ON $\mu$-STATISTICAL CONVERGENCE

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Abstract. The concept of $\mu$-statistical convergence at a point for measurable functions in measurable space with a measure is introduced in this work. This concept is a generalization of a similar idea about the sequence of numbers. We also introduce the concept of $\mu$-statistical fundamentality at a point, and the equivalence of these two concepts is proved. The concept of $\mu$-statistical convergence at a point generalizes the usual one of the limit of a function at a point.

1. Introduction

Apparently the concept of statistical convergence of a sequence of numbers, as the generalization of the classical concept of a limit of a sequence, was first introduced in [1]. In [2], [3], [4], [5] the basic properties of statistically convergent sequences have been studied. Later there has appeared much research with various generalizations of this concept (see [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16]). In [17], [18], [19] this concept was used in the approximation theory by positive operators.

We consider in this work a measurable space with a measure on the real axis. We introduce the concept of a $\mu$-statistical limit of measurable function at a point. Also is introduced the concept of a $\mu$-statistical fundamentality at a point, and the equivalence of these two concepts is proved. This concept is a generalization of a similar one of statistical convergence for a sequence.

2. $\mu$-statistical convergence

We will use the standard notation. $N$ will be the set of all positive integers; $R$ is the set of all real numbers; $\exists$ will mean “there exist(s)”; $\exists !$ will mean “there exists a unique”; $\Rightarrow$ will mean “it follows”; $\Leftrightarrow$ will mean equivalence.

Let $\mathcal{B}$ be a $\sigma$-algebra of Borel subsets in the segment $I \equiv [a, +\infty)$ and $(\mathcal{B}; \mu)$ be a measurable space with $\sigma$-finite measure $\mu : \mu (I) = +\infty$, where $a \in R$ is some number. Consider $\mathcal{B}$-measurable function $f : I \to R$. The measure of the set $e \in \mathcal{B}$ will be denoted by $|e|$, i.e. $|e| = \mu (e)$. Let $M \in \mathcal{B}$ be some set.

Definition 2.1. We say that the infinitely remote point $\infty$ is a point $\mu$-statistical ($\mu$-stat) density for $M$ if and only if

$$\lim_{x \to \infty} \frac{|M \cap I_x|}{|I_x|} = 1,$$

where $I_x \equiv [a, x]$.
Let \( \varepsilon > 0 \) be an arbitrary number and let
\[
A_\varepsilon^f = \{ x \in I : |f(x) - A| \geq \varepsilon \},
\]
where \( A \in \mathbb{R} \) is some number.

**Definition 2.2.** We say that \( f \) has a \( \mu \)-stat \((\mu\text{-statistical})\) limit \( A \) at infinity if and only if
\[
\lim_{x \to \infty} \frac{|A_\varepsilon^f \cap I_x|}{|I_x|} = 0, \quad \forall \varepsilon > 0,
\]
and this limit will be denoted as \( \mu\text{-stat} \lim_{x \to \infty} f(x) = A \).

Let \( M^c \equiv I \setminus M \) be a complement of a set \( M \subset I \). From the relation \( I_x = \lfloor A_\varepsilon^f \cap I_x \rfloor \cup \lfloor (A_\varepsilon^f)^c \cap I_x \rfloor \) it directly follows that the condition (2.1) is equivalent to the following one:
\[
\lim_{x \to \infty} \frac{|(A_\varepsilon^f)^c \cap I_x|}{|I_x|} = 1, \quad \forall \varepsilon > 0.
\]

Let us show that if there exists a \( \mu\text{-stat} \) limit, then it is unique. Let \( A \neq B \) and \( B \) also be a \( \mu\text{-stat} \) limit for \( f \). Take \( 0 < \varepsilon < \frac{1}{2} |A - B| \). Then it is clear that \( (A_\varepsilon^f)^c \cap (B_\varepsilon^f)^c = \emptyset \). Consequently, from
\[
\left[ (A_\varepsilon^f)^c \cap I_x \right] \cup \left[ (B_\varepsilon^f)^c \cap I_x \right] \subset I_x,
\]
it follows that
\[
\left| \left[ (A_\varepsilon^f)^c \cap I_x \right] \right| + \left| \left[ (B_\varepsilon^f)^c \cap I_x \right] \right| \leq |I_x|,
\]
i.e.
\[
\frac{|(A_\varepsilon^f)^c \cap I_x|}{|I_x|} + \frac{|(B_\varepsilon^f)^c \cap I_x|}{|I_x|} \leq 1.
\]
Passing to the limit as \( x \to \infty \) and using (2.2), we arrive at a contradiction.

It is easy to see that if \( \lim_{x \to \infty} f(x) = A \), then \( \exists \mu\text{-stat} \lim \text{ and } \mu\text{-stat} \lim_{x \to \infty} f(x) = A \). The converse is not true. Let \( \mu \) be a Lebesgue measure. Consider the Dirichlet function
\[
D(x) = \begin{cases} 
0, & x \in I \setminus Q_I, \\
1, & x \in Q_I,
\end{cases}
\]
where \( Q_I \) are rational numbers from \( I \). It is absolutely obvious that \( \mu\text{-stat} \lim_{x \to \infty} D(x) = 0 \), and \( \lim_{x \to \infty} D(x) \) does not exist.

From
\[
\lambda \neq 0 : \{ x : |\lambda f(x) - \lambda A| \geq \varepsilon \} \Leftrightarrow \{ x : |f(x) - A| \geq \frac{\varepsilon}{|\lambda|} \},
\]
it directly follows that
\[
\mu - \text{st} \lim_{x \to \infty} (\lambda f(x)) = \lambda (\mu - \text{st} \lim_{x \to \infty} f(x)).
\]
Let
\[
\mu - \text{st} \lim_{x \to \infty} f(x) = A; \mu - \text{st} \lim_{x \to \infty} g(x) = B.
\]
Then it is clear that
\[
\{ x : |f(x) + g(x) - A - B| \geq \varepsilon \}
\subset \left[ \{ x : |f(x) - A| \geq \frac{\varepsilon}{2} \} \cup \{ x : |g(x) - B| \geq \frac{\varepsilon}{2} \} \right].
\]
Consequently
\[ | \{ x : | f + g - A - B | \geq \varepsilon \} \cap I_x | \leq \left| \left\{ x : | f - A | \geq \frac{\varepsilon}{2} \right\} \cap I_x \right| + \left| \left\{ x : | g - B | \geq \frac{\varepsilon}{2} \right\} \cap I_x \right|. \]

Hence we obtain
\[ \mu-st \lim_{x \to \infty} (f(x) + g(x)) = A + B. \]

Thus, \( \mathcal{B} \) measurable functions with \( \mu-st \lim \) at infinity form a linear space, and we denote this space by \( \mathcal{B}_{st} \). The set of all subsets of \( I \), for which the infinity is the point of \( \mu-st \) density, will be denoted by \( I_{st}^\infty \). Assume that we have

\[ (2.3) \lim_{x \to \infty} x \in M(f(x)) = A, \ M \in I_{st}^\infty. \]

This implies that for \( \forall \varepsilon > 0, \exists x_0^\varepsilon \in I: \)
\[ | f(x) - A | < \varepsilon, \ \forall x \in [M \cap (x_0^\varepsilon, \infty)]. \]

It is absolutely obvious that for \( \forall t \in I : \)
\[ \lim_{x \to \infty} \frac{| M_t \cap I_x |}{| I_x |} = 1, \ \text{where} \ M_t \equiv M \cap I_t, I_t \equiv [a, t], \]
if \( |I| = \mu(I) = +\infty \). The latter follows from the \( \sigma \)-finiteness of the measure \( \mu \).

Consequently
\[ \lim_{x \to \infty} \frac{| M_t^C \cap I_x |}{| I_x |} = 0, \ \forall t \in I. \]

It is absolutely clear that \( (A_f^\xi)^C \supset M_0^\xi \). As a result, we obtain
\[ \frac{| (A_f^\xi)^C \cap I_x |}{| I_x |} \geq \frac{| M_0^\xi \cap I_x |}{| I_x |} \to 1, \ x \to \infty. \]

Hence we obtain
\[ (2.4) \mu-st \lim_{x \to \infty} f(x) = A. \]

Thus, if the relation \( (2.3) \) holds, then the relation \( (2.4) \) is true.

Now, let the relation \( (2.4) \) hold. Hence, for \( \forall \varepsilon > 0 : \)
\[ \lim_{x \to \infty} \frac{| (A_f^\xi)^C \cap I_x |}{| I_x |} = 1. \]

Put
\[ M_n \equiv \left\{ x \in I : | f(x) - A | < \frac{1}{n} \right\}, \ n \in N. \]

It is clear that
\[ (2.5) \lim_{x \to \infty} \frac{| M_n \cap I_x |}{| I_x |} = 1, \ \forall n \in N, \]
and \( M_1 \supset M_2 \supset \ldots \). Take \( \forall x_1 \in M_1 \) and let \( \xi_1 = x_1 \). From \( (2.5) \) we obtain that \( \exists x_2 > \xi_1, x_2 \in M_2; \ \xi_2 = \max \{ x_1, 1 \} : \)
\[ \frac{| M_2 \cap I_x |}{| I_x |} \geq \frac{1}{2}, \ \forall x \geq x_2. \]

Continuing this process, we obtain a sequence \( \{ x_n \}_{n \in N} : x_n \in M_n, \ x_1 < x_2 \ldots, \) and
\[ \frac{| M_n \cap I_x |}{| I_x |} \geq \frac{n - 1}{n}, \ \forall x \geq x_n \geq \xi_n. \]
where $\xi_n = \max \{x_{n-1}; n-1\}$. Put

$$A_0 = [a, x_1], \ A_n = [x_n, x_{n+1}) \cap M_n, \ \forall n \in N.$$ 

Let $M = \bigcup_{n=0}^{\infty} A_n$. Consider $M_{n_0} \cap I_x$, where $x \in [x_{n_0}, x_{n_0+1})$ is some point. Let $y \in M_{n_0} \cap I_x \Rightarrow |f(y) - A| < \frac{1}{n_0}$. It is clear that $\exists k_0 \in \{0; \ldots; n_0\}: y \in [x_{k_0}, x_{k_0+1})$, where $x_0 = a$. We have

$$|f(y) - A| < \frac{1}{k_0} \Rightarrow y \in A_{k_0} \Rightarrow y \in M \cap I_x.$$ 

Consequently, $M_{n_0} \cap I_x \subset M \cap I_x$ and

$$\frac{|M \cap I_x|}{|I_x|} \geq \frac{|M_{n_0} \cap I_x|}{|I_x|} \geq \frac{n_0 - 1}{n_0}.$$ 

As a result

$$\lim_{x \to \infty} \frac{|M \cap I_x|}{|I_x|} = 1.$$ 

Take $\forall \varepsilon > 0$. Since $x_n \to \infty$ as $n \to \infty$, we have $\exists n_0 : \frac{1}{n_0} \leq \varepsilon$. Let $x \geq x_{n_0}$ and $x \in M$. Consequently, $x \in A_{k_0}$, for some $k_0 \geq n_0$. We have

$$|f(x) - A| < \frac{1}{k_0} \leq \frac{1}{n_0} \leq \varepsilon.$$ 

Thus, $M \in I_{st}^\infty$ and

$$\lim_{x \to \infty, x \in M} f(x) = A.$$ 

So we get the validity of

**Theorem 2.3.** Let $\mu(I) = +\infty$, and $f : I \to R$ be some $\mathcal{B}$-measurable function. Then $f$ has a $\mu$-stat limit (2.3) if and only if there exists a set $M \in I_{st}^\infty$ such that relation (2.3) holds.

We introduce the following

**Definition 2.4.** The sequence $\{a_n\}_{n \in N} \subset I$ is said to have a $st$ lim at infinity if

$$\lim_{n \to \infty} \frac{r(A_n \cap e_n)}{n} = 0, \ \forall \varepsilon > 0,$$

where $e_n \equiv \{1; \ldots; n\}$, $A_n \equiv \{k \in N : |a_k| < \varepsilon\}$ and $r(M)$ is the number of elements $M$. For brevity’s sake, this limit will be denoted as $a_n \xrightarrow{s\text{-}t} \infty$, $n \to \infty$.

Now let’s give another definition of $st$ lim for function $f$ at infinity. First, we introduce the following concept of $st$ lim for a sequence.

**Definition 2.5.** We say that $st \lim_{n \to \infty} a_n = a$ if and only if

$$\lim_{n \to \infty} \frac{r(a_n \cap e_n)}{n} = 0, \ \forall \varepsilon > 0,$$

where $a_n \equiv \{k \in N : |a_k - a| \geq \varepsilon\}$.

This limit will be denoted as $a_n \xrightarrow{s\text{-}t} a$, $n \to \infty$.

And now we give the following

**Definition 2.6.** We say that the function $f : I \to R$ has an $st$ lim at infinity equal to $A$ if $f(a_n) \xrightarrow{s\text{-}t} A$ with $\forall \{a_n\}_{n \in N} \subset I : a_n \xrightarrow{s\text{-}t} \infty$, and it will be denoted as $st \lim_{x \to \infty} f(x) = A$. 
The question then arises: are Definitions 2.2 and 2.6 equivalent to each other? Consider the following function:

\[ f(x) = \begin{cases} n, & x = n \in \mathbb{N}, \\ x^{-1}, & x \in I_0 \setminus \mathbb{N}, \end{cases} \]

where \( I_0 \equiv [1, +\infty) \). Let \( \mu \) be the Lebesgue measure on \( I_0 \). It is absolutely obvious that \( f \) has \( \mu \)-st \( \lim \) equal to zero at infinity in the sense of Definition 2.2, but does not have \( st \lim \) in the sense of Definition 2.6.

Now let \( \exists st \lim_{x \to \infty} f(x) = A \). Consequently, for \( \forall \{a_n\}_{n \in \mathbb{N}} \subset I : a_n \xrightarrow{st} \infty \) we have \( f(a_n) \xrightarrow{st} A \). Assume that the relation \( \mu - st \lim_{x \to \infty} f(x) = A \) does not hold. This means that there exists \( \varepsilon_0 > 0 \) such that the relation

\[ \lim_{x \to \infty} \frac{|A_{\varepsilon_0}^f \cap I_x|}{|I_x|} = 0 \]

does not hold. Thus, \( \exists \delta_0 > 0 \) and \( \exists \{x_n\}_{n \in \mathbb{N}} \subset E : x_1 < x_2 < \ldots, x_n \to \infty \):

\[ \frac{|A_{\varepsilon_0}^f \cap I_{x_n}|}{|I_{x_n}|} \geq \delta_0, \]

i.e.

\[ |A_{\varepsilon_0}^f \cap I_{x_n}| \geq \delta_0 |I_{x_n}| \to \infty, \; n \to \infty. \]

Then \( \exists \{n_k\} : \)

\[ |A_{\varepsilon_0}^f \cap I_{x_{n_k}}| < |A_{\varepsilon_0}^f \cap I_{x_{n_k+1}}|, \]

\[ |A_{\varepsilon_0}^f \cap I_{x_{n_k}}| \to +\infty, \; n \to \infty. \]

It immediately follows that

\[ \exists \{a_k\} : a_k \in A_{\varepsilon_0}^f \cap I_{x_{n_k}}, \; a_k \in I_{x_{n_k}} \setminus I_{x_{n_k-1}}, \; a_k \to \infty. \]

Consequently, \( |f(a_k) - A| \geq \varepsilon_0 \), and we obtain that \( a_k \xrightarrow{st} \infty \), but the relation \( f(a_k) \xrightarrow{st} A \) does not hold. Thus, the following statement is true.

**Statement 2.7.** Let \( \mu(I) = +\infty \). If \( \exists st \lim f(x) \), then \( \exists \mu-st \lim_{x \to \infty} f(x) \) and \( \mu-st \lim_{x \to \infty} f(x) = st \lim_{x \to \infty} f(x) \). The converse is generally not true.

### 3. \( \mu \)-statistical fundamentality

Let us define the concept of \( \mu \)-statistical fundamentality at infinity.

**Definition 3.1.** We say that the function \( f : I \to \mathbb{R} \) is \( \mu \)-stat fundamental at infinity if \( \forall \varepsilon > 0, \; \exists x_\varepsilon \in I : \)

\[ \lim_{x \to \infty} \frac{|X_\varepsilon \cap I_x|}{|I_x|} = 0, \]

where \( X_\varepsilon \equiv \{x \in I : |f(x) - f(x_\varepsilon)| \geq \varepsilon\} \).

Let \( \exists \mu-st \lim_{x \to \infty} f(x) = A \). Then by Theorem 2.3 \( \exists M \in I_{\text{st}}^\infty : \lim_{x \in M} f(x) = A \). Consequently, for \( \forall \varepsilon > 0, \; \exists x_\varepsilon \in M : \)

\[ |f(x) - A| < \varepsilon, \; \forall x \geq x_\varepsilon, \; x \in M. \]

We have

\[ |f(x) - f(x_\varepsilon)| \leq |f(x) - A| + |f(x_\varepsilon) - A| < \varepsilon, \; \forall x \in M, \; x \geq x_\varepsilon. \]
and, consequently,

\[ \lim_{x \to \infty} \frac{|X^c \cap I_x|}{|I_x|} = 1, \]

and this in turn is equivalent to \(3.1\). Thus, if \(\exists \mu \text{-st } \lim_{x \to \infty} f(x),\) then \(f\) is \(\mu\text{-st}\) fundamental at infinity.

Let \(M_1, M_2 \in I^\infty_{st}\). We have

\[ M_1 \cap M_2 = M_1 \cup M_2 \setminus [(M_2 \setminus M_1) \cup (M_1 \setminus M_2)]. \]

Consequently

\[ (3.2) \quad M_1 \cap M_2 \cap I_x = [(M_1 \cup M_2) \cap I_x] \setminus [(M_2 \setminus M_1) \cup (M_1 \setminus M_2)] \cap I_x. \]

As \((M_2 \setminus M_1) \cup (M_1 \setminus M_2)\) \(\cap I_x = ((M_2 \setminus M_1) \cap I_x) \cup ((M_1 \setminus M_2) \cap I_x),\) taking into account relations

\[ (M_2 \setminus M_1) \cap I_x \subset M^c_1 \cap I_x \]

\[ \Rightarrow \frac{|(M_2 \setminus M_1) \cap I_x|}{|I_x|} \leq \frac{|M^c_1 \cap I_x|}{|I_x|} \to 0, \quad x \to \infty, \]

\[ \frac{|(M_1 \setminus M_2) \cap I_x|}{|I_x|} \to 0, \quad x \to \infty, \]

we obtain that

\[ \frac{|((M_2 \setminus M_1) \cup (M_1 \setminus M_2)) \cap I_x|}{|I_x|} \to 0, \quad x \to \infty. \]

From \((M_1 \cap I_x) \subset (M_1 \cup M_2) \cap I_x\) and \(M_1 \in I^\infty_{st}\) it follows that

\[ \frac{|(M_1 \cup M_2) \cap I_x|}{|I_x|} \to 1, \quad x \to \infty, \]

and, consequently, \(M_1 \cup M_2 \in I^\infty_{st}\). Then from \((3.2)\) we directly obtain

\[ \frac{|M_1 \cap M_2 \cap I_x|}{|I_x|} = \frac{|(M_1 \cup M_2) \cap I_x|}{|I_x|} - \frac{|((M_2 \setminus M_1) \cup (M_1 \setminus M_2)) \cap I_x|}{|I_x|} \to 1, \quad x \to \infty, \]

i.e. \(M_1 \cap M_2 \in I^\infty_{st}\). As a result, we get the validity of

**Lemma 3.2.** If \(M_2 \in I^\infty_{st}, \quad k = 1, 2,\) then \(M_1 \cap M_2 \in I^\infty_{st}\).

Now, to the contrary, assume that \(f\) is \(\mu\text{-st}\) fundamental at infinity. Then for \(\varepsilon_1 = 1, \exists x_1 \in I:\)

\[ \lim_{x \to \infty} \frac{|X^c_1 \cap I_x|}{|I_x|} = 0, \]

i.e.

\[ X^c_1 \equiv \{ x \in I : |f(x) - f(x_1)| < 1 \} \in I^c_{st}. \]

Similarly, for \(\varepsilon_2 = \frac{1}{k}, \exists x_2 \in I : X^c_2 \in I^\infty_{st},\) where

\[ X^c_k \equiv \left\{ x \in I : |f(x) - f(x_2)| < \frac{1}{k} \right\}. \]
By Lemma 3.2 we obtain $X_f^c \cap X_g^c \equiv Y_1 \in I_{st}^\infty$. Assume $\tilde{Y}_1 = \{ f(x) : x \in Y_1 \}$. It is absolutely clear that

$$\tilde{Y}_1 \subset I_2 \equiv \left[ f(x_2) - \frac{1}{2}, f(x_2) + \frac{1}{2} \right].$$

Similarly we define $X_f^c \equiv \{ x \in I : |f(x) - f(x_4)| < \frac{1}{4} \}$ and consider $Y_2 = Y_1 \cap X_f^c$. Again by Lemma 3.2 we have $Y_2 \subset I_{st}^\infty$. Let

$$\tilde{Y}_2 \subset I_4 \equiv \left[ f(x_4) - \frac{1}{4}, f(x_4) + \frac{1}{4} \right] \cap I_2.$$

Continuing this process, we obtain a sequence of segments $I_{2n}$ and sets $\tilde{Y}_n \subset I_{2n}$ with the following properties:

$$I_{2n} \supset I_{2n+1} \supset \ldots \supset d(I_{2n}) \leq \frac{1}{2n-1},$$

$$\tilde{Y}_n \equiv \{ f(x) : x \in Y_n \} \subset I_{2n},$$

$$Y_{n-1} \equiv X_{n-1}^c \cap X_n^c ; Y_n \in I_{st}^\infty,$$

$$\tilde{Y}_n \equiv \{ f(x) : x \in Y_n \},$$

where $d(E)$ is the length of the segment $E$.

Obviously, $\exists! A \in \bigcap_{n=1}^{\infty} I_{2n}$. Let us show that $\mu-st \lim_{x \to \infty} f(x) = A$. Take $\forall \varepsilon > 0$. It is clear that $\exists n_0 \in N : I_{2n} \subset (A - \frac{\varepsilon}{2}, A + \frac{\varepsilon}{2})$, $\forall n \geq n_0$. Thus,

$$\tilde{Y}_{n_0} \subset I_{2n_0} \equiv \left\{ f(x_{2n_0}) - \frac{1}{2n_0}, f(x_{2n_0}) + \frac{1}{2n_0} \right\} \cap I_{2n_0-1}.$$

On the other hand

$$\tilde{Y}_{n_0} \equiv \{ f(x) : x \in Y_{n_0} \},$$

and, due to the structure of $Y_{n_0} \in I_{st}^\infty$, where $Y_{n_0} \in Y_{n_0-1} \cap X_{2n_0}^c$, we have $X_{2n_0}^c \equiv \{ x \in I : |f(x) - f(x_{2n_0})| < \frac{1}{2n_0} \}$. Choose $n_0$ from the condition $\frac{1}{2n_0} < \frac{\varepsilon}{2}$. We have

$$|f(x) - A| \leq |f(x) - f(x_{2n_0})| + |f(x_{2n_0}) - A| < |f(x) - f(x_{2n_0})| + \frac{\varepsilon}{2}.$$

It directly follows that

$$x : |f(x) - f(x_{2n_0})| < \frac{1}{2n_0} \subset (A_f^c \equiv A_f^c,$$

i.e. $X_{2n_0}^c \subset (A_f^c \equiv A_f^c$. As $X_{2n_0}^c \in I_{st}^\infty$, it is clear that $(A_f^c \equiv I_{st}^\infty$. Thus, the relation $\mu-st \lim_{x \to \infty} f(x) = A$ is true. So the following theorem is valid.

**Theorem 3.3.** Let $\mu(I) = +\infty$. Then the function $f : I \to R$ is $\mu$-stat fundamental at infinity if and only if $\exists \mu-st \lim_{x \to \infty} f(x)$.

We give the following

**Definition 3.4.** The functions $f; g : I \to R$ are called $\mu$-stat equivalent at infinity if $I_{f,g} \subset I_{st}^\infty$, where $I_{f,g} = \{ x \in I : f(x) = g(x) \}$ (denoted as $f \sim g$).

Assume that $\exists \mu-st \lim_{x \to \infty} f(x) = A$. Then, by Theorem 2.3, $\exists M \in I_{st}^\infty$ such that $\lim_{x \in M} f(x) = A$. Define

$$g(x) \equiv \begin{cases} f(x), & x \in M, \\ A, & x \in M^c. \end{cases}$$

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It is absolutely clear that \( M \subset I_{f,g} \), i.e., \( f \nsim g \). Obviously
\[
(3.3) \quad \lim_{x \to \infty} g(x) = A.
\]

To the contrary, it is clear that if (3.3) holds and \( f \nsim g \), then \( \exists \mu \text{-st} \lim_{x \to \infty} f(x) = A \). Thus, the following theorem is true.

**Theorem 3.5.** Let \( \mu(I) = +\infty \). Then for the \( \mathcal{B} \)-measurable function \( f : I \to \mathbb{R} \) the following statements are equivalent to each other:

a) \( \exists \mu \text{-st} \lim_{x \to \infty} f(x) \);

b) \( f \) is \( \mu \text{-stat} \) fundamental at infinity;

c) \( \exists g : f \nsim g \) and \( \exists \lim_{x \to \infty} g(x) \).

4. \( \mu \text{-stat} \) continuity

Let \( R \) be the real axis, \( \mathcal{B} \) be a \( \sigma \)-algebra of all Borel sets in \( R \) and \( (R; \mathcal{B}; \mu) \) be some space with \( \sigma \)-finite measure \( \mu \) and \( \mu((-\infty, a)) = \mu((a, +\infty)) = +\infty \), \( \forall a \in R \). Let \( E_0 \subset \mathcal{B} \) be some set, and let \( f : E_0 \to R \) be some \( (R; \mathcal{B}) \)-measurable function. Let \( x_0 \) be a limit point of \( E_0 \). Let
\[
E_{x_0}^0 \equiv \left\{ x : x_0 + \frac{1}{x} \in E_0 \right\}.
\]
Put
\[
I_{x_0}^t \equiv \begin{cases} a, & (t - x_0)^{-1}, \frac{1}{t-x_0} \geq a, \\ (t - x_0)^{-1}, & \frac{1}{t-x_0} < a, \end{cases}
\]
where \( a \in R \) is an arbitrary fixed point. We introduce the following concept.

**Definition 4.1.** We say that the function \( f \) has a \( \mu \text{-stat} \) (\( \mu \)-statistical) limit at \( x_0 \), if
\[
\lim_{t \to x_0} \frac{|A_{x_0}^t \cap I_{x_0}^t|}{|I_{x_0}^t|} = 0, \quad \varepsilon > 0,
\]
where
\[
A_{x_0}^t = \left\{ x \in E_{x_0}^0 : |f(x_0 + \frac{1}{x}) - A| \geq \varepsilon \right\}.
\]
This will be denoted as \( \mu \text{-st} \lim_{x \to x_0} f(x) = A \).

It is absolutely clear that if \( \exists \lim_{x \to x_0} f(x) \), then \( \exists \mu \text{-st} \lim_{x \to x_0} f(x) \), and these two are equal. Moreover, if \( \exists \mu \text{-st} \lim_{x \to x_0} f_k(x), k = 1, 2, \) then there are the following limits:
\[
\mu \text{-st} \lim_{x \to x_0} (\lambda f_1(x)) = \lambda \left( \mu \text{-st} \lim_{x \to x_0} f_1(x) \right), \quad \forall \lambda \in \mathbb{R};
\]
\[
\mu \text{-st} \lim_{x \to x_0} (f_1 + f_2) = \mu \text{-st} \lim_{x \to x_0} f_1 + \mu \text{-st} \lim_{x \to x_0} f_2.
\]
Before we continue, we need some more concepts. Let \( M \in \mathcal{B} \) and \( x_0 \in R \) be a limit for \( M \). Put
\[
M_{x_0} \equiv \left\{ x \in R : x_0 + \frac{1}{x} \in M \right\}.
\]
**Definition 4.2.** We say that $x_0$ is a point of $\mu$-stat density for the set $M$ if
\[
\lim_{t \to x_0} \frac{|M_{x_0} \cap I_{t}^x|}{|I_{t}^x|} = 1.
\]

By $\mathcal{R}_{st_{x_0}}$ we denote the subset $\mathcal{R}$ for the elements of which the point $x_0$ is a point of $\mu$-stat density.

Using the results of previous section, we immediately obtain the validity of the following statements.

**Theorem 4.3.** Let $\mu(E_{x_0}^0) = +\infty$, and let $f : E_0 \to R$ be a function. Then $\exists \mu$-st\linebreak $\lim_{x \to x_0} f(x)$ if and only if $\exists M \in \mathcal{R}_{st_{x_0}} : \exists \lim_{x \to x_0} f(x)$, and these limits are equal.

Let us introduce the concept of $\mu$-stat fundamentality at the point $x_0$. Let a Borel measurable function $f : E_0 \to R$ be given and $x_0$ be a limit point of $E_0$. Let
\[
A^{\varepsilon} = \{x \in E_0 : |f(x) - f(x_0)| \geq \varepsilon\}
\]
and
\[
A^{\varepsilon}_{x_0} = \left\{x \in R : x_0 + \frac{1}{x} \in A^{\varepsilon}\right\}.
\]

**Definition 4.4.** We say that the function $f : E_0 \to R$ is $\mu$-stat fundamental at $x_0$, if for $\forall \varepsilon > 0$, $\exists x_0 \in E_0$:
\[
\lim_{t \to x_0} \frac{|A^{\varepsilon}_{x_0} \cap I_{t}^x|}{|I_{t}^x|} = 0.
\]

**Definition 4.5.** The functions $f, g : E_0 \to R$ are called $\mu$-stat equivalent at $x_0$, if $I_{t,g} \in E_{st_{x_0}}$.

Using Theorem 3.5, we obtain the following

**Theorem 4.6.** Let $\mu (E_{x_0}^0) = +\infty$, and let $f : E_0 \to R$ be a given Borel measurable\linebreak function $f : E_0 \to R$. Then the following statements are equivalent to each other:
\begin{enumerate}[(a)]
\item $\exists \mu$-st\linebreak $\lim_{x \to x_0} f(x)$;
\item $f$ is $\mu$-stat fundamental at $x_0$;
\item $\exists g : f \sim g$ at $x_0$.
\end{enumerate}

Quite naturally we come to the following

**Definition 4.7.** Let $f : E_0 \to R$ and $x_0 \in E_0$ be a limit point of $E_0$. $f$ is said to be $\mu$-stat continuous at $x_0$ in $E_0$, if $\mu$-st\linebreak $\lim_{x \to x_0} f(x) = f(x_0)$. $f$ is said to be $\mu$-stat continuous in the set $M \subset E_0$, if it is $\mu$-stat continuous at every point in $M$.

It is absolutely clear that if $f$ and $g$ are $\mu$-stat continuous at $x_0$ (in $M$), then $\lambda f$ and $(f + g)$ are also $\mu$-stat continuous at $x_0$ (in $M$). Denote by $C^{st}(M)$ the set of all $\mu$-stat continuous functions in $M$. Then $C^{st}(M)$ is a linear space.

**Example 4.8.** Let $E \equiv [1, +\infty)$, $\mathcal{B}$ be a $\sigma$-algebra of Borel subsets of $E$. As $\mu$, we take a discrete measure
\[
\mu(A) = \sum_{k=1}^{\infty} \chi_A(k),
\]
where $\chi_A(\cdot)$ is the characteristic function of the set $A \in \mathcal{B}$. Then it is not difficult to see that in this case the $\mu$-statistical limit of the function $f : E \to R$ at infinity coincides with the statistical limit of the corresponding sequence $\{f(n)\}_{n \in N}$.
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REFERENCES


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