SOME DIRECTED SUBSETS OF C*-ALGEBRAS
AND SEMICONTINUITY THEORY

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ABSTRACT. The main result concerns a \(\sigma\)--unital C*-algebra \(A\), a strongly lower semicontinuous element \(h\) of \(A^{**}\), the enveloping von Neumann algebra, and the set of self-adjoint elements \(a\) of \(A\) such that \(a \leq h - \delta 1\) for some \(\delta > 0\), where \(1\) is the identity of \(A^{**}\). The theorem is that this set is directed upward. It follows that if this set is non-empty, then \(h\) is the limit of an increasing net of self-adjoint elements of \(A\). A complement to the main result, which may be new even if \(h = 1\), is that if \(a\) and \(b\) are self-adjoint in \(A\), \(a \leq h\), and \(b \leq h - \delta 1\) for \(\delta > 0\), then there is a self-adjoint \(c\) in \(A\) such that \(c \leq h\), \(a \leq c\), and \(b \leq c\).

Let \(A\) be a C*-algebra and denote by \(A^{**}\) its bidual, the enveloping von Neumann algebra. In [1], C. Akemann and G. Pedersen discussed several classes of self-adjoint elements of \(A^{**}\) as analogues of semicontinuous functions on topological spaces.

For a subset \(S\) of \(A^{**}\), \(S_{sa}\) denotes \(\{h \in S : h^* = h\}\), \(S_+\) denotes \(\{h \in S ; h \geq 0\}\), and \(S^-\) denotes the norm closure of \(S\). And for \(S \subset A^{**}\), \(S_m\) denotes the set of \((\sigma\text{-weak})\) limits of bounded increasing nets in \(S\), \(S_m\) denotes the set of limits of bounded decreasing nets in \(S\), and \(S^\sigma\) denotes the set of limits of bounded increasing sequences in \(S\). Also \(\tilde{A}\) denotes \(A + \mathbb{C}1\), where \(1\) is the identity of \(A^{**}\), \(LM(A) = \{T \in A^{**} : TA \subset A\}\), \(QM(A) = \{T \in A^{**} : ATA \subset A\}\), \(K\) is the C*-algebra of compact operators on a separable infinite dimensional Hilbert space, and \(M_2\) is the C*-algebra of \(2 \times 2\) matrices. Finally, \(S(A)\) denotes the state space of \(A\), and states of \(A\) are also considered as (normal) states of \(A^{**}\).

Three of the classes considered in [1] are \((A_{sa}^{m})^-\), whose elements are called strongly lsc; \((A_{sa})^m\), whose elements are called middle lsc; and \((\tilde{A}_{sa})^m\), whose elements are called weakly lsc. Also \(h\) is called use in any sense if \(-h\) is lsc in that sense. For example, \((\tilde{A}_{sa})^m\) is the set of weakly usc elements. Akemann and Pedersen showed in [1] that these three kinds of semicontinuity can all be different. But they also considered the class \(A_{sa}^m\), and left open the question whether \(A_{sa}^m\) always equals \((A_{sa}^m)^-\).

In [2 Corollary 3.25] I showed that \(A_{sa}^m\) does equal \((A_{sa}^m)^-\) if \(A\) is separable. Whether this is so for general \(A\) is still open, and I have no conjecture concerning this. I think the main obstacle to proving the affirmative answer is that it is difficult to construct increasing nets in non-commutative C*-algebras. I know two ways of doing this. In cases where sequences suffice, one can use recursive constructions. Or one can find a subset of \(A_{sa}\) which is a directed set in the natural ordering. The first method was used in the proof of [2 Corollary 3.25], and the second method has

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been used many times. For example, it is used in the construction of approximate identities of $C^\ast$-algebras, and it is used several times in [1]. It seems to me that the best hope for proving $A_{sa}^m = (A_{sa}^m)^\ast$ when $A$ is non-separable is to prove that a suitable subset of $A_{sa}^m$ is directed upward.

For $h$ in $(A_{sa}^m)^\ast$ consider the following sets:

Let $A_1 = \{a \in A_{sa} : a \leq h\}$.

Let $A_2 = \{a \in A_{sa} : \varphi(a) < \varphi(h), \forall \varphi \in S(A)\}$.

If $h \geq 0$, let $A_3 = \{a \in A_+ : a \leq \theta h$ for some $\theta$ in $(0, 1)\}$.

If $\varphi(h) > 0, \forall \varphi \in S(A)$, let $A_4 = \{a \in A_+ : \exists \theta$ in $(0, 1)$ such that $\varphi(a) < \varphi(\theta h), \forall \varphi \in S(A)\}$.

Let $A_5 = \{a \in A_{sa} : a \leq h - \delta 1$ for some $\delta > 0\}$.

Examples will be given at the end of this note to show that, even if $A$ is separable, none of $A_1, A_2, A_3, A_4$ need be directed upward. Counterexamples for $A_1$ were already given in [2, 3.23 and 5.6]. The main result of this note is that $A_5$ is directed upward if $A$ is $\sigma$-unital. However, this does not imply that $A_{sa}^m = (A_{sa}^m)^\ast$ for $A$ $\sigma$-unital, since there are some choices of $h$ for which $A_5$ is empty.

**Proposition 1.** Let $A$ be a $C^\ast$-algebra and $a, b \in A_+$ such that $\|a\| \leq 1$ and $\|b\| < 1$. Then there is $c$ in $A$ such that $a \leq c, b \leq c$, and $\|c\| \leq 1$.

**Proof.** Let $c = b + (1 - b)^{1 \over 2}[(1 - b)^{-1 \over 2}(a - b)(1 - b)^{-1 \over 2}] + (1 - b)^{1 \over 2}$. Since $(1 - b)^{-1 \over 2}(a - b)(1 - b)^{-1 \over 2} \leq (1 - b)^{-1 \over 2}(1 - b)(1 - b)^{-1 \over 2} = 1$, then $0 \leq t \leq 1$ if $t = \{(1 - b)^{-1 \over 2}(a - b)(1 - b)^{-1 \over 2}\}$. Therefore $0 \leq (1 - b)^{1 \over 2}t(1 - b)^{1 \over 2} \leq (1 - b)$, and hence $b \leq c \leq 1$. Also since $t \geq (1 - b)^{-1 \over 2}(a - b)(1 - b)^{-1 \over 2}$, then $c \geq b + (a - b) \geq a$. □

**Lemma 2.** Let $A$ be a separable stable $C^\ast$-algebra, $h \in (A_{sa}^m)^\ast$, and let $A = \{a \in A_{sa} : h - a \geq \delta 1$ for some $\delta > 0\}$. Then $A$ is directed upward. Also if $a \in A_{sa}, a \leq h$, and $b \in A$, then there is $c$ in $A_{sa}$ such that $c \leq h, a \leq c$, and $b \leq c$.

**Proof.** Assume $A \neq \emptyset$. Since the hypothesis and conclusion are unaffected if $h$ is replaced by $h - a, a \in A_{sa}$, we may assume $h \geq \epsilon 1$ for some $\epsilon > 0$. By [2, Theorem 4.4(a)] $h = TT^\ast$ for some $T$ in $LM(A)$. Therefore $T^\ast T \in QM(A)$ and $\sigma(T^\ast T)$ omits the interval $(0, \epsilon)$. Let $p$ denote the range projection of $T^\ast T$. Then $p$ is closed (cf. [2, Proposition 2.44(b)]). Let $T' = (T^\ast T)^{-1}T^\ast$, where the inverse is computed in $pA^\ast p$. Then $TT' = 1$ and $TT' = p$.

We claim that $TA = A$. To see this, note first that $T = Tp$ and hence $TA = (Tp)(pA)$. Since $pA$ is norm closed by [3] and $(Tp)(pA) \geq \epsilon p$, it follows that $TA$ is norm closed in $A$. Therefore it is sufficient to show that $TA$ is $\sigma$-weakly dense in $A^\ast$. But the $\sigma$-weak closure of $TA$ includes $TA^\ast$, and $TA^\ast \supset TT' A^\ast = A^\ast$.

Now it is clear that $TT^\ast = (TA)(AT^\ast) = (TA)(TA)^\ast = A \cdot A = A$. If $TT^\ast \in A$, then since $TT^\ast \geq TA T^\ast + \delta 1$, we have $T'(TT^\ast)T^\ast \geq (T^\ast T) a(TT^\ast)^{1 \over 2} + \delta T^\ast T^\ast = p \delta + \delta(TT^\ast)^{-1}$. Since $T'(TT^\ast)T^\ast = p$ and $(TT^\ast)^{-1} \geq \eta p$ for some $\eta > 0$, this implies that $p \geq p \delta + \delta p$ for some $\delta' > 0$. Conversely, if $p \geq p \delta + \delta' p$ for $\delta' > 0$, then

$$h = TT^\ast = TpT^\ast \geq T(pap)T^\ast + \delta'(TT^\ast) = TaT^\ast + \delta h,$$

$$\geq TaT^\ast + \delta' \epsilon 1.$$ 

Therefore $TaT^\ast \in A$. By [2, Corollary 3.4] $\{x \in pA_{sa}p : x \leq (1 - \delta')p$ for some $\delta' > 0\} = \{yp : y \in A_{sa}$ and $y \leq (1 - \delta') 1$ for some $\delta' > 0\}$. It is known that the
last set is directed upward (cf. \cite[Theorem 1.4.2]{5}), so it follows that $A$ is directed upward. The last sentence is proved similarly using Proposition 1.

**Lemma 3.** Let $A$ be a separable $C^*$-algebra, $h \in (A_{sa}^m)^{-}$, and let $A = \{a \in A_{sa} : h - a \geq \delta 1$ for some $\delta > 0\}$. Then $A$ is directed upward. Also if $a \in A_{sa}, a \leq h$, and $b \in A$, then there is $c$ in $A_{sa}$ such that $c \leq h, a \leq c$, and $b \leq c$.

**Proof.** As in the proof of Lemma 2, we may assume $h \geq \epsilon 1$ for $\epsilon > 0$. Consider $B = A \otimes K$ and $\bar{h} = h \otimes 1$ in $B^{**}$. Obviously $\bar{h} \in (B_{sa}^m)^{-}$. Let $A = \{b \in B_{sa} : \bar{h} - b \geq \delta 1$ for some $\delta > 0\}$, and let $e$ be a rank one projection in $K$. If $a_1, a_2 \in A$, then $a_1 \otimes e, a_2 \otimes e \in \bar{A}$. By Lemma 2, $\exists b \in B_{sa}$ and $\delta > 0$ such that $a_i \otimes e \leq b \leq h - \delta 1$ for $i = 1, 2$. Therefore $a_i \otimes e \leq (1 \otimes e)b(1 \otimes e) \leq (1 \otimes e)\bar{h}(1 \otimes e) - \delta(1 \otimes e)$. Under the canonical isomorphism of $A$ with $(1 \otimes e)B(1 \otimes e)$, this is the desired conclusion. The last sentence is proved similarly.

The following is implicit in the proof of \cite[Theorem 3.24]{2}, but we provide a proof here.

**Lemma 4.** If $A$ is a $C^*$-algebra, $h \in (A_{sa}^m)^{-}$, and if $\{a_n\}$ is a countable subset of $A_{sa}$ such that $a_n \leq h, \forall n$, then there is $h_1 \in A_{sa}^n$ such that $h_1 \leq h$ and $a_n \leq h_1, \forall n$.

**Proof.** We construct recursively an increasing sequence $(b_n)$ in $A_{sa}$ such that $a_k \leq b_n + \frac{1}{n}1$ for $1 \leq k \leq n$ and $b_n \leq h, \forall n$. Then let $h_1 = \lim b_n$. Let $b_1 = a_1$. If $n > 1$ and $b_1, \ldots, b_{n-1}$ have been constructed, then by \cite[Corollary 3.17]{2} there is a function $f$ such that $\lim f(\epsilon) = 0$ and the following is true: If $\epsilon > 0, x \in A_{sa}$, and $b_{n-1} - \epsilon 1 \leq x \leq h + \epsilon 1$, then there is $y$ in $A_{sa}$ such that $b_{n-1} \leq y \leq h$ and $\|y - x\| < f(\epsilon)$. Choose $\epsilon$ in $(0, 1/2n)$ such that $f(\epsilon) < 1/2n$. By \cite[Theorem 3.24(b)]{2} there is $x$ in $A_{sa}$ such that $a_k - \epsilon 1 \leq x \leq h$ for $i = 1, \ldots, k$ and $b_{n-1} - \epsilon 1 \leq x$. Choose $b_n$ to be the $y$ indicated above, then $a_k \leq x + \frac{1}{2n}1 \leq y + \frac{1}{n}1$ for $k = 1, \ldots, n$.

**Theorem 5.** Let $A$ be a $\sigma$-unital $C^*$-algebra, $h \in (A_{sa}^m)^{-}$, and let $A = \{a \in A_{sa} : h - a \geq \delta 1$ for some $\delta > 0\}$. Then $A$ is directed upward. Also if $a \in A_{sa}, a \leq h$, and $b \in A$, then there is $c$ in $A_{sa}$ such that $c \leq h, a \leq c$, and $b \leq c$.

**Proof.** As in the proof of Lemma 2, we may assume $h \geq \epsilon 1$ for some $\epsilon > 0$. Consider $a_1, a_2 \in A_{sa}$ such that $a_i \leq h - \delta_i 1$ for $i = 1, 2$ and $\delta_i > 0$. Let $(\epsilon_n)$ be a sequential approximate identity of $A$. Then $(a_i + \delta_i \epsilon_n : i = 1, 2, n = 1, 2, \ldots)$ is a countable subset of $\{a \in A_{sa} : a \leq h\}$. Let $h_1$ in $A_{sa}^n$ be as in Lemma 4. If $h_1 = \lim b_m, b_m \in A_{sa}$, let $A_1$ be the separable $C^*$-algebra generated by $a_1, a_2$, the $\epsilon_n$'s, and the $b_m$'s. Thus $A_1^{**}$ and $A^{**}$ have the same identity element. Since $a_i + \delta_i \epsilon_n \leq h_1, n = 1, 2, \ldots$, then $a_i + \delta_i 1 \leq h_1$. Therefore by Lemma 3, there is $c$ in $(A_1)_{sa} \subset A_{sa}$ such that $a_i \leq c$ for $i = 1, 2$ and $c \leq h_1 - \delta 1 \leq h - \delta 1$ for some $\delta > 0$. The last sentence is proved similarly.

**Corollary 6.** Let $A$ be a $\sigma$-unital $C^*$-algebra and $h$ an element of $(A_{sa}^m)^{-}$ such that $h \geq a + \delta 1$ for some $a$ in $A_{sa}$ and some $\delta > 0$. Then $h \in A_{sa}^m$.

**Proof.** The theorem produces an increasing net in $A_{sa}$ whose limit is the supremum of $A$ in $A_{sa}^{**}$. To show that this supremum is $h$, we may assume $h \geq \epsilon 1$ for some $\epsilon > 0$, as in the proof of Lemma 2. Then by \cite[Proposition 3.5]{1}, $h^{-1} \in ((A_{sa})_m)^{-}$.

Next \cite[Proposition 3.1]{1} shows that there is a net $(h_i)$ in $A_{sa}$ such that $h_i \geq h^{-1}, \forall i$, and $h_i \rightarrow h^{-1} \sigma$-weakly. Finally \cite[Lemma 3.2]{4} implies that $h_i^{-1} \rightarrow h \sigma$-strongly.
(So far we have just used the argument for [1] Theorem 3.3] in a slightly different context.) Now for each $i$, $h_i^{-1} = \delta_i^1 + a_i$ for some $\delta_i > 0$ and $a_i \in A_{sa}$. If $e \in A_+$ and $\|e\| < 1$, then $\delta_i e + a_i \in A$, and the supremum of these elements, for fixed $i$, is $h_i^{-1}$. So the supremum of $A$ is indeed equal to $h$, whenever $A \neq \emptyset$.

**Examples and Remarks.** (i) There are a separable unital $C^*$-algebra $A$, a positive $h \in (A_{sa})^-$, $a_+, a_- \in A_+$, and $\theta \in (0, 1)$ such that $a_i \leq \theta h$ and there does not exist $b$ in $A_{sa}$ with $a_+ \leq b \leq h$. This shows that neither $A_1$ nor $A_3$ is directed upward.

Let $A$ be the algebra of convergent sequences in $\mathbb{M}_2$. Then $A^{**}$ can be identified with the set of bounded collections, $h = \{h_n\}_{1 \leq n \leq \infty}$ with each $h_n \in \mathbb{M}_2$. Here $h \in A$ if and only if $h_n \rightarrow h_{\infty}$. Define $h$ in $(A_{sa})^-$ by $h_n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ for $n < \infty$ and $h_{\infty} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Let $(a_\pm)_n = \begin{pmatrix} \theta/2 & \pm n^{-\frac{1}{2}} \theta/2 \\ \pm n^{-\frac{1}{2}} \theta/2 & n^{-1} \theta/2 \end{pmatrix} = \theta h_n^{\frac{1}{2}} \begin{pmatrix} 1/2 & \pm 1/2 \\ \pm 1/2 & 1/2 \end{pmatrix}$ for $n < \infty$ and $(a_\pm)_\infty = \begin{pmatrix} \theta/2 & 0 \\ 0 & 0 \end{pmatrix}$, where $\theta$ is to be determined. If $a_\pm \leq b \leq h$, then $b_n = h_n^{\frac{1}{2}} \begin{pmatrix} \alpha_n & \beta_n \\ \beta_n & \gamma_n \end{pmatrix} h_n^{\frac{1}{2}}$ where

$$\theta \begin{pmatrix} 1/2 & \pm 1/2 \\ \pm 1/2 & 1/2 \end{pmatrix} \leq \begin{pmatrix} \alpha_n & \beta_n \\ \beta_n & \gamma_n \end{pmatrix} \leq 1.$$ 

Note that the larger of $|\beta_n \pm \theta/2|$ is at least $\theta/2$. Therefore $(\theta/2)^2 \leq (\alpha_n - \theta/2)(\gamma_n - \theta/2)$. Since $\gamma_n \leq 1$, then $\alpha_n - \theta/2 \geq \frac{\theta^2/4}{1 - \theta/2}$. For suitable $\theta$, this implies that $\alpha_n \geq 3/4, \forall n$. Since $\lim \alpha_n = \frac{1}{2}$, this is a contradiction.

(ii) It is impossible to have a separable unital counterexample for $A_2$ or $A_4$, but a slight variation of the above gives a separable non-unital counterexample. Keeping the above notation, let $B$ be the hereditary $C^*$-subalgebra of $A$ consisting of sequences with limit of the form $\begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}$. Then $a_\pm \in B$ and $h \in B^{**} \subset A^{**}$. Also $\varphi(h) > 0, \forall \varphi \in S(B)$. If $\theta < \theta' < 1$, then $\varphi(\theta'h - a_\pm) > 0, \forall \varphi \in S(B)$. Therefore neither $A_2$ nor $A_4$ is directed upward.

(iii) For an arbitrary $C^*$-algebra $A$, if $h \in (A_{sa})^-$ and $h \geq 0$, then $q$, the range projection of $h$, is open (cf. [2] Proposition 2.44(a)]). Let $B$ be the hereditary $C^*$-subalgebra supported by $q$. Then $h \in B^{**} \subset A^{**}$, $\varphi(h) > 0, \forall \varphi \in S(B)$, and $h \in (B_{sa})^-$ (2 Proposition 2.14]). Clearly if $a \in A_+$ and $a \leq h$, then $a \in B$.

If $A$ is separable, then it follows from [2] Theorem 3.24(a)] that $h \geq e$ for some strictly positive element $e$ of $B$. Therefore if $a \in A_4$, then $\theta h - a \geq f$ for some strictly positive element $f$ of $B$.

There are some variants on $A_2, A_3, A_4$ that I could have included on the list near the beginning of this paper, and this remark shows why I didn’t bother.

(iv) The arguments for Lemmas 2 and 3 apply more generally and can prove the following:

Let $A$ be a separable $C^*$-algebra and $h$ a positive strongly lsc element of $A^{**}$ whose spectrum omits $(0, \epsilon)$ for some $\epsilon > 0$. Then for $B$, as in the previous remark, there are a closed projection $p$ in $(A \otimes K)^{**}$ and a complete order isomorphism $\varphi : B \rightarrow p(A \otimes K)p$ such that $\varphi^{**}(h) = p$. Here $\varphi^{**}$ is the natural extension of $\varphi$ to the biduals, and in particular for $b \in B_{sa}$, $b \leq h$ if and only if $\varphi(b) \leq p$. 


References


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