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ROGERS AND SHEPHARD INEQUALITY FOR THE ORLICZ DIFFERENCE BODY

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ABSTRACT. The inequalities involving the volume of convex bodies play an important role in convex geometry. In this paper, the Rogers and Shephard inequality for the Orlicz difference body of a convex body in the two-dimensional case is established.

In the last century, the classical Brunn-Minkowski theory was developed by Minkowski, Blaschke, Fenchel and others. The Minkowski addition and the volume (or mixed volumes) are extremely important and powerful concepts in convex geometry; it has been widely applied to various other areas of mathematics. For more details about the Minkowski theory one can refer to [34] and the references within.

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in Euclidean n-space, $\mathbb{R}^n$. Let $\mathcal{K}^n_o$ denote the set of convex bodies containing the origin in their interiors. For $K \in \mathcal{K}^n$, let $h_K = h(K,\cdot) : S^{n-1} \to \mathbb{R}$ denote the support function of K; i.e., for $u \in \mathbb{S}^{n-1}$, $h_K(u) = h(K;u) = \max\{u{\cdot}x : x \in K\}$, where $u \cdot x$ denotes the standard inner product in $\mathbb{R}^n$. The set $\mathcal{K}^n$ will be viewed as equipped with the usual Hausdorff metric, d, defined by $d(K,L) = |h_K - h_L|_\infty$, where $|\cdot|_\infty$ is the sup (*or* max) norm on the space of continuous function on the unit sphere, $C(S^{n-1})$. We recall that a more generally Minkowski sum of K and $L \in \mathcal{K}^n$ is

$$K + L = \{z \in \mathbb{R}^n | z = x + y, x \in K, y \in L\}.$$

One of the most interesting aspects in convex geometry are the geometric inequalities involving the volume, surface area, mean width and other geometric invariants. An important family of inequalities are those leading to estimate the volume of a special body associated with a convex body (for example the difference body or the reflection body) in terms of the volume of the body itself. A remarkable inequality

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of this type inequality is the classical Rogers and Shephard inequality (see [31]) which assert that for all $K \in \mathcal{K}^n$,

$$|K + (-K)| \leq \binom{2n}{n} |K|,$$

and equality holds if and only if K is a simplex. Here $|\cdot|$ denotes the n-dimensional volume of K. The body $K + (-K)$ is called the difference body of K and it is the Minkowski sum of K and its reflected body with respect to the origin, $-K$.

Another inequality due to Rogers and Shephard [32] concerning the volume of a convex hull of K and $-K \in \mathcal{K}_o^n$ is

$$|conv(K \cup (-K))| \leq 2^n |K|,$$

where equality holds if and only if K is a simplex with one vertex at the origin.

In the 1960s, Firey (see [12, 34]) introduced the p-Firey combination of convex bodies (now we call it the Firey-Minkowski combination or L_p Minkowski combination) which depends on the parameter $p \geq 1$. The L_p Minkowski combination can be seen as an extension of the Minkowski combination. It provides analysis tools for geometric problems and new connections between convex geometry and analysis. The L_p Brunn-Minkowski theory achieves great developments and expands rapidly after the emergence of the remarkable papers of Lutwak [22, 23]. See [5, 6, 8, 11, 13, 15, 17, 18, 21, 24–26, 29, 35, 37] for more references on L_p Brunn-Minkowski theory.

For $K, L \in \mathcal{K}_o^n$, the L_p Minkowski combination $K +_p L \in \mathcal{K}_o^n$ is defined by

$$h_{K+_pL}(u)^p = h_K(u)^p + h_L(u)^p, \quad u \in \mathbb{R}^n. \tag{0.1}$$

For $p = \infty$, we define

$$h_{K+_\infty L}(u) = \lim_{p\to\infty} h_{K+_pL}(u) = \max\{h_K(u), h_L(u)\}, \quad u \in \mathbb{R}^n.$$

It is shown in [1] that the volume of the p-difference body satisfies

$$|K +_p (-K)| \leq c_{2,p}|K|,$$

where K is a convex body in two-dimensional space, and $c_{2,p}$ is a constant depending on p and is given by

$$c_{2,p} = 2\left(1 + (p-1)\int_0^{\pi/2} \frac{\sin^{p-2} t \cos^{p-2} t}{(\sin^p t + \cos^p t)^{2\frac{p-1}{p}}} dt\right), \qquad 1 < p < +\infty. \tag{0.2}$$

Recently, beginning with the ground-breaking articles of Lutwak, Yang, Zhang and Haberl [16, 27, 28], a more wide extension of the L_p Brunn-Minkowski theory, which is called the Orlicz Brunn-Minkowski theory, emerged three years ago. In these articles, the Orlicz Petty projection inequality and the Orlicz Busemann-Petty centroid inequality were established, which led to the development of Orlicz Brunn-Minkowski theory. See, e.g., [2–4, 9, 10, 14, 16, 19, 39, 40] about the Orlicz Brunn-Minkowski theory. Very recently, in [14], the Orlicz Minkowski linear combination was introduced, and the Orlicz Brunn-Minkowski inequality and the Orlicz Minkowski inequality were established by Gardner, Hug, and Weil.

In view of the importance of the difference body in convex geometry, we are tempted to consider the naturally posed problem in the booming Orlicz Brunn-Minkowski theory: what is the maximum of the volume of the Orlicz difference body?

Our main task in this paper is to obtain the maximum of the volume of the Orlicz difference body in planar case $n = 2$.

Let $\varphi : [0, \infty) \to [0, \infty)$ be a convex function with $\varphi(0) = 0$ and $\varphi(1) = 1$; the set of such φ is denoted by $\mathcal{C}$. In view of Lemma 1.2, we have

$$K +_\varphi L \subseteq K + L \quad \text{and} \quad K +_\varphi L \subseteq \frac{1}{\varphi^{-1}(1/2)} conv(K \cup L).$$

If we take $L = -K$, we have

$$K +_\varphi (-K) \subseteq K + (-K)$$

and

$$K +_\varphi (-K) \subseteq \frac{1}{\varphi^{-1}(1/2)} conv(K \cup (-K)).$$

So by the above formulas, we roughly know

$$|K +_\varphi (-K)| \leq \min \left\{ \left(\begin{array}{c} 2n \\ n \end{array} \right), \left(\frac{2}{\varphi^{-1}(1/2)} \right)^n \right\} |K|.$$

What is the best constant $c_{n,\varphi}$, depending on n and φ, such that

$$|K +_\varphi (-K)| \leq c_{n,\varphi} |K|?$$

In this paper we solve this problem in the two-dimensional case.

Theorem 0.1. *Let $\varphi \in \mathcal{C}$. There exists a constant $c_{2,\varphi}$ such that*

$$|K +_\varphi (-K)| \leq c_{2,\varphi} |K| \tag{0.3}$$

for all $K \in \mathcal{K}_o^n$. If K is a triangle with one vertex at the origin, then equality holds.

The paper is organized as follows, In section 1 we introduce the Orlicz linear combination and show some of its properties. The shadow system and related results on the Orlicz Minkowski linear combination are given in section 2. The proof that the volume of the Orlicz difference body attains its maximum at triangle and an explicit expression of $c_{2,\varphi}$ is presented in section 3.

1. Orlicz Minkowski linear combination

Let $\mathcal{C}$ be the class of convex, strictly increasing functions $\varphi : [0, \infty) \to [0, \infty)$ satisfying $\varphi(0) = 0$ and $\varphi(1) = 1$. Here the normalization is a matter of convenience and other choices are possible. It is not hard to conclude that $\varphi \in \mathcal{C}$ is continuous on $[0, \infty)$.

Let $K, L \in \mathcal{K}_o^n$, $\varphi \in \mathcal{C}$. The Orlicz linear combination $K +_\varphi L$ is the convex body with support function

$$h_{K+_\varphi L}(x) = \inf \left\{ \lambda > 0 : \varphi\left(\frac{h_K(x)}{\lambda} \right) + \varphi\left(\frac{h_L(x)}{\lambda} \right) \leq 1 \right\}.$$

Since φ is strictly increasing, then

$$\lambda \to \varphi\left(\frac{h_K(x)}{\lambda} \right) + \varphi\left(\frac{h_L(x)}{\lambda} \right)$$

is strictly decreasing. So, equivalently, $h_{K+_\varphi L}(u_0) = \lambda_0$ if and only if

$$\varphi\left(\frac{h_K(u_0)}{\lambda_0} \right) + \varphi\left(\frac{h_L(u_0)}{\lambda_0} \right) = 1. \tag{1.1}$$

When $\varphi(t) = t^p (p \geq 1)$, the Orlicz linear combination is precisely the L_p Minkowski combination $K +_p L$.

By the definition of Orlicz linear combination, we have the following lemma.

Lemma 1.1. *Let $\varphi \in \mathcal{C}$ and $K_i, L_i, K, L \in \mathcal{K}_o^n$. Suppose $K_i \to K$ and $L_i \to L$; then*

$$K_i +_\varphi L_i \to K +_\varphi L. \tag{1.2}$$

This lemma is proved in [38]. For the sake of completeness here we present the proof of this lemma.

Proof. Let $h_{K_i +_\varphi L_i} = \lambda_i$ and $h_{K+_\varphi L} = \lambda$. In order to prove $K_i +_\varphi L_i \to K +_\varphi L$, we only need to prove $\lambda_i \to \lambda$. By the definition of Orlicz linear combination we have

$$\varphi\left(\frac{h_{K_i}}{\lambda_i}\right) + \varphi\left(\frac{h_{K_i}}{\lambda_i}\right) = 1.$$

Since φ is increasing, then

$$1 = \varphi\left(\frac{h_{K_i}}{\lambda_i}\right) + \varphi\left(\frac{h_{K_i}}{\lambda_i}\right) \leq 2\varphi\left(\frac{h_{K_i} + h_{L_i}}{\lambda_i}\right).$$

Then we have

$$\lambda_i \leq \frac{h_{K_i} + h_{L_i}}{\varphi^{-1}(\frac{1}{2})}.$$

Since $h_{K_i} \to h_K$ and $h_{L_i} \to h_L$, then there exists a constant R such that $\lambda_i \leq R$, so λ_i is bounded from above. Let $\{\overline{\lambda}_i\}$ be a subsequence of $\{\lambda_i\}$, and suppose that $\overline{\lambda}_i \to \lambda_0$. Since φ is continuous, we have

$$\varphi\left(\frac{h_K}{\lambda_0}\right) + \varphi\left(\frac{h_L}{\lambda_0}\right) = \lim_{i\to 0}\left[\varphi\left(\frac{h_{K_i}}{\overline{\lambda}_i}\right) + \varphi\left(\frac{h_{L_i}}{\overline{\lambda}_i}\right)\right] = 1,$$

which implies $\lambda_0 = \lambda$, and so we complete the proof. $\square$

Note that φ is increasing on $[0, \infty)$ and the normalization $\varphi(1) = 1$, by (1.1), we have

$$h_K(u) \leq h_{K+_\varphi L}(u), \qquad h_L(u) \leq h_{K+_\varphi L}(u), \qquad \text{for all } u \in S^{n-1}.$$

This means

$$K \subseteq K +_\varphi L, \qquad L \subseteq K +_\varphi L.$$

Moreover, for φ_1 and φ_2 in $\mathcal{C}$, if $\varphi_1 \leq \varphi_2$, we have the following lemma.

Lemma 1.2. *Let φ and φ_1, φ_2 in $\mathcal{C}$ be convex functions, for $K, L \in \mathcal{K}_o^n$; then*
(i). If $\varphi_1 \leq \varphi_2$ for all $x \in [0, 1]$, then $K +_{\varphi_1} L \subseteq K +_{\varphi_2} L$,
(ii). $K +_\varphi L \subseteq K + L$,
(iii). $conv(K \cup L) \subseteq K +_\varphi L$,
(iv). $K +_\varphi L \subseteq \frac{1}{\varphi^{-1}(1/2)} conv(K \cup L)$.

Proof. In order to prove (i), we only need to prove

$$h_{K+_{\varphi_1} L}(u) \leq h_{K+_{\varphi_2} L}(u), \qquad \text{for all } u \in S^{n-1}.$$

Let $h_{K+_{\varphi_1} L}(u) = \lambda_1$ and $h_{K+_{\varphi_2} L}(u) = \lambda_2$. By formula (1.1), we have

$$\varphi_1\left(\frac{h_K(u)}{\lambda_1}\right) + \varphi_1\left(\frac{h_L(u)}{\lambda_1}\right) = 1, \quad \varphi_2\left(\frac{h_K(u)}{\lambda_2}\right) + \varphi_2\left(\frac{h_L(u)}{\lambda_2}\right) = 1.$$

Since $\varphi_1 \le \varphi_2$,

$$\varphi_2\left(\frac{h_K(u)}{\lambda_1}\right) + \varphi_2\left(\frac{h_L(u)}{\lambda_1}\right) \ge 1,$$

which means $\lambda_1 \le \lambda_2$, that is, $K +_{\varphi_1} L \subseteq K +_{\varphi_2} L$.

(ii). Let Id denote the identity function on $[0,1]$, since φ is convex on $[0,1]$, for $x \in [0,1]$. We have

$$\varphi(x) = \varphi(x \cdot 1 + (1-x)0) \le x\varphi(1) + (1-x)\varphi(0),$$

so we have $\varphi(x) \le x$, which means that $\varphi \le Id$. On the other hand, when $\varphi(t) = Id$, the Orlicz Minkowski linear combination $K +_\varphi L$ is precisely the Minkowski sum $K + L$. Now by (i) we have $K +_\varphi L \subseteq K + L$.

(iii). Note that $h_{conv(K\cup L)}(u) = \max\{h_K(u), h_L(u)\}$. Then

a). If $h_{conv(K\cup L)}(u) = h_K(u)$ for some $u \in \Omega$, then

$$\varphi\left(\frac{h_K(u)}{h_{conv(K\cup L)}(u)}\right) + \varphi\left(\frac{h_L(u)}{h_{conv(K\cup L)}(u)}\right) = \varphi(1) + \varphi\left(\frac{h_L(u)}{h_K(u)}\right) \ge 1.$$

b). If $h_{conv(K\cup L)}(u) = h_L(u)$ for some $u \in S^{n-1}/\Omega$, then

$$\varphi\left(\frac{h_K(u)}{h_{conv(K\cup L)}(u)}\right) + \varphi\left(\frac{h_L(u)}{h_{conv(K\cup L)}(u)}\right) = \varphi(1) + \varphi\left(\frac{h_K(u)}{h_L(u)}\right) \ge 1.$$

So we obtain $\varphi\left(\frac{h_K(u)}{h_{conv(K\cup L)}(u)}\right) + \varphi\left(\frac{h_L(u)}{h_{conv(K\cup L)}(u)}\right) \ge 1$. By the definition of the Orlicz Minkowski linear combination we have

$$h_{conv(K\cup L)}(u) \le h_{K+_\varphi L}(u).$$

(iv). Since

$$\varphi\left(\frac{h_K(u)}{h_{K+_\varphi L}(u)}\right) + \varphi\left(\frac{h_L(u)}{h_{K+_\varphi L}(u)}\right) = 1,$$

by the increasing of φ, we have

$$1 \le \varphi\left(\frac{\max\{h_K(u), h_L(u)\}}{h_{K+_\varphi L}(u)}\right) + \varphi\left(\frac{\max\{h_K(u), h_L(u)\}}{h_{K+_\varphi L}(u)}\right).$$

Then,

$$h_{K+_\varphi L}(u) \le \frac{1}{\varphi^{-1}(1/2)} \max\{h_K(u), h_L(u)\}.$$

So we have $K +_\varphi L \subseteq \frac{1}{\varphi^{-1}(1/2)} conv(K \cup L)$ and we complete the proof. □

If we take $\varphi(t) = t^p$ $(p \ge 1)$ in Lemma 1.2, it is exactly the p-sum obtained by Firey [12].

2. Shadow system and linear parameter systems

A shadow system along the direction $v \in S^{n-1}$ is a family of convex bodies $K_t \subset \mathbb{R}^n$ such that

$$K_t = conv\{x + \alpha(x)tv : x \in A \subset \mathbb{R}^n\}, \tag{2.1}$$

where *conv* denotes convex hull, A is an arbitrary bounded set of points, α is a bounded function on A that is called the speed function, and t belongs to an interval I of the real axis. The notation of a shadow system was introduced by Rogers and Shephard [33]. In [36] Shephard shows that a shadow system can be seen as the family of projections of an $(n+1)$-dimensional convex body $\widetilde{K} \subseteq \mathbb{R}^{n+1}$ onto the hyperplane $\{e_{n+1}^{\perp}\}$, which here we identify with $\mathbb{R}^n$, along the direction $e_{n+1} - tu$. Here u varies in $\{e_{n+1}^{\perp}\}$, and the shadow system is said to be originated from the $(n+1)$-dimensional body $\widetilde{K}$.

A parallel chord movement is a special of a shadow system. More precisely, fix $K \in \mathcal{K}^n$ and a direction $v \in \mathbb{R}^n$. We move each chord of K parallel to v in that direction with a certain speed, and we consider the union of these chords as the parameter varies such that the union is convex.

The following result is due to Rogers and Shephard [32, 36].

Proposition 2.1. *$\{K_t : t \in I\}$ is a shadow system in $\mathbb{R}^n$ if and only if there exist a convex body $\widetilde{K} \in \mathbb{R}^{n+1}$ and $v \in \{e_{n+1}^{\perp}\}$ such that for every $t \in I$, K_t is the projection of $\widetilde{K}$ onto the hyperplane $\{e_{n+1}^{\perp}\}$ along the direction $e_{n+1} - tv$.*

By Proposition 2.1 we know that a body $\widetilde{K}$ which generates a shadow system K_t can be explicitly written as

$$\widetilde{K} = conv\{x + \alpha(x)e_{n+1} : x \in K\}.$$

Moreover, Campi and Gronchi showed in [7] that the support function of $\widetilde{K}$ and K_t have the following relation:

$$h_{K_t}(u) = h_{\widetilde{K}}(u + t\langle u \cdot v\rangle e_{n+1}), \quad u \in \mathbb{R}^n. \tag{2.2}$$

A remarkable result about the volume of a shadow system is due to Shephard (see [36]). This result was largely used by Campi and Gronchi [5–8], Li and Leng [20] and Chen, Zhou, and Yang [10].

Lemma 2.2. *Every mixed volume involving n shadow systems along the same direction is a convex function of the parameter. In particular, the volume $V(K_t)$ and all quermassintegrals $W_i(K_t)$, $i = 1, 2, \cdots, n$, of a shadow system are convex functions of t.*

Assume that the speed function is constant on each chord parallel to v, i.e., $\alpha(x) = \beta(x|v^{\perp})$, where $x|v^{\perp}$ is the projection of x onto $\{v^{\perp}\}$ and β is a function defined on the orthogonal projection of K onto $\{v^{\perp}\}$. So, a parallel chord movement along the direction v is a family of convex bodies K_t in $\mathbb{R}^n$ defined by

$$K_t = \{z + \beta(z|v^{\perp})tv : z \in K\}, \tag{2.3}$$

where β is a continuous real function on $v^{\perp}$ and the parameter t runs in an interval of the real axis, say $t \in I$. In other words, to each chord of $K = K_0$ parallel to v we assign a speed vector $\beta(x)v$, where x is the projection of the chord onto $v^{\perp}$. Then let the chords move for a time t and denote their union by K_t.

Notice that if $\{K_t : t \in I\}$ is a parallel chord movement, then via Fubini's Theorem, one deduces that the volume of K_t is independent of t.

Another special instance is the movement related to Steiner symmetrization. For a direction v let

$$K = \{x + yv \in \mathbb{R}^n : x \in K|v^{\perp}, y \in \mathbb{R},\, f(x) \le y \le g(x)\}. \tag{2.4}$$

Here f and $-g$ are convex functions on $K|v^{\perp}$. Take $\beta(x) = -(f(x) + g(x))$ in (2.3) and $t \in [0, 1]$ such that $K_0 = K$ and $K_1 = K^v$, where K^v is the reflection of K in the hyperplane $v^{\perp}$, and $K_{1/2}$ is the Steiner symmetrization of K with respect to $v^{\perp}$.

A basic feature of a shadow system, that will be used later, regards the Orlicz Minkowski linear combination expressed by the following theorem.

Theorem 2.3. *If $\{K_t : t \in I\}$ and $\{L_t : t \in I\}$ are shadow systems along the same direction v, then $\{K_t +_\varphi L_t : t \in I\}$ is still a shadow system along v.*

The proof of Theorem 2.3 is a straightforward consequence of Proposition 2.1 and the following lemma.

Lemma 2.4. *Let $\{K_t : t \in I\}$ and $\{L_t : t \in I\}$ be linear parameter systems along the same direction v and let $\widetilde{K}$ and $\widetilde{L}$ be the $(n+1)$-dimensional convex bodies which generate K_t and L_t, respectively, defined as in (2.2). Hence for all $t \in I$, $\{K_t +_\varphi L_t : t \in I\}$ is the projection of $\widetilde{K} +_\varphi \widetilde{L}$ onto the hyperplane $\{e_{n+1}^\perp\}$ along the direction $e_{n+1} - tv$.*

Proof. Let $\overline{u} = u + t\langle u \cdot v\rangle e_{n+1}$. By the definition of the Orlicz Minkowski linear combination one has

$$\varphi\left(\frac{h_{\widetilde{K}}(\overline{u})}{h_{\widetilde{K}+_\varphi\widetilde{L}}(\overline{u})}\right) + \varphi\left(\frac{h_{\widetilde{L}}(\overline{u})}{h_{\widetilde{K}+_\varphi\widetilde{L}}(\overline{u})}\right) = 1. \tag{2.5}$$

By formula (2.2) we have

$$\varphi\left(\frac{h_{K_t}(u)}{h_{\widetilde{K}+_\varphi\widetilde{L}}(\overline{u})}\right) + \varphi\left(\frac{h_{L_t}(u)}{h_{\widetilde{K}+_\varphi\widetilde{L}}(\overline{u})}\right) = 1.$$

On the other hand, we have

$$\varphi\left(\frac{h_{K_t}(u)}{h_{K_t+_\varphi L_t}(u)}\right) + \varphi\left(\frac{h_{L_t}(u)}{h_{K_t+_\varphi L_t}(u)}\right) = 1.$$

Note that $\lambda \to \varphi\left(\frac{h_K(x)}{\lambda}\right) + \varphi\left(\frac{h_L(x)}{\lambda}\right)$ is strictly decreasing on $[0, \infty)$. We have

$$h_{K_t+_\varphi L_t}(u) = h_{\widetilde{K}+_\varphi\widetilde{L}}(\overline{u})$$

by Proposition 2.1, which means $\{K_t +_\varphi L_t : t \in I\}$ is the projection of $\widetilde{K} +_\varphi \widetilde{L}$ onto the hyperplane $\{e_{n+1}^\perp\}$ along the direction $e_{n+1} - tv$. $\square$

3. The proof of the Rogers and Shephard inequality for the Orlicz difference body

In this section we will prove the Rogers and Shephard inequality for the Orlicz difference body. Let us consider the function $F_\varphi(K)$ defined on $\mathcal{K}_o^n$:

$$F_\varphi(K) = \frac{|K +_\varphi (-K)|}{|K|}.$$

Now the maximum of $F_\varphi(K)$ in $\mathcal{K}_o^n$ is the best constant $c_{2,\varphi}$ in Theorem 0.1.

We will use the shadow system to find a maximum for the functional $F_\varphi(K)$ in the planar case.

Theorem 3.1. *If $\{K_t : t \in I\}$ is a parallel chord movement defined in (2.3), along the direction v with speed function β, then $F_\varphi(K_t)$ is a convex function of the parameter t.*

Proof. By Fubini's Theorem, it is easy to see that the volume $|K_t|$ is independent of t. By Lemma 2.2 and Theorem 2.3, the convexity of $F_\varphi(K_t)$ follows the convexity of the $|K_t +_\varphi (-K_t)|$. □

Now let us turn to the problem of finding the maximizers of the functional $F_\varphi(K)$.

Theorem 3.2. *For $n = 2$, the maximum of $F_\varphi(K)$ in the class of convex bodies is attained when K is a triangle.*

Proof. First, the continuity of $|K +_\varphi (-K)|$ on $\mathcal{K}_o^n$ implies the continuity of $F_\varphi(K)$ on $\mathcal{K}_o^n$. It is sufficient to show that triangles are the maximizers of $F_\varphi(K)$ in the class of all polygons.

Let P be a polygon with m vertices. If $m \geq 4$, call v_0, v_1, v_2, v_3, v_4 five consecutive vertices of P (if $m = 4$ take $v_0 = v_4$) and u the direction of $v_3 - v_1$. The shadow system $\{P_t : t \in [t_0, t_1]\}$, with $(t_0 < 0 < t_1)$ along u, with speed 1 at v_2 and 0 at all the other vertices, is a parallel chord movement if we choose $[t_0, t_1]$ as the largest interval such that the area of P_t is constant for all $t \in [t_0, t_1]$. In fact, let l be a line parallel to u and through point v_2, and let t_0 be the coordinate on the line l for the intersection point of the line l with the line that contains v_0 and v_1, and let t_1 be the coordinate on the line l for the intersection point of the line l with the line that contains v_3 and v_4. We notice that P_{t_0} and P_{t_1} have $m-1$ vertices, and the consecutive vertices of P_{t_0} and P_{t_1} are $t_0, v_3, \cdots, v_m$ and $t_1, v_4, \cdots, v_m, v_1$ respective. By Theorem 3.1, we have

$$F_\varphi(P) < \max\{F_\varphi(P_{t_0}), F_\varphi(P_{t_1})\}.$$

If the number of vectors for P_{t_0} (or P_{t_1}) is still larger than 4, iteration of this will proceed on P_{t_0} (or P_{t_1}) until the number of vectors is 3. So we obtain the fact that triangles are the only maximizers of $F_\varphi(P)$ in the class of polygons and we complete the proof. □

Now we are in position to compute the constant $c_{2,\varphi}$. We choose a triangle $\triangle_2$ as a maximizer with vertices at origin, (1,0) and (0,1). Namely,

$$c_{2,\varphi} = \frac{|\triangle_2 +_\varphi (-\triangle_2)|}{|\triangle_2|}.$$

To express the value of $c_{2,\varphi}$, we use the parametrization of the boundary of a convex body in terms of its support function.

Let $\omega = e^{i\theta} \in S^1$. The support function of $\triangle_2 +_\varphi (-\triangle_2)$ is:

(i). When $0 \leq \theta < \frac{\pi}{2}$;

$$h_{\triangle_2 +_\varphi (-\triangle_2)}(\omega) = \begin{cases} \cos\theta, & \text{if } 0 \leq \theta < \frac{\pi}{4}; \\ \sin\theta, & \text{if } \frac{\pi}{4} \leq \theta < \frac{\pi}{2}. \end{cases}$$

(ii). When $\frac{\pi}{2} \leq \theta < \pi$, the support function $h_{\triangle_2 +_\varphi (-\triangle_2)}$ satisfies:

$$\varphi\left(\frac{\sin\theta}{h_{\triangle_2 +_\varphi (-\triangle_2)}(\omega)}\right) + \varphi\left(\frac{-\cos\theta}{h_{\triangle_2 +_\varphi (-\triangle_2)}(\omega)}\right) = 1. \tag{3.1}$$

(iii). When $\pi \leq \theta < 2\pi$, by the symmetry of $\triangle_2 +_\varphi (-\triangle_2)$,

$$h_{\triangle_2 +_\varphi (-\triangle_2)}(e^{i\theta}) = h_{\triangle_2 +_\varphi (-\triangle_2)}(e^{i(\theta - \pi)}).$$

In order to write this simply, let $r(\theta) = h_{\triangle_2 +_\varphi (-\triangle_2)}(e^{i\theta})$ when $\frac{\pi}{2} \leq \theta < \pi$. By a simple computation we have

$$|\triangle_2 +_\varphi (-\triangle_2)| = 2\left(\frac{1}{2} + \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi}[r^2(\theta) - (r'(\theta))^2]d\theta\right)$$
$$= 1 + \int_{\frac{\pi}{2}}^{\pi}[r^2(\theta) - (r'(\theta))^2]d\theta.$$

So we have

$$c_{2,\varphi} = \frac{|\triangle_2 +_\varphi (-\triangle_2)|}{|\triangle_2|} = 2 + 2\int_{\frac{\pi}{2}}^{\pi}[r^2(\theta) - (r'(\theta))^2]d\theta,$$

where $r(\theta)$ is determined by (3.1).

If $\varphi(t) = t^p$, it returns to the p difference body, and then $r^p(\theta) = \sin^p\theta + (-\cos\theta)^p$ for $\frac{\pi}{2} \leq \theta < \pi$, and

$$r^2(\theta) - (r'(\theta))^2 = \sin^{2p}\theta + (-\cos\theta)^{2p} - \sin^2\theta(-\cos\theta)^{2p-2}$$
$$- \cos^2\theta\sin^{2p-2}\theta + 4\sin^p\theta(-\cos\theta)^p.$$

When $p = 1$, it is the classical difference body, and

$$r^2(\theta) - (r'(\theta))^2 = -4\sin\theta\cos\theta.$$

The integral of $r^2(\theta) - (r'(\theta))^2$ on $[\frac{\pi}{2}, \pi]$ is

$$\int_{\frac{\pi}{2}}^{\pi}[r^2(\theta) - (r'(\theta))^2]d\theta = 2.$$

So the Rogers and Shephard inequality in $\mathbb{R}^2$ is

$$|K + (-K)| \leq 6|K|.$$

This result is proved by Rademacher [30], and then extended by Rogers and Shephard [31].

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