

## SELF MAPS OF $\mathbb{HP}^n$ VIA THE UNSTABLE ADAMS SPECTRAL SEQUENCE

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ABSTRACT. We use obstruction theory based on the unstable Adams spectral sequence to construct self maps of finite quaternionic projective spaces. As a result, a conjecture of Feder and Gitler regarding the classification of self maps up to homology is proved in two new cases.

### 1. INTRODUCTION

Let  $\mathbb{HP}^n$  denote  $n$ -dimensional quaternionic projective space and recall that  $H^*(\mathbb{HP}^n; \mathbb{Z}) = \mathbb{Z}[u]/u^{n+1}$  with  $|u| = 4$ . Given a self map  $f$  of  $\mathbb{HP}^n$ , the *degree* of  $f$  is the integer  $\deg(f)$  such that  $f^*(u) = \deg(f)u$ .

The homotopy classification of self maps of  $\mathbb{HP}^\infty$  is well known: self maps are classified by their degrees and the allowable degrees are zero and the odd square integers [Ms]. The situation for finite projective spaces is more complicated. It is not true in general that self maps are classified by their degrees (see [MR, IMO, GS]), and even the set of possible degrees is unknown. We will write

$$\mathbf{R}_n = \{\deg(f) \in \mathbb{Z} : f : \mathbb{HP}^n \rightarrow \mathbb{HP}^n\}$$

for the set of possible degrees. Alternatively,  $\mathbf{R}_n$  can be described as the set of homotopy classes of self maps of  $S^3$  which admit an  $A_n$  structure (see [St]).

There is a conjectural description of  $\mathbf{R}_n$  due to Feder and Gitler. Consider the congruences

$$(1) \quad C_n : \quad \prod_{i=0}^{n-1} (k - i^2) \equiv 0 \pmod{\begin{cases} (2n)! & \text{if } n \text{ is even,} \\ \frac{(2n)!}{2} & \text{if } n \text{ is odd,} \end{cases}}$$

and define the sets of integers

$$\mathbf{FG}_n = \{k \in \mathbb{Z} \mid k \text{ is a solution of } C_1, \dots, C_n\}.$$

In [FG], Feder and Gitler computed the restrictions imposed by  $K$ -theory and Adams operations on the degrees of self maps of  $\mathbb{HP}^n$ : they are precisely that the degree should be in  $\mathbf{FG}_n$ . Therefore  $\mathbf{R}_n \subset \mathbf{FG}_n$ .

**Conjecture 1.1** (Feder and Gitler).  $\mathbf{R}_n = \mathbf{FG}_n$ .

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This is trivial for  $n = 1$ . A proof for  $n = 2$  is contained in [AC], and for  $n = 3$  in [MG2]. Feder and Gitler showed in [FG] that the integers in  $\mathbf{FG}_\infty$  are precisely the odd square integers and zero so the conjecture also holds for  $n = \infty$ . In fact, self maps of  $\mathbb{HP}^\infty$  with these degrees were first constructed by Sullivan in [Su], motivating Feder and Gitler's work. McGibbon has also proved that the conjecture holds stably, in a suitable sense (see [MG1, Theorem 3.5]).

The solutions to the congruences (1) are more easily described one prime at a time. It turns out (see Proposition 2.1) that the  $p$ -local integers which satisfy  $C_1, \dots, C_n$  consist of the  $p$ -adic squares (if  $p = 2$  they must also be units or zero) together with all multiples of  $p^{e(p,n)}$  for a certain function  $e$  described in (2). The former can all be realized as degrees of self maps of the  $p$ -localization  $\mathbb{HP}_{(p)}^\infty$  by work of Rector (see Proposition 2.2), so to prove the conjecture it is enough to build self maps of  $\mathbb{HP}_{(p)}^n$  with degree an arbitrary multiple of  $p^{e(p,n)}$ .

The obstruction to extending a self map of  $\mathbb{HP}^n$  to  $\mathbb{HP}^{n+1}$  is an element in  $\pi_{4n+2}S^3$  (see (5)). We will use some of the information available on  $\pi_*S^3$  to analyse this obstruction and produce self maps of  $\mathbb{HP}_{(p)}^n$  with degree any multiple of  $p^{f(p,n)}$  where  $f$  is a different function described in (12) (see Theorem 4.3). In general,  $e(p,n) < f(p,n)$  but for  $n \leq 5$  the two functions agree which implies the following result.

**Theorem 1.2.** *The Feder-Gitler conjecture holds for  $n \leq 5$ .*

This is qualitatively stronger evidence for the conjecture than had previously been obtained as the obstruction group  $\pi_{4n-2}S^3$  is detected by  $K$ -theory and Adams operations (i.e. by the Adams  $e$ -invariant) only for  $n \leq 3$ . For  $n \geq 3$  it is easy to produce self maps of  $\mathbb{HP}^n$  with degree in  $\mathbf{FG}_{n+1}$  which do not extend one stage. Theorem 1.2 is proved by inductively building maps of high Adams filtration thus ensuring their obstructions to extension are detected by the  $e$ -invariant.

The paper is organized as follows. In section 2 we describe the  $p$ -local solutions to the congruences (1) and quote results of Rector which reduce the Feder-Gitler conjecture to producing maps with degree any multiple of  $p^{e(p,n)}$ . In section 3, we gather the necessary information on the unstable Adams spectral sequence for  $\mathbb{HP}^\infty$ . In section 4 we show that the degree of a self map of  $\mathbb{HP}^n$  is in  $\mathbf{FG}_{n+1}$  if and only if the Adams  $e$ -invariant of its obstruction to extension vanishes. The results quoted in section 3 then allow us to construct self maps of  $\mathbb{HP}_{(p)}^n$  of degree any multiple of  $p^{f(p,n)}$  and thus prove Theorem 1.2.

## 2. THE LOCAL VERSION OF THE CONJECTURE

We will write  $|k|_p = \sup\{l: p^l|k\}$  for the  $p$ -adic valuation of  $k$  and  $\lfloor x \rfloor$  for the greatest integer  $\leq x$ . In this section we reduce the Feder-Gitler conjecture to constructing self maps of the localizations  $\mathbb{HP}_{(p)}^n$  of degree  $k$  for all  $k$  with  $|k|_p \geq e(p,n)$  for a certain function  $e(p,n)$ .

First note that, by cellular approximation, the inclusion of  $\mathbb{HP}^n$  in  $\mathbb{HP}^\infty$  induces a bijection  $[\mathbb{HP}^n, \mathbb{HP}^n] \rightarrow [\mathbb{HP}^n, \mathbb{HP}^\infty]$ . As  $\mathbb{HP}_{(0)}^\infty = K(\mathbb{Q}, 4)$ , it follows easily from Sullivan's arithmetic square that in order to produce a self map of  $\mathbb{HP}^n$  of degree  $k$ , it suffices to do so after localizing at each prime  $p$  (or, in fact, at each prime  $p \leq 2n - 1$ ).

We will write  $\mathbf{R}_{n,p} = \{\deg(f) \in \mathbb{Z}_{(p)} \mid f : \mathbb{HP}_{(p)}^n \rightarrow \mathbb{HP}_{(p)}^n\}$  and  $\mathbf{FG}_{n,p}$  for the set of solutions in  $\mathbb{Z}_{(p)}$  of the congruences (1). Let  $\mathbf{D}_p$  denote the set defined by

$$\mathbf{D}_p = \begin{cases} \{k \in \mathbb{Z}_{(2)} \mid k = 0 \text{ or } k \equiv 1 \pmod{8}\} & \text{if } p = 2, \\ \{k \in \mathbb{Z}_{(p)} \mid k = u^2 \text{ for some } u \in \hat{\mathbb{Z}}_p\} & \text{if } p > 2. \end{cases}$$

It will soon emerge that  $\mathbf{D}_p = \mathbf{R}_{\infty,p} = \mathbf{FG}_{\infty,p}$ . Note that  $\mathbf{D}_2$  consists of the 2-adic unit squares in  $\mathbb{Z}_{(2)}$  together with 0. We define the function

$$(2) \quad e(p, n) = \begin{cases} 0 & \text{if } n = 1, \\ \#\{k \leq n \mid k = p^j \text{ or } k = (\frac{p+1}{2})p^{j-1} \text{ for some } j \geq 1\} & \text{if } p > 2, \\ 1 + 2\lfloor \log_2(n) \rfloor & \text{otherwise.} \end{cases}$$

**Proposition 2.1.** *Let  $p$  be a prime and  $n < \infty$ . Then*

$$\mathbf{FG}_{n,p} = \mathbf{D}_p \cup p^{e(p,n)}\mathbb{Z}_{(p)}.$$

*Proof.* The sets  $\mathbf{D}_p$  are the closure of  $\mathbf{R}_{\infty} = \{0, 1, 9, 25, \dots\}$  in  $\mathbb{Z}_{(p)}$  with the  $p$ -adic topology. Since the solution set of each congruence is closed we conclude that  $\mathbf{D}_p \subset \mathbf{FG}_{n,p}$  for every  $n, p$ . We will conclude the proof in the case  $p = 2$  (the case when  $p$  is odd is similar).

If  $k \in \mathbb{Z}_{(2)}$  is a solution of  $C_2$ , then either  $k \equiv 1 \pmod{8}$ , or  $k = 2^3l$  for some  $l \in \mathbb{Z}_{(2)}$  so it suffices to consider solutions  $k$  with  $|k|_2 \geq 3$ . For these, since  $|(4m+2)!/2|_2 = |(4m)!|_2$  it suffices to consider the congruences  $C_{2m}$ . So let  $|k|_2 \geq 3$ . Factoring out units in  $\mathbb{Z}_{(2)}$  we see that  $k$  is a solution of  $C_{2m}$  if and only if

$$\prod_{i=0}^{m-1} (k - (2i)^2) \equiv 0 \pmod{2^{(4m)!|_2}}.$$

Writing  $k = 4l$  and noting that  $|(4m)!|_2 = 2m + |(2m)!|_2$ , this becomes

$$\prod_{i=0}^{m-1} (l - i^2) \equiv 0 \pmod{2^{(2m)!|_2}}$$

so if  $k = 4l \in \mathbf{FG}_{2m,2}$ , then  $l \in \mathbf{FG}_{m,2}$ . Since  $l \notin \mathbb{Z}_{(2)}^\times$ , assuming the result is true for all  $n < 2^d$  and letting  $2^d \leq 2m < 2^{d+1}$ , it follows by induction that  $\mathbf{FG}_{2m,2} - \mathbf{D}_2 \subset 2^{2d+1}\mathbb{Z}_{(2)}$ .

It remains to check that for  $2^d \leq 2m < 2^{d+1}$  we have  $k \in \mathbf{FG}_{2m,2}$  as long as  $|k|_2 \geq 2d + 1$ . That is, for  $j < 2^d$  and  $|k|_2 \geq 2d + 1$ , the following equivalent inequalities hold:

$$\begin{aligned} \left| \prod_{i=0}^{2j-1} (k - i^2) \right|_2 &\geq |(4j)!|_2, \\ |k|_2 + \sum_{i=1}^{2j-1} 2|i|_2 &\geq 2j + j + \dots + \lfloor j/2^{d-1} \rfloor, \\ |k|_2 + 2(j + \dots + \lfloor j/2^{d-1} \rfloor) - 2\lfloor 2j \rfloor_2 &\geq 2j + j + \dots + \lfloor j/2^{d-1} \rfloor, \\ |k|_2 + j + \dots + \lfloor j/2^{d-1} \rfloor &\geq 2j + 2\lfloor 2j \rfloor_2. \end{aligned}$$

The last inequality is easy to check writing  $2j = 2^r v$  with  $v$  odd, noticing that  $r \leq d$  and  $v < 2^{d-r+1}$ . □

As  $e(p, n)$  tends to  $\infty$  with  $n$ , this means that  $\mathbf{FG}_{\infty,p} = \mathbf{D}_p$ . The following result of Rector implies that in fact we have  $\mathbf{R}_{\infty,p} = \mathbf{FG}_{\infty,p} = \mathbf{D}_p$  and, therefore, that in order to prove Conjecture 1.1 it suffices to construct self maps of  $\mathbb{HP}^n$  with degrees all multiples of  $p^{e(p,n)}$ . We note that, although the statement on [Re, p.103] in the case when  $p$  is odd concerns only the units in  $\mathbf{D}_p$ , the argument given applies verbatim to all of  $\mathbf{D}_p$ .

**Proposition 2.2** (Rector). *If  $k \in \mathbf{D}_p$ , then there is a self map of  $\mathbb{HP}_{(p)}^\infty$  of degree  $k$ .*

3. THE UNSTABLE ADAMS SPECTRAL SEQUENCE FOR  $\mathbb{HP}^\infty$

Let  $\mathcal{U}$  denote the category of unstable modules over the mod  $p$  Steenrod algebra. If  $M \in \mathcal{U}$ ,  $U(M)$  denotes the free unstable algebra generated by  $M$  and for  $P \in \mathcal{U}$  a free module, we write  $K(P)$  for the product of mod  $p$  Eilenberg-MacLane spaces such that  $H^*(K(P); \mathbb{F}_p) = U(P)$ . Recall from [HM2] (see also [Ha, Section 7.2]) that if  $X$  is a simply connected space such that  $H^*(X; \mathbb{F}_p) = U(M)$  for some  $M \in \mathcal{U}$  of finite type, then given a free resolution

$$0 \longleftarrow M \longleftarrow P_0 \longleftarrow P_1 \longleftarrow P_2 \longleftarrow \dots$$

of  $M$  with  $P_s$  of finite type and 0 in degree  $\leq s + 1$ , one can construct a tower of principal fibrations over  $X$ , called an *Adams resolution for  $X$*

$$(3) \quad \begin{array}{ccc} \vdots & & \\ \downarrow p_2 & & \\ X\{2\} & \xrightarrow{j_2} & K(\Omega^2 P_2) \\ \downarrow p_1 & & \\ X\{1\} & \xrightarrow{j_1} & K(\Omega P_1) \\ \downarrow p_0 & & \\ X\{0\} = X & \xrightarrow{j_0} & K(P_0) \end{array}$$

where  $\Omega$  denotes the algebraic loop functor (the left adjoint to suspension). The corresponding homotopy spectral sequence is the Massey-Peterson version of the unstable Adams spectral sequence. For a simply connected space of finite type this spectral sequence converges strongly and

$$E_2^{s,t}(X) = \text{Ext}_{\mathcal{U}}^s(M, \Sigma^t \mathbb{F}_p) \Rightarrow \pi_{t-s} X \otimes \widehat{\mathbb{Z}}_p.$$

The *Adams filtration* of a map  $f : W \rightarrow X$  is the largest  $s$  such that  $f$  factors through  $X\{s\}$ . It's easy to check that  $f$  has Adams filtration  $s$  if and only if it can be factored as a composite of  $s$  maps (of spaces) inducing 0 on mod  $p$  homology, but not as a composite of  $s + 1$  such maps. The following result is an immediate consequence of the definition of the homotopy spectral sequence.

**Lemma 3.1.** *Let  $X$  be a simply connected  $p$ -local space of finite type with  $H^*(X; \mathbb{F}_p) = U(M)$  and let  $E_r^{s,t}(X) \Rightarrow \pi_{t-s} X \otimes \widehat{\mathbb{Z}}_p$  denote the Massey-Peterson spectral*

sequence associated to a minimal resolution of  $M$  (so that  $d_1 = 0$ ). Then

- (a) If  $E_2^{s,n+s} = 0$  for  $s \geq r$ , then  $\pi_n X\{r\} = 0$ .
- (b) Suppose  $0 \leq r < s$  and the differentials  $d_k : E_k^{u,n+1+u} \rightarrow E_k^{u+k,n+u+k}$  are 0 for  $r \leq u < s$  and  $k \geq 2$ . Then the map  $\pi_n X\{s\} \rightarrow \pi_n X\{r\}$  is injective.

The algebra  $H^*(\mathbb{HP}^\infty; \mathbb{F}_p)$  is only of the form  $U(M)$  when  $p = 2$ , but as pointed out to me by John R. Harper, this problem can be overcome by replacing  $\mathbb{HP}^\infty$  with the homotopy fiber

$$\overline{\mathbb{HP}^\infty} = \text{hofib} \left( \mathbb{HP}^\infty \xrightarrow{\alpha} K(\mathbb{Z}/p, 4) \right)$$

where  $\alpha$  denotes a generator of  $H^4(\mathbb{HP}^\infty; \mathbb{F}_p)$  (when  $p = 2$  and we use a minimal resolution,  $\overline{\mathbb{HP}^\infty} = \mathbb{HP}^\infty\{1\}$ ). Indeed, it follows easily from the Serre spectral sequence of the fibration

$$K(\mathbb{Z}/p, 3) \rightarrow \overline{\mathbb{HP}^\infty} \rightarrow \mathbb{HP}^\infty$$

that, denoting by  $F_3$  the free unstable module on a class of degree 3 and by  $\overline{F_3}$  the kernel of the canonical map  $F_3 \rightarrow \Sigma^3 \mathbb{F}_p$ , we have  $H^*(\overline{\mathbb{HP}^\infty}; \mathbb{F}_p) = U(\overline{F_3})$ . It follows from the long exact sequence in Ext that the  $E_2$  terms of the Massey-Peterson spectral sequences for  $S^3$  and  $\overline{\mathbb{HP}^\infty}$  are related by

$$(4) \quad E_2^{s,t}(\overline{\mathbb{HP}^\infty}) = \text{Ext}_{\mathcal{U}}^s(\overline{F_3}, \Sigma^t \mathbb{F}_p) \cong \text{Ext}_{\mathcal{U}}^{s+1}(\Sigma^3 \mathbb{F}_p, \Sigma^t \mathbb{F}_p) = E_2^{s+1,t}(S^3).$$

We will now invoke some well-known information about the spectral sequence for  $S^3$  (cf. [Ta, CM]), from which it will follow that Figures 1 and 2 describe the portion of the  $E_2$  term of the spectral sequence for  $\overline{\mathbb{HP}^\infty}$  along the vanishing line. Vertical lines represent multiplication by  $p$  and the slanted lines composition with  $\eta$  if the prime is 2 and with  $\alpha_1$  if the prime is odd. If  $p$  is an odd prime, we set  $q = 2(p - 1)$ .

**Theorem 3.2** (Adams, Mahowald, Miller, Harper-Miller). *Consider Figures 1,2.*

- (a) Above the lines and classes shown, the  $E_2$  term vanishes.
- (b) The classes in dimensions  $qk + 3$  with filtration  $k - 1$  for  $p$  odd and the classes in dimension  $8k + 7$  with filtration  $\geq 4k + 1$  for  $p = 2$  detect elements in  $\pi_* \overline{\mathbb{HP}^\infty}$  which under the isomorphism  $\pi_* \overline{\mathbb{HP}^\infty} \rightarrow \pi_{*-1} S^3$  correspond to stable classes which are detected by the (stable, real) Adams  $e$ -invariant.<sup>1</sup>

*Proof.* Recall from [Ma2] and [HM1] that for each prime there is a bigraded complex  $(\Lambda(3), d)$  such that

$$E_2^{s,t}(S^3) = H^{s,t-3} \Lambda(3).$$

There is a short exact sequence of complexes

$$0 \longrightarrow \Lambda(1) \longrightarrow \Lambda(3) \longrightarrow W(1) \longrightarrow 0$$

which induces a split short exact sequence on homology.  $\Lambda(1)$  is a complex with 0 differential which corresponds to the  $\mathbb{Z}$ -tower in  $t - s = 3$ .

Let  $M \otimes \Lambda$  denote the  $E_1$  term of the stable Adams spectral sequence for the mod  $p$  Moore spectrum given by the  $\Lambda$ -algebra. The main results of [Ma2] and [HM1] state that there is a map of complexes  $W(1) \longrightarrow M \otimes \Lambda$  inducing an isomorphism on homology for  $t - s < 5s - 16$  for  $p = 2$  and  $t - s < (p + 1)qs - (p + 2)q$  for

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<sup>1</sup>See (11) below for the definition of the  $e$ -invariant.

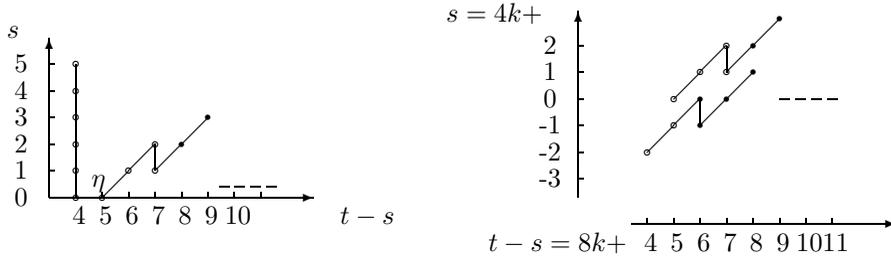


FIGURE 1.  $E_2^{(s,t)}(\overline{\mathbb{H}\mathbb{P}^\infty})$  for  $p = 2$ .

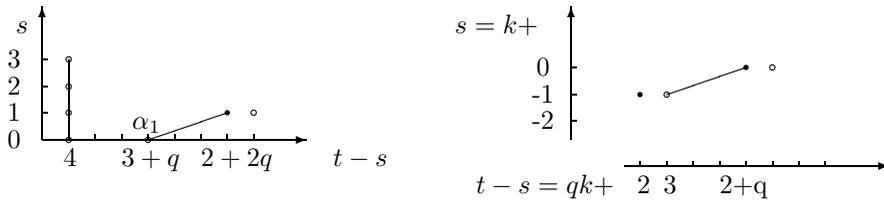


FIGURE 2.  $E_2^{(s,t)}(\overline{\mathbb{H}\mathbb{P}^\infty})$  for  $p$  odd.

$p$  odd. This reduces (a) to a statement about the stable  $E_2$  term of the Moore spectrum except in low dimensions where it is easily checked directly by computing the homology of the complex  $\Lambda(3)$ .

The  $E_2$  term of the Moore spectrum is computed in the required range in [Mi] for  $p$  odd and in [Ma1] for  $p = 2$ . It agrees with the description given in Figures 1 and 2. In view of (4), this completes the proof of (a).

In order to prove (b), let  $M^n$  denote the mod  $p$  Moore space with top cell in dimension  $n$  and  $i : S^{n-1} \rightarrow M^n$  denote the inclusion of the bottom cell. Let  $A : M^{n+r} \rightarrow M^n$  with  $r = 8$  if  $p = 2$  and  $r = q$  if  $p > 2$  be a map inducing an isomorphism in  $K$ -theory (see [Ad, pp. 65-68]). Recall that, by construction, when  $p = 2$ , the map  $A$  is the composite of 4 maps inducing 0 on mod 2 homology.

Suppose first that  $p$  is odd. Let  $\tilde{\alpha}$  denote an extension of a generator  $\alpha \in \pi_{2p}S^3$  to  $M^{2p+1}$ . Then by [Ad, Proposition 12.7],  $\alpha_k = \tilde{\alpha} \circ A^{k-1} \circ i \in \pi_{qk+2}S^3$  survives to a stable class which is detected by the  $e$ -invariant. The image of  $\Sigma\alpha_k$  in  $\pi_{qk+3}\mathbb{H}\mathbb{P}^\infty$  under the inclusion  $S^4 \hookrightarrow \mathbb{H}\mathbb{P}^\infty$  lifts uniquely to a class  $\overline{\Sigma\alpha_k} \in \pi_{qk+2}\overline{\mathbb{H}\mathbb{P}^\infty}$ . Since  $\alpha_k$  has Adams filtration at least  $k$ ,  $\overline{\Sigma\alpha_k}$  has Adams filtration at least  $k - 1$  and so it must be represented in  $E_2(\overline{\mathbb{H}\mathbb{P}^\infty})$  by the class in bidegree  $(t - s, s) = (qk + 3, k - 1)$ .

Next let  $p = 2$ . Let  $\mu \in \pi_{12}S^3$  be the unique element with Adams filtration 5 (corresponding to the leftmost class of the top lightning flash in Figure 1, for  $k = 1$ ). The stabilization map  $\pi_{12}S^3 \rightarrow \pi_9S^0$  is injective (see [To]) and cannot decrease Adams filtration. It follows that  $\mu$  suspends to the class  $\mu_9 = \overline{\eta} \circ A \circ i$  in  $\pi_9S^0$  constructed in [Ad, Theorem 12.13] (there is only one class of filtration 5 in the 9-stem; see for instance [MT, p. 202]). Let  $\tilde{\mu}$  be an extension of  $\mu$  to  $M^{13}$  chosen to have Adams filtration 5, which is possible as the map  $S^3\{5\} \rightarrow S^3$

determined by a minimal Adams resolution is injective on  $\pi_{12}$  by Lemma 3.1(b). Then  $\tilde{\mu}_k = \tilde{\mu} \circ A^{k-1} \circ i \in \pi_{8k+4}S^3$  has Adams filtration at least  $4k + 1$ . Lemma 3.1 implies that  $S^3\{4k + 1\} \rightarrow S^3$  and  $S^3\{4k + 2\} \rightarrow S^3\{4k + 1\}$  are injective on  $\pi_{8k+4}$  from which it is easy to check that the Toda bracket  $\langle \tilde{\mu}_k, 2, \eta \rangle \in \pi_{8k+6}S^3$  has Adams filtration at least  $4k + 2$ . [Ad, Proposition 12.18] shows that this bracket survives to a stable class of order 4 detected by the  $e$ -invariant so the bracket must be represented on  $E_2(S^3)$  by the class in bidegree  $(t - s, s) = (8k + 6, 4k + 2)$ . As in the case when  $p$  is odd, this bracket corresponds to an element in  $\pi_{8k+7}\overline{\mathbb{HP}^\infty}$  with Adams filtration  $4k + 1$  represented in  $E_2(\overline{\mathbb{HP}^\infty})$  by the class in bidegree  $(t - s, s) = (8k + 7, 4k + 1)$ . This completes the proof of (b).  $\square$

4. CONSTRUCTION OF MAPS

Let  $\nu : S^{4n+3} \rightarrow \mathbb{HP}^n$  denote the Hopf map. The cofiber sequence

$$S^{4n+3} \xrightarrow{\nu} \mathbb{HP}^n \rightarrow \mathbb{HP}^{n+1}$$

shows that the obstruction to extending a map  $f : \mathbb{HP}^n \rightarrow \mathbb{HP}^\infty$  to  $\mathbb{HP}^{n+1}$  is

$$(5) \quad o(f) := [f \circ \nu] \in \pi_{4n+3}\mathbb{HP}^\infty.$$

Alternatively, one can regard the obstruction as an element in  $\pi_{4n+3}\mathbb{HP}^{n+1}$ , in which case, it is given by the formula

$$(6) \quad o(f) := [f \circ \nu] - \deg(f)^{n+1}[\nu] \in \pi_{4n+3}\mathbb{HP}^{n+1}$$

as a simple cohomology calculation shows. Note that, as the equivalence  $S^3 \rightarrow \Omega\mathbb{HP}^\infty$  factors through  $\Omega S^4$ , any element of  $\pi_*\mathbb{HP}^\infty$  (in particular the obstruction to extension) factors canonically through the bottom cell of  $\mathbb{HP}^\infty$ . We will still write  $o(f)$  for the corresponding element in  $\pi_{4n+3}S^4$ .

We will write  $KO(X), K(X), KSp(X)$  for the reduced  $K$ -groups of a space  $X$ . The Atiyah-Hirzebruch spectral sequence implies that  $K^0(\mathbb{HP}^n) = \mathbb{Z}[x]/x^{n+1}$ . In [FG], Feder and Gitler show that the Adams operations on  $K(\mathbb{HP}^n)$  are determined by the formula

$$(7) \quad \psi^l(x) = l^2x + \dots + \frac{2l^2(l^2 - 1) \dots (l^2 - (n - 1)^2)}{(2n)!}x^n.$$

Moreover, they show that  $\phi : K^0(\mathbb{HP}^n) \otimes \mathbb{R} \rightarrow K^0(\mathbb{HP}^n) \otimes \mathbb{R}$  is a map of rings commuting with Adams operations iff  $\phi = \psi^l$  (i.e.  $\phi$  is determined by (7)) for some real or purely imaginary number  $l$ . In particular, given a self map  $f$  of  $\mathbb{HP}^n$ , the induced homomorphism on  $K$ -theory is  $f^* = \psi^{\sqrt{\deg(f)}}$ .

Let  $\mathcal{A}$  denote Adams' category of finitely generated abelian groups with real Adams operations [Ad, Section 6].

**Lemma 4.1.** *Let  $f$  be a self map of  $\mathbb{HP}^n$  of degree  $k$ . Then  $k \in \mathbf{FG}_{n+1}$  if and only if there is a self map  $\phi$  of  $KO(\Sigma^4\mathbb{HP}^{n+1})$  in  $\mathcal{A}$  such that the following diagram commutes:*

$$(8) \quad \begin{array}{ccccccc} 0 & \longrightarrow & KO(S^{4n+8}) & \longrightarrow & KO(\Sigma^4\mathbb{HP}^{n+1}) & \longrightarrow & KO(\Sigma^4\mathbb{HP}^n) \longrightarrow 0 \\ & & \downarrow k^{n+1} & & \downarrow \phi & & \downarrow \Sigma^4 f^* \\ 0 & \longrightarrow & KO(S^{4n+8}) & \longrightarrow & KO(\Sigma^4\mathbb{HP}^{n+1}) & \longrightarrow & KO(\Sigma^4\mathbb{HP}^n) \longrightarrow 0. \end{array}$$

*Proof.* As  $\psi^{-1}$  acts trivially,  $K(\mathbb{H}P^n)$ , is an object of  $\mathcal{A}$ . For each real or purely imaginary number  $l$ , let  $\psi_n^l: K(\mathbb{H}P^n) \otimes \mathbb{R} \rightarrow K(\mathbb{H}P^n) \otimes \mathbb{R}$  denote the map of real vector spaces with Adams operations determined by (7) and multiplicative extension. If  $\varphi$  is a map of vector spaces such that

$$(9) \quad \begin{array}{ccccccc} 0 & \longrightarrow & K(S^{4n+4}) \otimes \mathbb{R} & \longrightarrow & K(\mathbb{H}P^{n+1}) \otimes \mathbb{R} & \longrightarrow & K(\mathbb{H}P^n) \otimes \mathbb{R} \longrightarrow 0 \\ & & \downarrow k^{n+1} & & \downarrow \varphi & & \downarrow f^* = \psi_n^{\sqrt{k}} \\ 0 & \longrightarrow & K(S^{4n+4}) \otimes \mathbb{R} & \longrightarrow & K(\mathbb{H}P^{n+1}) \otimes \mathbb{R} & \longrightarrow & K(\mathbb{H}P^n) \otimes \mathbb{R} \longrightarrow 0 \end{array}$$

commutes, then we can write (using the monomial basis for  $K(\mathbb{H}P^{n+1})$ )

$$\varphi = \begin{bmatrix} \psi_n^{\sqrt{k}} & 0 \\ w & k^{n+1} \end{bmatrix} = \psi_{n+1}^{\sqrt{k}} + \begin{bmatrix} 0 & 0 \\ v & 0 \end{bmatrix}$$

for some  $v \in \mathbb{R}^n$ . The map  $\varphi$  commutes with  $\psi_{n+1}^l$  iff  $v\psi_n^l = l^{2n+2}v$ . But in view of (7),  $l^{2n+2}$  is not an eigenvalue of  $\psi_n^l$  (except for  $l \in \{-1, 0, 1\}$ ) and hence  $\varphi = \psi_{n+1}^{\sqrt{k}}$ . In particular,  $\varphi$  must be a ring homomorphism. Conversely,  $\varphi = \psi_{n+1}^{\sqrt{k}}$  gives a map making (9) commute.

The integer  $k$  belongs to  $\mathbf{FG}_{n+1}$  iff  $\varphi(x)$  is in the image of the restriction map  $KSp(\mathbb{H}P^n) \rightarrow K(\mathbb{H}P^n)$  (i.e.  $\varphi(x) = a_1x + \dots + a_nx^n$  with  $a_i \in \mathbb{Z}$  and  $a_{2i+1} \in 2\mathbb{Z}$ ). Since  $\varphi$  is a ring homomorphism this is equivalent to saying that the matrix  $[a_{ij}]$  representing  $\varphi$  in the monomial basis has

$$(10) \quad a_{ij} \in \mathbb{Z} \quad \text{and} \quad a_{(2i+1)(2j+1)} \in 2\mathbb{Z}.$$

Given  $M$  in  $\mathcal{A}$ , let  $\Sigma^4 M$  denote the object in  $\mathcal{A}$  with underlying abelian group  $M$  and  $\psi_{\Sigma^4 M}^l = l^2 \psi_M^l$ . Note that when  $M$  is torsion free, the self maps of  $M$  and  $\Sigma^4 M$  in  $\mathcal{A}$  are the same. There is a natural inclusion in  $\mathcal{A}$

$$\Sigma^4 K(\mathbb{H}P^n) \xrightarrow{j} KO(\Sigma^4 \mathbb{H}P^n)$$

determined by the zigzag

$$\Sigma^4 K(\mathbb{H}P^n) \xleftarrow{\Sigma^4 c} \Sigma^4 KO(\mathbb{H}P^n) \xleftarrow{p} KO(\Sigma^4 \mathbb{H}P^n)$$

where  $c$  denotes the complexification map and  $p$  is given by multiplication with the generator of  $KO^{-4}(\ast)$ . The map  $j \otimes \mathbb{R}$  is an isomorphism of vector spaces with Adams operations and hence, upon tensoring with  $\mathbb{R}$ , the existence of a map  $\phi \otimes \mathbb{R}$  making (8)  $\otimes \mathbb{R}$  commute is guaranteed. Moreover we must have  $\phi \otimes \mathbb{R} = (j \otimes \mathbb{R}) \circ \varphi \circ (j \otimes \mathbb{R})^{-1}$ .

One easily checks that the integrality condition (10) on the matrix representation of  $\varphi$  precisely corresponds to the integrality of  $\phi \otimes \mathbb{R}$ , i.e., to the existence of the map  $\phi$  in the statement. This completes the proof.  $\square$

Recall from [Ad] that the (real)  $e$ -invariant is a group homomorphism

$$(11) \quad Z \xrightarrow{e} \text{Ext}_{\mathcal{A}}(KO(X), KO(S^{j+1}))$$

where  $Z = \{\alpha \in \pi_j(X) \mid KO(\alpha) = 0\}$ .  $e(\alpha)$  is defined to be the Yoneda class of the extension

$$0 \rightarrow KO(S^{j+1}) \rightarrow KO(X \cup_{\alpha} e^{j+1}) \rightarrow KO(X) \rightarrow 0.$$

If  $X$  is a sphere there is a natural identification of the target of  $e$  with a subgroup of  $\mathbb{Q}/\mathbb{Z}$ . The stable  $e$ -invariant of  $\beta \in \pi_{k+l} S^k$  is defined by suspending  $\beta$  to a

class  $\beta' \in \pi_{8m+i}S^{8m}$  for some  $m$  and then setting  $e^s(\beta) = e(\beta') \in \mathbb{Q}/\mathbb{Z}$ . This is independent of the choice of  $m$ .

**Proposition 4.2.** *Let  $f$  be a self map of  $\mathbb{HP}^n$  and let  $o(f) \in \pi_{4n+3}S^4$  denote its obstruction to extension. Then  $\deg(f) \in \mathbf{FG}_{n+1}$  if and only if the stable  $e$ -invariant of  $o(f)$  vanishes.*

*Proof.* Let  $k = \deg(f)$ . By Lemma 4.1,  $k \in \mathbf{FG}_{n+1}$  iff the map of extensions (8) exists. Writing  $\circ$  for the Yoneda product in the category of extensions, and  $\Sigma^4\nu : S^{4n+3} \rightarrow \Sigma^4\mathbb{HP}^n$  for the attaching map of the top cell of  $\Sigma^4\mathbb{HP}^{n+1}$ , the existence of (8) is translated into the following equalities in  $\text{Ext}_{\mathcal{A}}$ :

$$\begin{aligned} e(\Sigma^4\nu) \circ \Sigma^4 f^* &= k^{n+1} \circ e(\Sigma^4\nu), \\ e(\Sigma^4(f \circ \nu)) &= e(k^{n+1}\Sigma^4\nu), \\ e(\Sigma^4 o(f)) &= 0 \in \text{Ext}_{\mathcal{A}}(KO(\Sigma^4\mathbb{HP}^n), KO(S^{4n+8})), \end{aligned}$$

where we have used that the  $e$ -invariant sends compositions of maps to Yoneda products, is a group homomorphism and (6). Since  $o(f)$  factors through the inclusion of the bottom cell of  $\mathbb{HP}^n$ , letting  $X = \Sigma^4\mathbb{HP}^n \cup_{o(f)} e^{4n+8}$ , we have a map of extensions

$$\begin{array}{ccccccc} 0 & \longrightarrow & KO(S^{4n+8}) & \longrightarrow & KO(X) & \longrightarrow & KO(\Sigma^4\mathbb{HP}^n) \longrightarrow 0 \\ & & \downarrow = & & \downarrow & & \downarrow i^* \\ 0 & \longrightarrow & KO(S^{4n+8}) & \longrightarrow & KO(S^8 \cup_{\Sigma^4 o(f)} e^{4n+8}) & \longrightarrow & KO(S^8) \longrightarrow 0 \end{array}$$

where  $i$  denotes the inclusion of the bottom cell. We conclude that  $e^s(o(f)) \circ i^* = e(\Sigma^4 o(f))$ . The long exact sequence in  $\text{Ext}_{\mathcal{A}}$  induced by the short exact sequence

$$0 \longrightarrow KO(\Sigma^4(\mathbb{HP}^n/\mathbb{HP}^1)) \longrightarrow KO(\Sigma^4\mathbb{HP}^n) \xrightarrow{i^*} KO(S^8) \longrightarrow 0$$

together with the easily checked fact that  $\text{Hom}_{\mathcal{A}}(KO(\Sigma^4(\mathbb{HP}^n/\mathbb{HP}^1)), KO(S^{4n+8})) = 0$  show that composition with  $i^*$  is injective which concludes the proof.  $\square$

We are now ready to prove our main result. Consider the function

$$(12) \quad f(p, n) = \begin{cases} e(p, n) & \text{for } (p = 2, n \leq 5) \text{ or } (p = 3, n \leq 6) \\ & \text{or } \left( p > 3, n \leq 1 + \frac{(p-1)^2}{2} \right), \\ 4\lfloor \frac{n}{2} \rfloor - 3 & \text{for } p = 2, n > 5, \\ \lceil \frac{2n-2}{p-1} \rceil - 1 & \text{otherwise,} \end{cases}$$

where  $\lceil x \rceil$  denotes the least integer  $\geq x$ . Theorem 1.2 is now an immediate consequence of the following result.

**Theorem 4.3.**  $p^{f(p,n)}\mathbb{Z}_{(p)} \subset \mathbf{R}_{n,p}$ .

For a fixed  $p$ , the function  $e(p, n)$  (defined in (2)) grows logarithmically while  $f(p, n)$  grows linearly. Thus there remains a large gap between Theorem 4.3 and Conjecture 1.1. For the convenience of the reader, we include Table 1 comparing the values of  $e(p, n)$  and  $f(p, n)$  for small values of  $p$  and  $n$ .

The two inputs needed for the proof of Theorem 4.3 are the (local versions of) Proposition 4.2 and the information about the unstable Adams spectral sequence contained in Theorem 3.2.

TABLE 1. Comparison of  $e(p, n)$  and  $f(p, n)$  for  $p \leq 5$  and  $n \leq 10$ .

| $n$ | $e(2, n)$ | $f(2, n)$ | $e(3, n)$ | $f(3, n)$ | $e(5, n)$ | $f(5, n)$ |
|-----|-----------|-----------|-----------|-----------|-----------|-----------|
| 1   | 0         | 0         | 0         | 0         | 0         | 0         |
| 2   | 3         | 3         | 1         | 1         | 0         | 0         |
| 3   | 3         | 3         | 2         | 2         | 1         | 1         |
| 4   | 5         | 5         | 2         | 2         | 1         | 1         |
| 5   | 5         | 5         | 2         | 2         | 2         | 2         |
| 6   | 5         | 9         | 3         | 3         | 2         | 2         |
| 7   | 5         | 9         | 3         | 5         | 2         | 2         |
| 8   | 7         | 13        | 3         | 6         | 2         | 2         |
| 9   | 7         | 13        | 4         | 7         | 2         | 2         |
| 10  | 7         | 17        | 4         | 8         | 2         | 4         |

Using exclusively Proposition 4.2, we see that the  $p$ -local version of Conjecture 1.1 holds for  $n$  as long as the obstruction groups  $\pi_{4j-2}S_{(p)}^3$  are detected by the  $e$ -invariant for  $j \leq n$ . As a consequence of the work of Toda [To] and Adams [Ad] (precise references will be given in the proof below) this happens in the following ranges:  $n \leq 3$  for  $p = 2$ ,  $n \leq 6$  for  $p = 3$  and  $n \leq 1 + \frac{(p-1)^2}{2}$  for  $p > 3$ . The range for  $p > 3$  can be roughly doubled using the techniques of this paper (cf. Remark 4.4(a) below) but we will not pursue this here. The strength of the  $e$ -invariant thus accounts for the first line in the definition of  $f(p, n)$  except for the cases when  $p = 2$  and  $n = 4, 5$ .

The remaining cases in the definition of  $f(p, n)$  come from putting together Proposition 4.2 and the description of the unstable Adams spectral sequence along the vanishing line in Theorem 3.2. The slopes of the functions  $n \mapsto f(p, n)$  are determined by the vanishing line of the unstable Adams spectral sequence but, crucially for the cases when  $p = 2$  and  $n = 4, 5$ , the fact that the classes along the vanishing line are detected by the  $e$ -invariant effectively lowers the  $y$ -intercept of the vanishing line. At odd primes, this “lowering of the vanishing line” is not enough to prove the  $p$ -local Feder-Gitler conjecture in any cases other than those settled by the method of the previous paragraph. Nevertheless, it permits the construction of a number of maps for higher values of  $n$ .

*Proof of Theorem 4.3.* The obstruction to extending a map  $\mathbb{H}\mathbb{P}^{n-1} \rightarrow \mathbb{H}\mathbb{P}_{(p)}^\infty$  lies in  $\pi_{4n-1}\mathbb{H}\mathbb{P}_{(p)}^\infty = \pi_{4n-2}S_{(p)}^3$ . For  $p = 2$ , this group is detected by the  $e$ -invariant for  $n \leq 3$ . This is just the case when  $p = 2$  and  $k = 0$  of Theorem 3.2(b). For  $p = 3$ , the obstruction group is detected by the  $e$ -invariant for  $n \leq 6$ . This follows from the fact that  $\pi_{4n-2}S_{(3)}^3 = \mathbb{Z}/3$  for  $2 \leq n \leq 6$  (see [To, Theorems 13.4,13.9,13.10]) as Theorem 3.2(b) gives a  $\mathbb{Z}/3$  summand in these groups which is detected by the  $e$ -invariant. Similarly, for  $p > 3$ , the obstruction group is detected by the  $e$ -invariant for  $n \leq 1 + \frac{(p-1)^2}{2}$  (but see also Remark 4.4(a)). Indeed,  $\pi_{4n-2}S_{(p)}^3$  is either  $\mathbb{Z}/p$  or 0 in this range of values of  $n$  by [To, Theorem 13.4] and, when it is not 0, Theorem 3.2(b) provides a  $\mathbb{Z}/p$  summand in the group which is detected by the  $e$ -invariant. By (the  $p$ -local version of) Proposition 4.2, the proof of the theorem when  $p = 2, n \leq 3$  or  $p = 3, n \leq 6$  or  $p > 3, n \leq 1 + \frac{(p-1)^2}{2}$  is complete.

First note that, in the remaining cases,  $f(p, n) \geq e(p, n) \geq 2$ . Pick an Adams resolution for  $\overline{\mathbb{H}P^\infty}$  constructed from a minimal resolution. Then the Adams covers  $\overline{\mathbb{H}P^\infty}\{s\}$  are 3-connected,  $\pi_4(\overline{\mathbb{H}P^\infty}\{s\}) = \mathbb{Z}_{(p)}$  and the maps  $\overline{\mathbb{H}P^\infty}\{s\} \rightarrow \mathbb{H}P^\infty$  induce multiplication by  $p^{s+1}$  on  $\pi_4$ . Thus, in order to prove the theorem, it is enough to provide a map  $\mathbb{H}P^n \rightarrow \overline{\mathbb{H}P^\infty}\{f(p, n) - 1\}$  of degree  $k$  on the bottom cell for an arbitrary  $k \in \mathbb{Z}_{(p)}$ . Such a map will be obtained by extending a map  $\mathbb{H}P^1 \rightarrow \overline{\mathbb{H}P^\infty}\{f(p, n) - 1\}$  of degree  $k$ .

Consider first the case when  $p = 2$  and suppose  $n$  is even. Then for  $k < n - 1$  the obstruction to extension of the map  $\mathbb{H}P^k \rightarrow \overline{\mathbb{H}P^\infty}\{f(2, n) - 1\}$  to  $\mathbb{H}P^{k+1}$  lies in the group  $\pi_{4k+3}\mathbb{H}P^\infty\{f(2, n) - 1\}$ . By Lemma 3.1(a) and the vanishing line of Theorem 3.2(a) this group is 0 so the map extends. Thus we have a map  $\mathbb{H}P^{n-1} \rightarrow \overline{\mathbb{H}P^\infty}\{f(2, n) - 1\}$ . Since the Hopf map  $S^{4n-1} \rightarrow \mathbb{H}P^{n-1}$  has Adams filtration 1, the composite  $S^{4n-1} \rightarrow \overline{\mathbb{H}P^\infty}\{f(2, n) - 1\}$  factors through  $\overline{\mathbb{H}P^\infty}\{f(2, n)\}$ . Let  $\gamma : S^{4n-1} \rightarrow \overline{\mathbb{H}P^\infty}\{f(2, n)\}$  denote this obstruction class. By Theorem 3.2(b), the image  $\gamma'$  of  $\gamma$  in  $\pi_{4n-1}\mathbb{H}P^\infty$  is detected by the  $e$ -invariant (it is in the summand of order 4 corresponding to one of the top lightning flashes). Since  $\gamma'$  is the obstruction to extension of a map  $\mathbb{H}P^{n-1} \rightarrow \mathbb{H}P^\infty$  with degree in  $\mathbf{F}G_{n,2}$  (because  $f(p, n) \geq e(p, n)$ ), Proposition 4.2 implies that  $\gamma' = 0$ . Since the map  $\pi_{4n-1}\overline{\mathbb{H}P^\infty}\{f(2, n)\} \rightarrow \pi_{4n-1}\mathbb{H}P^\infty$  is injective by Lemma 3.1(b), we conclude that  $\gamma = 0$  and hence our map extends to a map  $\mathbb{H}P^n \rightarrow \overline{\mathbb{H}P^\infty}\{f(2, n) - 1\}$  of degree  $k$ . This completes the proof of the case when  $p = 2$  and  $n$  is even.

The case when  $p = 2$  and  $n$  is odd follows, since  $f(2, 2m + 1) = f(2, 2m)$  and, for  $n = 2m + 1$ , the map  $\mathbb{H}P^{2m} \rightarrow \overline{\mathbb{H}P^\infty}\{f(2, 2m) - 1\}$  described in the previous paragraph extends one more stage. Indeed, its obstruction to extension factors through  $\overline{\mathbb{H}P^\infty}\{f(2, 2m)\}$  and hence vanishes by Lemma 3.1(a) (using the vanishing line of Theorem 3.2(a)). This completes the proof for  $p = 2$ .

For  $p$  odd, the reasoning is identical: suppose  $j \geq 2$  and we want to extend a map  $\mathbb{H}P^1 \rightarrow \overline{\mathbb{H}P^\infty}\{j - 1\}$  of degree  $k \in \mathbb{Z}_{(p)}$ . By Lemma 3.1(a) and the vanishing line of Theorem 3.2(a) there is no obstruction to extending it to  $\mathbb{H}P^n$  for  $n \leq \frac{j(p-1)}{2}$ . For  $n = \frac{j(p-1)}{2} + 1$  there is an obstruction but, since the Hopf map has Adams filtration 1, the obstruction factors through  $\overline{\mathbb{H}P^\infty}\{j\}$  and hence vanishes. By Theorem 3.2 there are no further obstructions until  $n = \frac{(j+1)(p-1)}{2} + 1$ . At that point, there is an obstruction, which again factors through  $\overline{\mathbb{H}P^\infty}\{j\}$ . This obstruction is detected by the  $e$ -invariant and hence, by Proposition 4.2, vanishes as long as  $j \geq e\left(p, \frac{(j+1)(p-1)}{2} + 1\right)$ . This last condition holds because  $j \geq 2$  and hence we obtain a map  $\mathbb{H}P^{\frac{(j+1)(p-1)}{2} + 1} \rightarrow \mathbb{H}P_{(p)}^\infty$  of degree  $kp^j$ .

For a given  $n$ , the least  $j$  such that  $n \leq \frac{(j+1)(p-1)}{2} + 1$  is  $\lceil \frac{2n-2}{p-1} \rceil - 1$ . In the range of values of  $n$  under consideration (namely  $n \geq 7$  for  $p = 3$  and  $n > 1 + \frac{(p-1)^2}{2}$  for  $p > 3$ ) we have  $f(p, n) = \lceil \frac{2n-2}{p-1} \rceil - 1 \geq 2$ . We conclude that there are maps of degree  $kp^{f(p,n)} : \mathbb{H}P^n \rightarrow \mathbb{H}P_{(p)}^\infty$  for all  $k \in \mathbb{Z}_{(p)}$ . This completes the proof.  $\square$

*Remark 4.4.* For  $p > 3$ , Theorem 4.3 can be improved using the following observations:

- (a) For  $n < \frac{(2p+1)(p-1)}{2}$  all classes in  $\pi_{4n-2}(S_{(p)}^3)$  are detected by the  $e$ -invariant, therefore, the conjecture holds at odd primes for these values of  $n$ . This follows from the results of Mahowald and Harper-Miller (cf. proof of Theorem 3.2)

together with calculations with the  $\Lambda$ -algebra in low degrees. See [G, Corollary 2.9] for a proof.

- (b) Let  $g(p, n) = \sup\{k : p^k \mathbb{Z}_{(p)} \subset \mathbf{R}_{n,p}\}$ . Using (6), it is easy to show that, for  $f, g$  self maps of  $\mathbb{H}\mathbb{P}^n$ , the obstruction to extension of the composite is  $o(fg) = \deg(g)^n o(f) + \deg(f) o(g)$  (see [G, Proposition 4.6(b)]). Since (by Proposition 2.2) there is a self map of  $\mathbb{H}\mathbb{P}_{(p)}^n$  of degree  $p^2$ , Selick's exponent theorem [Se] implies that

$$(13) \quad g(p, n+1) \leq 2 + g(p, n).$$

For  $p > 3$ , the first positive value of the function  $n \mapsto f(p, n) - e(p, n)$  is  $p - 3$  (with the improved range of Remark 4.4(a), it would be  $2p - 3$ ). When this number is greater than 2, we can apply (13) to improve on Theorem 4.3 in a very limited range of values of  $n$ .

*Remark 4.5.* The first open case of Conjecture 1.1 is  $n = 6$ . In view of Theorem 4.3, the only prime that needs to be considered is  $p = 2$ . The relevant obstruction group is  $\pi_{22} S_{(2)}^3 = \mathbb{Z}/2 \oplus \mathbb{Z}/4$  (cf. [To, Theorem 12.9]). As the  $\mathbb{Z}/4$  summand corresponds to a class detected by the  $e$ -invariant, in order to settle the conjecture for  $n = 6$  it suffices to find an effective way of detecting the remaining  $\mathbb{Z}/2$  summand. This  $\mathbb{Z}/2$  is generated by an unstable  $v_1$ -periodic class appearing in the bottom lightning flash in Figure 1. At an odd prime, the first value of  $n$  for which the conjecture is open is  $n = \frac{(2p+1)(p-1)}{2}$ . The obstruction group for this value of  $n$  is  $\pi_{(2p+1)q-1} S_{(p)}^3 = \mathbb{Z}/p \oplus \mathbb{Z}/p$ . One of the  $\mathbb{Z}/p$  summands is detected by the  $e$ -invariant whilst the other is not  $v_1$ -periodic (see [G, Corollary 2.9] for more details).

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