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PERIODIC ORBITS WITH PRESCRIBED ABBREVIATED ACTION

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ABSTRACT. We show that for every convex Lagrangian quadratic at infinity there is a real number w_0 such that for every $w > w_0$ the Lagrangian has a periodic orbit with abbreviated action w.

1. INTRODUCTION

Let M be a closed connected smooth Riemannian manifold. Let

$$L : TM \to \mathbb{R}$$

be a smooth convex Lagrangian. This means that L restricted to each T_xM has positive definite Hessian. We shall also assume that L is quadratic at infinity, that is, there exists $R > 0$ such that for each $x \in N$ and $|v|_x > R$, $L(x, v)$ has the form

$$L(x,v) = \frac{1}{2}|v|_x^2 + \theta_x(v) - V(x),$$

where θ is a smooth 1-form on M and $V : M \to \mathbb{R}$ a smooth function.

Let Λ be the set of absolutely continuous curves $x : [0, 1] \to M$, $x(0) = x(1)$, such that $\dot{x}$ has finite L^2-norm. It is well known [Pal63], [Con06]) that Λ has a Hilbert manifold structure compatible with the Riemannian metric on M.

Recall that the free-time action

$$\mathcal{A}_L : \mathbb{R}^+ \times \Lambda \to \mathbb{R}$$

is given by

$$\mathcal{A}_L(b, x) = \int_0^1 b\, L(x(t), \dot{x}(t)/b)\, dt.$$

If the Lagrangian is quadratic at infinity, arguments in Proposition 3.1 in [AS09b] show that $\mathcal{A}_{L+k}$ is a C^1-function with locally Lipschitz derivative.

Recall that the abbreviated action of $y : [0, b] \to M$ is

$$\int_0^b L_v(y(t), \dot{y}(t))\dot{y}(t)\, dt.$$

The abbreviated action is the functional involved in Maupertuis principle in which orbits with energy k can be regarded as critical points of the abbreviated action restricted to holonomic curves with energy k. Equivalently, orbits with energy

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k arise as critical points of the free-time action $\mathcal{A}_{L+k}$. In particular, periodic orbits with energy larger than Mañé's critical value are obtained in [Con06]. (For other results and related approaches see [BT98, Con06, CMP04, Mer10, Nov82, Taĭ91, Taĭ92, AS10, AS09a, AS09b, CFP10, Gin96, GG09b, GG09a, GK99, BM06, MS05] and the references therein.)

In this note we take the inverse viewpoint defining a functional $\mathcal{E}_w$ depending on a real parameter w whose critical points correspond to periodic orbits having abbreviated action w and energy equal to the critical value of the functional $\mathcal{E}_w$. Using this functional, we show the existence of periodic orbits with large enough abbreviated action. Define $\mathcal{E}_w : \mathbb{R}^+ \times \Lambda \to \mathbb{R}$ by

$$\mathcal{E}_w(b, x) = \frac{w - \mathcal{A}_L(b, x)}{b}.$$

Proposition 5 shows that critical points of $\mathcal{E}_w$ correspond to periodic orbits of abbreviated action w. It can also be noted that results by Mañé directly imply that the supremum of $\mathcal{E}_0$ is Mañé's critical value (see Proposition 4). Periodic orbits with large enough abbreviated action are obtained by Proposition 5 and the following

Theorem 1. *There is w_0 such that $\mathcal{E}_w$ has critical points for every $w > w_0$.*

2. Periodic orbits with prescribed abbreviated action

2.1. **Abbreviated action.** Recall that the *energy* $E_L : TM \to \mathbb{R}$ is defined by

$$E_L(x, v) = \frac{\partial L}{\partial v}(x, v).v - L(x, v).$$

Since L is autonomous, E_L is a first integral of the flow Euler-Lagrange flow ϕ_t.

Critical points of $\mathcal{A}_{L+k}$ correspond to periodic orbits with energy k:

Proposition 2. *If (b, x) is a critical point of $\mathcal{A}_{L+k}$, then $y : [0, b] \to M$ given by $y(t) = x(t/b)$ is a solution of the Euler-Lagrange equation of L with energy k (see* [AM78, CDI97, CI99, AS09b]*).*

It is useful to define the average energy function

$$e : \mathbb{R}^+ \times \Lambda \to \mathbb{R}$$

by

$$e(b, x) = \int_0^1 E_L(x(t), \dot{x}(t)/b) = \frac{1}{b}\int_0^b E_L(y(s), \dot{y}(s))\, ds.$$

We state the following straightforward remark for future reference.

Remark 3.

$$\frac{\partial \mathcal{A}_L}{\partial b}(b, x) = -e(b, x).$$

Recall that the abbreviated action of $y : [0, b] \to M$ is

$$\int_0^b L_v(y(t), \dot{y}(t))\dot{y}(t)\, dt$$

and so we define $\mathcal{W} : \mathbb{R}^+ \times \Lambda \to \mathbb{R}$ by $\mathcal{W}(b, x) = \mathcal{A}_L(b, x) + be(b, x)$. Observe that the abbreviated action of y is $\mathcal{W}(b, x)$, being $y(t) = x(t/b)$.

Let $c(L)$ stand for Mañé's critical value, that is, $c(L)$ is the minimum of those $k \in \mathbb{R}$ such that $\mathcal{A}_{L+k}(b, x) \geq 0$ for all $(b, x) \in \mathbb{R}^+ \times \Lambda$ ([Mañ97]).

Proposition 4. $c(L) = \sup \mathcal{E}_0$.

Proof. This is an immediate consequence of Theorem 2-5.2 of [CI99] (which in turn is based on results from [Mañ96]) which shows that $-c(L)$ is the infimum of $\int d\mu$, where μ ranges among the measures of TM supported in closed curves of the form $t \to (y(t), \dot{y}(t))$. $\square$

Proposition 5. *If (b, x) is a critical point of $\mathcal{E}_w$, then $y : [0, b] \to M$ given by $y(t) = x(t/b)$ is a solution of the Euler-Lagrange equation with abbreviated action w and energy $\mathcal{E}_w(b, x)$.*

Proof. First note that

$$\frac{\partial \mathcal{E}_w}{\partial x} = -\frac{1}{b}\frac{\partial \mathcal{A}_L}{\partial x}.$$

Using Remark 3

$$\frac{\partial \mathcal{E}_w}{\partial b} = -\frac{w}{b^2} + \frac{\mathcal{W}(b, x)}{b^2} = \frac{e(b, x) - \mathcal{E}_w(b, x)}{b}.$$

This implies that if (b, x) is critical, then $\mathcal{W}(b, x) = w$, implying in turn $\mathcal{E}_w(b, x) = e(b, x)$. $\square$

3. Minimax principle

Definition 6. Let $f : X \to \mathbb{R}$ be a C^1 map where X is an open set of a Hilbert manifold. We say that f satisfies the *Palais-Smale* condition at level c if every sequence $\{x_n\}$ such that $f(x_n) \to c$ and $||d_{x_n} f|| \to 0$ as $n \to \infty$ has a converging subsequence.

Let X be an open set in a Hilbert manifold and $f : X \to \mathbb{R}$ is a $C^{1,1}$ map (i.e. the derivative of f is locally Lipschitz). The following version of the minimax principle (Proposition 7 below) is a particular case of Proposition 6.3 from [Con06] (which in turn is inspired by that of [HZ94] (see also [Str96])).

Observe that if the vector field $Y = -\nabla f$ is not globally Lipschitz, the gradient flow ψ_t of $-f$ is a priori only a local flow. Given $p \in X$, $t > 0$ define

$$\alpha(p) := \sup\{\, a > 0 \,|\, s \mapsto \psi_s(p) \text{ is defined on } s \in [0, a] \,\}.$$

We say that a function $\tau : X \to [0, +\infty)$ is an *admissible time* if τ is differentiable and $0 \le \tau(x) < \alpha(x)$ for all $x \in X$. Given and admissible time τ, and a subset $F \subset X$ define

$$F_\tau := \{\, \psi_{\tau(p)}(p) \,|\, p \in F \,\}.$$

Let $\mathcal{F}$ be a family of subsets $F \subset X$. We say that $\mathcal{F}$ is *forward invariant* if $F_\tau \in \mathcal{F}$ for all $F \in \mathcal{F}$ and any admissible time τ. Define

$$c(f, \mathcal{F}) = \inf_{F \in \mathcal{F}} \sup_{p \in F} f(p).$$

Proposition 7. *Let f be a $C^{1,1}$-function satisfying the Palais-Smale condition at level $c(f, \mathcal{F})$. Assume also that $\mathcal{F}$ is forward invariant under the gradient flow of $-f$. Suppose that $c = c(f, \mathcal{F})$ is finite and that there is ε such that the gradient flow is relatively complete in the set $[c - \varepsilon \le f \le c + \varepsilon]$. Then $c(f, \mathcal{F})$ is a critical value of f.*

4. Proof of Theorem 1

For the sequel in this section, we assume that M is simply connected. At the end we indicate how the non-simply connected case is treated.

Since M is closed, there is some non-trivial homotopy group $\pi_l(M) \neq 0$. Choose a non-trivial free homotopy class $0 \neq \sigma \in [S^l, M]$. A non-trivial homotopy class of maps $S^l \to M$ can be seen as a family of maps $\Gamma : S^{l-1} \to \mathbb{R}^+ \times \Lambda$ (see e.g. [Kli78, p. 37]). Let $\mathcal{G}$ be the set of all such maps corresponding to the homotopy class σ and let $\mathcal{F}$ be the family of sets $\Gamma(S^{l-1})$, $\Gamma \in \mathcal{G}$. Clearly $\mathcal{F}$ is a forward invariant family. Since the homotopy class σ is non-trivial, its length is bounded away from zero, i.e., (cf. [Kli78, Theorem 2.1.8, p. 37]):

$$inf_\Gamma \;\; max_s \;\; \ell(x(s)) = \rho > 0 \tag{1}$$

where $\Gamma(s) = (b(s), x(s))$.

For the sequel let

$$\alpha_k = c(\mathcal{A}_{L+k}, \mathcal{F}).$$

Let A, B be positive such that $L(x,v) > A|v|^2/2 - B$ for every (x,v) in TM. Take $k_0 > B$ and $k_0 > 2c(L)$ and set

$$w_0 = \alpha_{k_0}.$$

Lemma 8. *$c(-\mathcal{E}_w, \mathcal{F}) < -k_0$ for every $w > w_0$.*

Proof. By definition of α_{k_0} there is $\Gamma \in \mathcal{G}$, $\Gamma(s) = (b(s), x(s))$ such that

$$\mathcal{A}_L(b(s), x(s)) + k_0 b_s < w \qquad \text{for every } s \in S^{l-1}$$

i.e. $-\mathcal{E}_w(\Gamma(s)) < -k_0$ for every $s \in S^{l-1}$. □

Lemma 9. *For every $w > w_0$ there is $k > k_0$ such that $\alpha_k > w$.*

Proof. Let $\Gamma \in \mathcal{G}$ and $(b(s_0), x(s_0)) = \Gamma(s_0)$, such that $\ell(x(s_0)) \geq \rho$.

Let $k \geq k_0$ and $b > 0$. Then

$$\mathcal{A}_L(b, x(s_0)) + kb \geq A\frac{||\dot{x}(s_0)||^2_{L^2}}{2b} + (k-B)b \geq A\frac{\rho^2}{2b} + (k-B)b.$$

The function

$$b \to A\frac{\rho^2}{2b} + (k-B)b$$

takes its minimum at

$$b = \frac{\sqrt{A}\rho}{\sqrt{2(k-B)}},$$

therefore,

$$\max_s \mathcal{A}_{L+k}(\Gamma(s)) \geq \mathcal{A}_L(b(s_0), x(s_0)) + kb_0 \geq \rho\sqrt{2A(k-B)},$$

and hence $\alpha_k \geq \rho\sqrt{2A(k-B)}$. □

Remark 10. Observe that the proof of Lemma 9 also shows that $w_0 > 0$.

Lemma 11. *Let $k > k_0$ be such that $\alpha_{k/2} > w > w_0$. Then $c(-\mathcal{E}_w, \mathcal{F}) > -k$.*

Proof. Let $k > k_0$ be such that $\alpha_{k/2} > w$. Then, for every $\Gamma \in \mathcal{G}$, $\Gamma(s) = (b(s), x(s))$, there is $s_0 \in S^{l-1}$ such that

$$\mathcal{A}_L(b(s_0), x(s_0)) + \frac{k}{2} b(s_0) \geq \alpha_{k/2} > w.$$

Consequently, for every $\Gamma \in \mathcal{G}$ there is $s_0 \in S^{l-1}$ such that

$$-\mathcal{E}_w(\Gamma(s_0)) \geq -\frac{k}{2}$$

implying the lemma. □

Lemma 12. *Let $w > w_0$ and assume that $\{(b_n, x_n)\}$ is a Palais-Smale sequence of $-\mathcal{E}_w$ at level $c(-\mathcal{E}_w, \mathcal{F})$. Then b_n is bounded and bounded away from zero.*

Proof. Let $\{(b_n, x_n)\}$ be a Palais-Smale sequence such that $-\mathcal{E}_w(b_n, x_n) \to c(-\mathcal{E}_w, \mathcal{F})$. By contradiction assume that b_n converges to zero.

We have, by Remark 3,

$$\frac{\partial \mathcal{E}_w}{\partial b}(b_n, x_n) = \frac{e(b_n) - \mathcal{E}_w(b_n, x_n)}{b_n} \to 0$$

and hence

$$e(b_n) - \mathcal{E}_w(b_n, x_n) \to 0.$$

Let k be such that $\alpha_{k/2} > w$. By Lemma 11 we have $\mathcal{E}_w(b_n, x_n) \leq 2k$ for large enough n and hence $e(b_n, x_n) \leq 4k$ for large enough n. Let A_1, B_1, C, D be positive numbers such that

$$L(x, v) \leq \frac{A_1}{2}|v|^2 + B_1$$

and

$$E_L(x, v) \geq \frac{C}{2}|v|^2 - D$$

for all (x, v) in TM. Therefore,

$$C\frac{||\dot{x}_n||^2_{L^2}}{2b_n^2} - D \leq e(b_n, x_n) < 4k$$

for large enough n. Using again that b_n converges to zero we obtain

$$\frac{||\dot{x}_n||^2_{L^2}}{2b_n} \to 0.$$

On the other hand, since $-\mathcal{E}_w(b_n, x_n) \geq -2k$, we have that $w \leq \mathcal{A}_L(b_n, x_n) + 2kb_n$. Therefore,

$$0 < w \leq \mathcal{A}_L(b_n, x_n) + 2kb_n \leq A_1\frac{||\dot{x}_n||^2_{L^2}}{2b_n} + B_1 b_n + 2kb_n,$$

which is a contradiction, since the right-hand side converges to zero.

To see that b_n is bounded, observe that Lemma 8 gives

$$-\mathcal{E}_w(b_n, x_n) \leq -k_0/2$$

for large enough n. Therefore,

$$\mathcal{A}_L(b_n, x_n) + \frac{k_0}{2} b_n \leq w.$$

On the other hand,

$$\mathcal{A}_L(b_n, x_n) + \frac{k_0}{2} b_n = A_L(b_n, x_n) + c(L) b_n + (\frac{k_0}{2} - c(L)) b_n.$$

Recall that $\mathcal{A}_L(b_n, x_n) + c(L)b_n \geq 0$ by the definition of $c(L)$; therefore, b_n is bounded since $k_0 > 2c(L)$. $\square$

Lemma 13. *The flow of $\nabla\mathcal{E}_w$ is relatively complete in $k_0 \leq \mathcal{E}_w \leq k$ if $w > w_0$.*

Proof. By contradiction let $s \to \Gamma(s) = (b(s), x(s))$ be a flow semi-trajectory defined in the maximal interval $(0, a)$ and contained in $k_0 \leq \mathcal{E}_w \leq k$. Let $t_n \in (0, a)$ be a sequence converging to a. By the same argument as in Lemma 6.9 of [Con06], the sequence $\Gamma(t_n)$ is a Cauchy sequence, implying that the sequence $b(t_n)$ converges to $b_0 \in [0, \infty)$. If $b_0 > 0$, the sequence $\Gamma(t_n)$ converges and hence the flow trajectory could be extended. If $b_0 = 0$ there would be a sequence s_n converging to a, such that $b(s_n)$ converges to 0 and such that

$$\frac{db}{ds}(s_n) \leq 0.$$

Then

$$\frac{db}{ds}(s_n) = \frac{\partial \mathcal{E}_w}{\partial b}(b(s_n), x(s_n)) = \frac{e(b(s_n)) - \mathcal{E}_w(b(s_n), x(s_n))}{b(s_n)} \leq 0.$$

Since

$$k_0 \leq \mathcal{E}_w(b(s_n), x(s_n)) \leq k,$$

we conclude that $e(b(s_n), x(s_n))$ is bounded and hence we obtain the same contradiction as in the previous lemma. $\square$

Lemma 14. *$-\mathcal{E}_w$ satisfies the Palais-Smale condition at level $c(-\mathcal{E}_w, \mathcal{F})$.*

Proof. Let $\{(b_n, x_n)\}$ be a sequence in $\mathbb{R}^+ \times \Lambda$ such that $-\mathcal{E}_w(b_n, x_n) \to c(-\mathcal{E}_w, \mathcal{F})$ and such that $\nabla\mathcal{E}_w(b_n, x_n) \to 0$. By Lemma 12 the sequence b_n is bounded and bounded away from zero. Therefore,

$$\frac{\partial \mathcal{A}_L}{\partial x}(b_n, x_n) \to 0,$$

which allows us to apply the argument of Proposition 3.12 of [Con06] (see also [CIPP00] and [Ben86]) which implies that (b_n, x_n) has a converging subsequence in $\mathbb{R}^+ \times \Lambda$. $\square$

Proof of Theorem 1. Lemmas 13 and 14 allow us to apply Proposition 7, completing the proof in the simply connected case. If M is not simply connected, there exists σ, a non-trivial connected component of Λ. Let $\mathcal{F}$ be the family of sets $F = \{(b, x)\}$ such that $x \in \sigma$. Since all curves in σ have length bounded away from zero, equation (1) holds and this implies that all arguments of the simply connected case apply. $\square$

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