound, we assume only that  $u_0 \in L^2[0, L]$ . We use the notation of §2.1, and observe that for all  $t > 0$ , and integers  $k \geq 0$  and  $n \geq 1$ ,

$$(3.1) \quad \begin{aligned} \mathbb{E} \left[ (u_t^{(k+1)}, \phi_n)^2 \right] &\leq (P_t u_0, \phi_n)^2 + \lambda^2 L_\sigma^2 \int_0^t e^{-2\mu_n(t-s)} ds \int_0^L dy |\phi_n(y)|^2 \mathbb{E} \left( |u_s^{(k)}(y)|^2 \right) \\ &\leq (P_t u_0, \phi_n)^2 + \frac{2\lambda^2 L_\sigma^2}{L} \int_0^t e^{-2\mu_n(t-s)} \mathbb{E} \left( \|u_s^{(k)}\|_{L^2}^2 \right) ds, \end{aligned}$$

because  $|\phi_n(y)| \leq \sqrt{2/L}$ . See also the derivation of (2.5). Next, we apply the Gram–Schmidt procedure to introduce functions  $\phi_0, \phi_{-1}, \phi_{-2}, \dots \in L^2[0, L]$  such that the collection  $\{\phi_n\}_{n \in \mathbb{Z}}$  is a complete orthonormal system in  $L^2[0, L]$ . Since  $p_t(x, y) = p_t(y, x)$ , (2.1) and induction [on  $k$ ] together shows that  $(u_t^{(k+1)}, \phi_n) = 0$

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<sup>2</sup>This is precisely where the Dirichlet and Neumann problems differ: In the Dirichlet case,  $\inf_{y \in (0, L)} \int_0^L [p_r(x, y)]^2 dx = 0$ .

for  $n \leq 0$ . Therefore, we sum (3.1) over all integers  $n$  and apply the Parseval identity to find that

$$\mathbb{E} \left( \|u_t^{(k+1)}\|_{L^2}^2 \right) \leq \|P_t u_0\|_{L^2}^2 + \frac{2\lambda^2 L_\sigma^2}{L} \int_0^t \mathcal{M}(t-s) \mathbb{E} \left( \|u_s^{(k)}\|_{L^2}^2 \right) ds;$$

where  $\mathcal{M}(\tau) := \sum_{n=1}^\infty e^{-2\mu_n \tau} \leq \int_0^\infty e^{-2z^2 \pi^2 \tau / L^2} dz = L(8\pi\tau)^{-1/2}$  for all  $\tau > 0$ . Since  $P_t$  is non-expansive on  $L^2$ , the preceding two displays and (2.10) together yield

$$\begin{aligned} \mathbb{E} \left( \|u_t^{(k+1)}\|_{L^2}^2 \right) &\leq \|u_0\|_{L^2}^2 + \frac{\lambda^2 L_\sigma^2}{\sqrt{2\pi}} \int_0^t \mathcal{S}(t-s) \mathbb{E} \left( \|u_s^{(k)}\|_{L^2}^2 \right) ds \\ (3.2) \qquad \qquad \qquad &\leq \|u_0\|_{L^2}^2 \sum_{j=0}^{k+1} \left( \frac{\lambda^4 L_\sigma^4}{2\pi} \right)^{j/2} \mathcal{S}^{*(j)}(t), \end{aligned}$$

where the last inequality follows from recursion and the fact that  $u_t^{(0)}(x) := u_0(x)$ . Therefore, we may conclude from (2.11) and Fatou’s lemma that

$$(3.3) \qquad |\mathcal{E}_t(\lambda)|^2 \leq \|u_0\|_{L^2}^2 \sum_{j=0}^\infty \left( \frac{\lambda^4 L_\sigma^4 t}{2} \right)^{j/2} \frac{1}{\Gamma((j+2)/2)}.$$

If we add the preceding summands over all even integers  $j \geq 0$ , then we obtain  $\exp\{\lambda^4 L_\sigma^4 t/2\}$ . On the other hand, the same sum, once added over all odd integers  $j$ , yields

$$(3.4) \qquad \sum_{k=0}^\infty \left( \frac{\lambda^4 L_\sigma^4 t}{2} \right)^{k+(1/2)} \frac{1}{\Gamma(k+3/2)} \leq \left( \frac{\lambda^4 L_\sigma^4 t}{2} \right)^{1/2} \exp \left( \frac{\lambda^4 L_\sigma^4 t}{2} \right),$$

because  $\Gamma(k+3/2) \geq k!$  for all integers  $k \geq 0$ . We add the two bounds, once for odd and one for even  $j$ ’s, in order to obtain the upper bound of Theorem 1.1.  $\square$

**3.2. The upper bound for Theorem 1.2.** Just as one does in the Dirichlet case, the solution to (1.1) for the Neumann Laplacian can be realized as  $\lim_{k \rightarrow \infty} u_t^{(k)}(x)$ , where the  $u^{(k)}$ s are defined in (2.3) and  $p$  now denotes the Neumann kernel. Let us first record a simple estimate.

**Lemma 3.1.** *There exists a finite  $C$  such that, for all  $s > 0$ ,*

$$(3.5) \qquad \sup_{y \in (0,L)} p_{2s}(y, y) \leq 2(8\pi s)^{-1/2} + C.$$

*Proof.* It is easy to see that

$$(3.6) \qquad C := \sup_{y \in \mathbf{R}} \sup_{s > 0} \left[ 2 \sum_{n=1}^\infty \Gamma_{2s}(2nL) + \sum_{n \in \mathbf{Z}: |nL-y| \geq L/2} \Gamma_{2s}(2|nL-y|) \right]$$

is finite. Because there can be at most one choice of  $n \in \mathbf{Z}$  such that  $|nL-x| < L/2$  for each  $x \in [0, L]$  and  $\Gamma_{2s}(0) = (8\pi s)^{-1/2}$ , (2.8) implies that  $p_{2s}(y, y) \leq 2\Gamma_{2s}(0) + C$  for all  $s > 0$  and every  $y \in (0, L)$ . This has the desired result.  $\square$

Since  $\int_0^L p_t(x, y) dy = 1$  and  $\int_0^L [p_{t-s}(x, y)]^2 dx = p_{2(t-s)}(y, y)$ , we square and integrate (2.1) in order to find that, because  $\|P_t u_0\|_{L^2}^2 \leq \|u_0\|_{L^2}^2$ , we have for all  $k \geq 0$  and  $t > 0$ ,

$$\begin{aligned} \mathbb{E} \left( \|u_t^{(k+1)}\|_{L^2}^2 \right) &\leq \|u_0\|_{L^2}^2 + \lambda^2 L_\sigma^2 \int_0^t ds \int_0^L dy p_{2(t-s)}(y, y) \mathbb{E} \left( |u_s^{(k)}(y)|^2 \right) \\ (3.7) \qquad \qquad \qquad &\leq \|u_0\|_{L^2}^2 + 2\lambda^2 L_\sigma^2 \int_0^t \left( [8\pi(t-s)]^{-1/2} + C \right) \mathbb{E} \left( \|u_s^{(k)}\|_{L^2}^2 \right) ds; \end{aligned}$$

see Lemma 3.1. Instead of iterating this we use a method of [15] and define, for all  $\beta > 0$  and  $k \geq 0$ ,

$$(3.8) \qquad \qquad \qquad M^{(k)}(\beta) := \sup_{t \geq 0} \left[ e^{-\beta t} \mathbb{E} \left( \|u_t^{(k)}\|_{L^2}^2 \right) \right].$$

We multiply both sides of (3.7) by  $\exp(-\beta t)$  and optimize over  $t \geq 0$  in order to see that

$$\begin{aligned} M^{(k+1)}(\beta) &\leq \|u_0\|_{L^2}^2 + 2\lambda^2 L_\sigma^2 M^{(k)}(\beta) \sup_{t \geq 0} \int_0^t e^{-\beta(t-s)} \left( [8\pi(t-s)]^{-1/2} + C \right) ds \\ (3.9) \qquad \qquad \qquad &\leq \|u_0\|_{L^2}^2 + 2\lambda^2 L_\sigma^2 \left( [8\beta]^{-1/2} + C\beta^{-1} \right) M^{(k)}(\beta). \end{aligned}$$

We apply this with  $\beta := \lambda^4 L_\sigma^4 / (2\delta)$  where  $\delta \in (0, 1)$  is fixed. In this way we find that

$$(3.10) \qquad \qquad \qquad M^{(k+1)} \left( \frac{\lambda^4 L_\sigma^4}{2\delta} \right) \leq A + \left( \delta^{1/2} + \frac{4C\delta}{\lambda^2 L_\sigma^2} \right) M^{(k)} \left( \frac{\lambda^4 L_\sigma^4}{2\delta} \right).$$

where  $A := \|u_0\|_{L^2}^2$ . For all  $\varepsilon > 0$ , small enough to ensure that  $\delta^{1/2}(1 + \varepsilon) < 1$ , there exists  $\lambda_0 > 0$  such that the coefficient of  $M^{(k)}(\dots)$  in (3.10) is less than  $\delta^{1/2}(1 + \varepsilon)$  whenever  $\lambda > \lambda_0$ . Since  $M^{(0)}(\beta) = A$  for all  $\beta > 0$ , it follows that  $M^{(k)}(\lambda^4 L_\sigma^4 / 2\delta) \leq A / (1 - \delta^{1/2}(1 + \varepsilon)) := B$ , uniformly for all  $k \geq 0$  and  $\lambda > \lambda_0$ . Therefore, Fatou's lemma shows that for all  $\lambda > \lambda_0$  and  $t > 0$ ,

$$(3.11) \qquad \qquad \qquad |\mathcal{E}_t(\lambda)|^2 \leq e^{\lambda^4 L_\sigma^4 t / (2\delta)} \liminf_{k \rightarrow \infty} M^{(k)} \left( \frac{\lambda^4 L_\sigma^4}{2\delta} \right) \leq B \exp \left( \frac{\lambda^4 L_\sigma^4 t}{2\delta} \right).$$

This implies the upper bound of Theorem 1.2, since  $\delta \in (0, 1)$  is arbitrary. □

#### 4. A STOCHASTIC WAVE EQUATION

In this section we consider the non-linear stochastic wave equation

$$(4.1) \qquad \partial_t^2 w_t(x) = \partial_x^2 w_t(x) + \lambda \sigma(w_t(x)) \xi \qquad \text{for } x \in \mathbf{R} \text{ and } t > 0,$$

subject to non-random initial function  $w_0(x) \equiv 0$  and non-random non-negative initial velocity  $v_0 \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$  such that  $\|v_0\|_{L^2(\mathbf{R})} > 0$ . It is well known (see Dalang [11] for the general theory, as well as Dalang and Mueller [12], for the existence of an  $L^2$ -valued solution) that there exists a unique continuous solution  $w$  to (4.1) that satisfies the moment conditions,  $\sup_{t \in [0, T]} \sup_{x \in \mathbf{R}} \mathbb{E}(|w_t(x)|^k) + \sup_{t \in [0, T]} \mathcal{E}_t(\lambda) < \infty$ , for every  $k \in [2, \infty)$  and  $T > 0$ .

The main results of this section are the following bounds on the energy  $\mathcal{E}_t(\lambda) := \{\mathbb{E}(\|u_t\|_{L^2(\mathbf{R})}^2)\}^{1/2}$  of the solution to (4.1).

**Theorem 4.1.** *For every  $t > 0$ ,*

$$(4.2) \quad \ell_\sigma t / \sqrt{8} \leq \liminf_{\lambda \rightarrow \infty} \lambda^{-1} \log \mathcal{E}_t(\lambda) \leq \limsup_{\lambda \rightarrow \infty} \lambda^{-1} \log \mathcal{E}_t(\lambda) \leq L_\sigma t / \sqrt{8}.$$

*Remark 4.2.* In the case that  $\sigma(x) = \theta x$  for some constant  $\theta > 0$  (this is the hyperbolic Anderson model [12]), Theorem 4.1 implies that  $\lambda^{-1} \log \mathcal{E}_t(\lambda) \rightarrow \theta t / \sqrt{8}$  as  $\lambda \rightarrow \infty$ .  $\square$

Here and throughout, we define  $H(r) := r \min(1, r)$  for all  $r > 0$ . The preceding theorem requires the following elementary real-variable fact.

**Lemma 4.3.** *If  $g \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$  is non-negative and  $\|g\|_{L^2(\mathbf{R})} > 0$ , then there exist positive and finite constants  $A_1$  and  $A_2$  such that*

$$(4.3) \quad A_1 H(t) \leq \int_{-t}^t dz \int_{-t}^t dy (g * \tilde{g})(y - z) \leq A_2 H(t),$$

for all  $t > 0$ , where  $\tilde{g}(x) := g(-x)$  for all  $x \in \mathbf{R}$ .

*Proof.* Define  $h := g * \tilde{g}$  and note that: (i)  $h \geq 0$ ; (ii)  $h \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$  with  $\|h\|_{L^1(\mathbf{R})} = \|g\|_{L^1(\mathbf{R})}^2$  and  $h(0) = \|g\|_{L^2(\mathbf{R})}^2$ . Furthermore,  $h(x)$  is maximized at  $x = 0$ , thanks to well-known facts about continuous, positive-definite functions. Now we put these facts together in order to see that  $\int_{-t}^t dy \int_{-t}^t dz h(y - z) \leq 4h(0)t^2 \wedge 2t\|h\|_{L^1(\mathbf{R})} \leq \text{const} \cdot H(t)$ . This proves the upper bound. On the other hand,  $\int_{-t}^t dy \int_{-t}^t dz h(y - z) = (4 + o(1))t^2 h(0)$  as  $t \downarrow 0$  and  $\int_{-t}^t dy \int_{-t}^t dz h(y - z) = (2 + o(1))t\|h\|_{L^1(\mathbf{R})}$  as  $t \uparrow \infty$ . The lower bound follows from these observations, since  $h$  is non-negative.  $\square$

*Proof of Theorem 4.1.* The solution to the stochastic wave equation (4.1) can be written in mild form as follows:

$$(4.4) \quad w_t(x) = \frac{1}{2} W_t(x) + \frac{1}{2} \lambda \int_{(0,t) \times \mathbf{R}} \mathbf{1}_{[0,t-s]}(|x-y|) \sigma(w_s(y)) \xi(ds dy),$$

where  $W_t(x) := \int_{-t}^t v_0(x-y) dy$ . Therefore, the Walsh isometry for stochastic integrals assures us that

$$\mathbb{E}(|w_t(x)|^2) = \frac{1}{4} |W_t(x)|^2 + \frac{1}{4} \lambda^2 \int_0^t ds \int_{-\infty}^{\infty} dy \mathbf{1}_{[0,t-s]}(|x-y|) \mathbb{E}(|\sigma(w_s(y))|^2),$$

whence by Fubini's theorem and the fact that  $|\sigma(z)| \leq L_\sigma |z|$  for all  $z \in \mathbf{R}$ ,

$$(4.5) \quad |\mathcal{E}_t(\lambda)|^2 \leq \frac{1}{4} \|W_t\|_{L^2(\mathbf{R})}^2 + \frac{1}{2} \lambda^2 L_\sigma^2 \int_0^t (t-s) ds \int_{-\infty}^{\infty} dy \mathbb{E}(|w_s(y)|^2).$$

Since  $\|W_t\|_{L^2(\mathbf{R})}^2 = \int_{-t}^t dy \int_{-t}^t dz (v_0 * \tilde{v}_0)(y-z)$ , Lemma 4.3 ensures that

$$(4.6) \quad |\mathcal{E}_t(\lambda)|^2 \leq A_2 t^2 + \frac{1}{2} \lambda^2 L_\sigma^2 \int_0^t (t-s) |\mathcal{E}_s(\lambda)|^2 ds \quad (t > 0).$$

Since  $\sup_{t \geq 0} [t^2 e^{-\beta t}] = 4(e\beta)^{-2}$ , the preceding implies that, for all  $\beta > 0$  fixed, the quantity  $\mathcal{F}(\beta) := \sup_{t \geq 0} [e^{-\beta t} |\mathcal{E}_t(\lambda)|^2]$  solves

$$(4.7) \quad \begin{aligned} \mathcal{F}(\beta) &\leq 4A_2 (e\beta)^{-2} + \frac{1}{2} \lambda^2 L_\sigma^2 \mathcal{F}(\beta) \sup_{t \geq 0} \int_0^t (t-s) e^{-\beta(t-s)} ds \\ &\leq 4A_2 (e\beta)^{-2} + \lambda^2 L_\sigma^2 \mathcal{F}(\beta) (\sqrt{2}\beta)^{-2}. \end{aligned}$$

Let us choose and fix an arbitrary  $\delta \in (0, 1)$  and define  $\beta_* := \lambda L_\sigma / \sqrt{2(1 - \delta)}$ . Thus, (4.7) implies that  $\mathcal{F}(\beta_*) \leq 8A_2\delta^{-1}(e\lambda L_\sigma)^{-2}$ . This readily yields the upper bound of the theorem.

First we prove the corresponding lower bound for  $t \in [0, 1]$ . The derivation of (4.6) yields the following bound in the present context:

$$(4.8) \quad |\mathcal{E}_t(\lambda)|^2 \geq A_1 H(t) + \frac{1}{2} \lambda^2 \ell_\sigma^2 \int_0^t (t - s) |\mathcal{E}_s(\lambda)|^2 ds \quad (t \geq 0),$$

thanks to Lemma 4.3. Since  $H(t) = t^2$  for  $t \in [0, 1]$ , we iterate to obtain  $|\mathcal{E}_t(\lambda)|^2 \geq A_1 \sum_{j=0}^\infty (\lambda^2 \ell_\sigma^2 / 2)^j \mathcal{J}_j(t)$  ( $0 \leq t \leq 1$ ), where  $\mathcal{J}_0(t) := t^2$  and  $\mathcal{J}_j(t) := \int_0^t (t - s) \mathcal{J}_{j-1}(s) ds$  for  $j \geq 1$ . By induction,  $\mathcal{J}_j(t) = 2t^{2j+2} / (2j + 2)!$ . Therefore, it follows that, as long as  $0 < t \leq 1$ ,

$$(4.9) \quad |\mathcal{E}_t(\lambda)|^2 \geq 2A_1 \left( \frac{\lambda \ell_\sigma}{\sqrt{2}} \right)^{-2} \sum_{j=1}^\infty \left( \frac{\lambda \ell_\sigma t}{\sqrt{2}} \right)^{2j} \frac{1}{(2j)!}.$$

Because  $\sum_{j=1}^\infty r^{2j} / (2j)! \geq r^{-1} \sum_{j=1}^\infty r^{2j+1} / (2j + 1)!$  for all  $r > 0$  and  $0 < t \leq 1$ , it follows that

$$(4.10) \quad |\mathcal{E}_t(\lambda)|^2 \geq A_1 \left( \frac{\lambda \ell_\sigma t}{\sqrt{2}} \right)^{-3} \sum_{j=2}^\infty \left( \frac{\lambda \ell_\sigma t}{\sqrt{2}} \right)^j \frac{1}{j!},$$

for all  $\lambda$  sufficiently large. This proves the theorem when  $t \leq 1$ .

When  $t > 1$ , we appeal to (4.8) in order to see that  $|\mathcal{E}_t(\lambda)|^2 \geq A_1 H(1) + \frac{1}{2} \lambda^2 \ell_\sigma^2 \int_0^t (t - s) |\mathcal{E}_s(\lambda)|^2 ds$ . Since  $H(1) = 1$  and  $\int_0^t (t - s) ds = \frac{1}{2} \mathcal{J}_0(t)$ , the iteration and the already-proved part of Theorem 4.1 implies that

$$(4.11) \quad |\mathcal{E}_t(\lambda)|^2 \geq A_1 \sum_{j=0}^\infty \left( \frac{\lambda \ell_\sigma t}{\sqrt{2}} \right)^{2j} \frac{1}{(2j)!} \geq \frac{A_1}{2} \left( \frac{\lambda \ell_\sigma}{\sqrt{2}} \right)^{-1} \sum_{j=0}^\infty \left( \frac{\lambda \ell_\sigma t}{\sqrt{2}} \right)^j \frac{1}{j!}$$

for all  $\lambda$  sufficiently large. The  $t > 1$  case follows from these. □

**Postscript.** Recently, M. Foondun and M. Joseph [14] have proved, among other results, that the lower energy estimate of Theorem 1.1 is not sharp. That is, they prove that  $\liminf_{\lambda \rightarrow \infty} \lambda^{-4} \log \mathcal{E}_t(\lambda) > 0$  for all  $t > 0$ .

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REFERENCES

[1] M. Balázs, J. Quastel, and T. Seppäläinen, *Fluctuation exponent of the KPZ/stochastic Burgers equation*, J. Amer. Math. Soc. **24** (2011), no. 3, 683–708, DOI 10.1090/S0894-0347-2011-00692-9. MR2784327 (2012e:60165)

[2] P. H. Baxendale and B. L. Rozovskii, *Kinematic dynamo and intermittence in a turbulent flow*, Geophys. Astrophys. Fluid Dynam. **73** (1993), no. 1-4, 33–60, DOI 10.1080/03091929308203618. Magnetohydrodynamic stability and dynamos (Chicago, IL, 1992). MR1289018 (95e:76100)

- [3] Lorenzo Bertini and Nicoletta Cancrini, *The stochastic heat equation: Feynman-Kac formula and intermittence*, J. Statist. Phys. **78** (1995), no. 5-6, 1377–1401, DOI 10.1007/BF02180136. MR1316109 (95j:60093)
- [4] B. Blümich (1987). White noise nonlinear system analysis in nuclear magnetic resonance spectroscopy, *Progr. NMR Spectr.* **19**, 331–417.
- [5] Julian Fernández Bonder and Pablo Groisman, *Time-space white noise eliminates global solutions in reaction-diffusion equations*, Phys. D **238** (2009), no. 2, 209–215, DOI 10.1016/j.physd.2008.09.005. MR2516340 (2010f:35451)
- [6] René A. Carmona and S. A. Molchanov, *Parabolic Anderson problem and intermittency*, Mem. Amer. Math. Soc. **108** (1994), no. 518, viii+125, DOI 10.1090/memo/0518. MR1185878 (94h:35080)
- [7] Ivan Corwin, *The Kardar-Parisi-Zhang equation and universality class*, Random Matrices Theory Appl. **1** (2012), no. 1, 1130001, 76, DOI 10.1142/S2010326311300014. MR2930377
- [8] M. Cranston and S. Molchanov, *Quenched to annealed transition in the parabolic Anderson problem*, Probab. Theory Related Fields **138** (2007), no. 1-2, 177–193, DOI 10.1007/s00440-006-0020-7. MR2288068 (2008h:60066)
- [9] M. Cranston, T. S. Mountford, and T. Shiga, *Lyapunov exponent for the parabolic Anderson model with Lévy noise*, Probab. Theory Related Fields **132** (2005), no. 3, 321–355, DOI 10.1007/s00440-004-0346-y. MR2197105 (2007h:60053)
- [10] M. Cranston, T. S. Mountford, and T. Shiga, *Lyapunov exponents for the parabolic Anderson model*, Acta Math. Univ. Comenian. (N.S.) **71** (2002), no. 2, 163–188. MR1980378 (2004d:60162)
- [11] Robert C. Dalang, *Extending the martingale measure stochastic integral with applications to spatially homogeneous s.p.d.e.'s*, Electron. J. Probab. **4** (1999), no. 6, 29 pp. (electronic), DOI 10.1214/EJP.v4-43. MR1684157 (2000b:60132)
- [12] Robert C. Dalang and Carl Mueller, *Some non-linear S.P.D.E.'s that are second order in time*, Electron. J. Probab. **8** (2003), no. 1, 21 pp. (electronic), DOI 10.1214/EJP.v8-123. MR1961163 (2004a:60118)
- [13] Frank den Hollander, *Random polymers*, Lecture Notes in Mathematics, vol. 1974, Springer-Verlag, Berlin, 2009. Lectures from the 37th Probability Summer School held in Saint-Flour, 2007. MR2504175 (2010h:60265)
- [14] M. Foondun and M. Joseph (2014). Remarks on the nonlinear noise excitability of some stochastic heat equations, Preprint available at <http://arxiv.org/pdf/1402.0084.pdf>.
- [15] Mohammud Foondun and Davar Khoshnevisan, *Intermittence and nonlinear parabolic stochastic partial differential equations*, Electron. J. Probab. **14** (2009), no. 21, 548–568, DOI 10.1214/EJP.v14-614. MR2480553 (2010c:60183)
- [16] Jürgen Gärtner and Wolfgang König, *The parabolic Anderson model*, Interacting stochastic systems, Springer, Berlin, 2005, pp. 153–179, DOI 10.1007/3-540-27110-4.8. MR2118574 (2005k:82042)
- [17] J. D. Gibbon and E. S. Titi, *Cluster formation in complex multi-scale systems*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **461** (2005), no. 2062, 3089–3097, DOI 10.1098/rspa.2005.1548. MR2172218 (2006d:82080)
- [18] Stanley Kaplan, *On the growth of solutions of quasi-linear parabolic equations*, Comm. Pure Appl. Math. **16** (1963), 305–330. MR0160044 (28 #3258)
- [19] Mehran Kardar, *Replica Bethe ansatz studies of two-dimensional interfaces with quenched random impurities*, Nuclear Phys. B **290** (1987), no. 4, 582–602, DOI 10.1016/0550-3213(87)90203-3. MR922846 (89f:82058)
- [20] Kardar, M., G. Parisi, and Y.-C. Zhang (1986). *Dynamic scaling of growing interfaces*, Phys. Rev. Lett. **56**(9), 889–892.
- [21] Khoshnevisan, D. and K. Kim (2013). *Non-linear noise excitation of intermittent stochastic PDEs and the topology of LCA groups*, Ann. Probab., to appear, preprint available at <http://arxiv.org/pdf/1302.3266.pdf>.
- [22] Andrew J. Majda, *The random uniform shear layer: an explicit example of turbulent diffusion with broad tail probability distributions*, Phys. Fluids A **5** (1993), no. 8, 1963–1970, DOI 10.1063/1.858823. MR1228390 (94d:76041)
- [23] Benoit B. Mandelbrot, *The fractal geometry of nature*, W. H. Freeman and Co., San Francisco, CA, 1982. Schriftenreihe für den Referenten. [Series for the Referee]. MR665254 (84h:00021)

- [24] Stanislav A. Molchanov, *Ideas in the theory of random media*, Acta Appl. Math. **22** (1991), no. 2-3, 139–282, DOI 10.1007/BF00580850. MR1111743 (92m:82067)
- [25] John B. Walsh, *An introduction to stochastic partial differential equations*, École d'été de probabilités de Saint-Flour, XIV—1984, Lecture Notes in Math., vol. 1180, Springer, Berlin, 1986, pp. 265–439, DOI 10.1007/BFb0074920. MR876085 (88a:60114)
- [26] Ya. B. Zel'dovich, A. A. Ruzmaïkin, and D. D. Sokoloff, *The almighty chance*, World Scientific Lecture Notes in Physics, vol. 20, World Scientific Publishing Co., Inc., River Edge, NJ, 1990. Translated from the Russian by Anvar Shukurov. MR1141627 (93b:82001)

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