ON THE COLLECTION OF BAIRE CLASS ONE FUNCTIONS ON THE IRRATIONALS

ROMAN POL

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Abstract. The Baire class one fan over the irrationals \( \mathbb{N}^\mathbb{N} \) is the space \( S(\mathbb{N}^\mathbb{N}) = (\mathbb{N}^\mathbb{N} \times \mathbb{N}) \cup \{ \infty \} \), where basic neighbourhoods of \( \infty \) are epigraphs of the first Baire class functions \( f : \mathbb{N}^\mathbb{N} \to \mathbb{N} \), augmented by \( \infty \), and the remaining points are isolated; \( S(\mathbb{N}) = (\mathbb{N} \times \mathbb{N}) \cup \{ \infty \} \) is the standard countable sequential fan. We prove that \( S(\mathbb{N}^\mathbb{N}) \times S(\mathbb{N}) \) has countable tightness: in this product, whenever \( p \in A \), then \( p \in B \) for some countable \( B \subset A \).

1. Introduction

Let \( \mathbb{N} \) be the set of natural numbers in the real line \( \mathbb{R} \) and let \( \mathbb{N}^\mathbb{N} \) be the countable product of \( \mathbb{N} \), homeomorphic to the irrationals.

Given a metrizable space \( X \), the Baire class one fan over \( X \) is the space \( S(X) = (X \times \mathbb{N}) \cup \{ \infty \} \), where basic neighbourhoods of the point \( \infty \) are the epigraphs \( \{(x,n) : n \geq f(x)\} \) of the first Baire class functions \( f : X \to \mathbb{N} \), augmented by \( \infty \), and the points in \( X \times \mathbb{N} \) are isolated. Note that for any infinite discrete \( X \), \( S(X) \) is the standard sequential fan of cardinality \( |X| \), cf. [EGKTT].

Let us recall that a space \( E \) has countable tightness if, whenever \( x \in A \) in \( E \), there is a countable set \( B \subset A \) with \( x \in B \), cf. [Ar]. The aim of this note is to prove the following theorem.

Theorem 1.1. The product \( S(\mathbb{N}^\mathbb{N}) \times S(\mathbb{N}) \) of the Baire class one fan over the irrationals and the countable sequential fan has countable tightness.

G. Gruenhage [Gr] proved that for a discrete space \( X \), the product of the sequential fans \( S(X) \times S(\mathbb{N}) \) has countable tightness if and only if \( |X| < b \leq 2^\omega \), cf. [Kun], III.1.11.

We shall prove in fact that if \( X \) is a metrizable separable space all whose closed subspaces are Baire, then \( S(X) \times S(\mathbb{N}) \) has countable tightness (however, in contrast with discrete spaces, for each such uncountable \( X \), the fan \( S(X) \) is not sequential, cf. Comment 4.1).

Let us also mention that there are subspaces \( Y \) of \( \mathbb{R} \) such that \( S(Y) \times S(\mathbb{N}) \) fails to have countable tightness, cf. Comment 4.2.

Remark 1.2. Theorem 1.1 also implies the following statement: each set in \( (\mathbb{N}^\mathbb{N} \times \mathbb{R}) \times (\mathbb{N} \times \mathbb{R}) \) intersecting every product of the epigraph of a Baire class one function...
Let, cf. (1):

\[ f : \mathbb{N}^\mathbb{N} \to \mathbb{R} \]

and the epigraph of a function \( \varphi : \mathbb{N} \to \mathbb{N} \) contains a countable subset with this property.

In fact, for any function of Baire class one \( f : \mathbb{N}^\mathbb{N} \to \mathbb{R} \), if \( f_n : \mathbb{N}^\mathbb{N} \to \mathbb{R} \) are continuous functions converging pointwise to \( f \), \( g(x) = \min\{m \in \mathbb{N} : f_n(x) \leq m \text{ for all } n\} \) is a lower-semicontinuous function with \( f \leq g \).

### 2. The main lemma

The following notion comes in handy in our proof of Theorem 1.1.

**Definition 2.1.** Let \( X \) be a separable metrizable space. A set \( A \subset X \times \mathbb{N} \) is nowhere bounded on \( L \subset X \), if \( L \neq \emptyset \) and for each nonempty relatively open set \( W \in L \) and \( n \in \mathbb{N} \), \( A \cap (W \times [n, +\infty)) \neq \emptyset \).

Let

\( \pi_X : X \times \mathbb{N} \to X \), \( \pi_X(x, n) = x \),

be the projection. Then \( A \subset X \times \mathbb{N} \) is nowhere bounded on \( L \neq \emptyset \) if and only if all projections \( \pi_X(A \cap (L \times [n, +\infty)) \) are dense in \( L \).

**Lemma 2.2.** Let \( X \) be a separable metrizable space all whose closed subspaces are Baire and let \( A \subset X \times \mathbb{N} \) be nowhere bounded on \( L \subset X \). Then there is a countable set \( B \subset A \cap (L \times \mathbb{N}) \) intersecting every epigraph in \( X \times \mathbb{N} \) of Baire class one functions \( f : X \to \mathbb{N} \).

**Proof.** For each \( n \in \mathbb{N} \) one can choose a countable set \( B_n \subset (A \cap (L \times [n, +\infty)) \) such that \( \pi_X(B_n) \) is dense in \( L \), and let \( B = \bigcup_n B_n \).

Let \( f : X \to \mathbb{N} \) be any Baire class one function. Since \( F = L \) is Baire, \( f \) restricted to \( F \) has a point of continuity, cf. \([\text{Kur}], \S\ 34, \text{VII}, \) and therefore there is a nonempty relatively open set \( U \) in \( L \) such that \( \sup\{f(x) : x \in U\} \leq n \). Since \( \pi_X(B_n) \cap U \neq \emptyset \), one can pick \((x, m) \in B_n \) with \( x \in U \) and \( m \geq n \). Then \((x, m) \) is in the epigraph of \( f \).

The following lemma is a key observation in our reasonings. To get Theorem 1.1 we need this result only for zero-dimensional spaces, where a justification is somewhat simpler. However, we aim at Proposition 3.1, which is more general than Theorem 1.1.

**Lemma 2.3.** Let \( X \) be a separable metrizable space and \( A \subset X \times \mathbb{N} \) be such that \( \infty \in \overline{A} \text{ in the Baire class one fan } S(X) \) (cf. Section 1). Then there is a closed nonempty set \( K \subset X \) such that \( A \) is nowhere bounded on \( K \) and \( \infty \not\in \overline{A} \cap ((X \setminus K) \times \mathbb{N}) \).

**Proof.** Let, cf. (1),

\( Y_n = \pi_X(A \cap [n, +\infty)) \).

If each \( Y_n \) is dense in \( X \), \( A \) is nowhere bounded on \( X \) and one can take \( K = X \). Otherwise, we proceed by a (possibly transfinite) exhaustion process as follows.

We start from a nonempty open set \( U_1 \) disjoint from some \( Y_{n(1)} \). Assume that we have defined open sets \( U_\alpha \) in \( X \) and \( n(\alpha) \in \mathbb{N}, \alpha < \xi, \) where \( \xi \) is a countable ordinal, such that

\( U_\alpha \subset \overline{U_\beta} \text{ for } \alpha < \beta < \xi, \)

\( H_\alpha = U_\alpha \setminus \bigcup_{\beta < \alpha} U_\beta \neq \emptyset, \)

\( H_\alpha \cap Y_{n(\alpha)} = \emptyset, \text{ for } \alpha < \xi. \)
We claim that
\[ \alpha \notin \mathcal{A} \cap (\bigcup_{\alpha < \xi} U_\alpha \times \mathbb{N}). \]
To see this, let us fix a metric on $X$ bounded by 1, and let us consider the closed sets
\[ H_\alpha = \{ x \in H_\alpha : \frac{1}{n+1} \leq \text{dist} (x,X \setminus U_\alpha) \leq \frac{1}{n} \}, \]
and their open neighbourhoods
\[ W_\alpha = \{ x \in X : \text{dist} (x,H_\alpha) < \frac{1}{n+1} \} \subset U_\alpha. \]
For each $x \in X$ and fixed $\alpha < \xi$, $x$ belongs to $W_\alpha$ only for finitely many $n$, and for each fixed $n$, if $\alpha \neq \beta$, then dist $(H_\alpha,H_\beta) \geq \frac{1}{n+1}$.

Let $\tau : \{ \alpha : \alpha < \xi \} \to \mathbb{N}$ be an injection and let
\[ V_\alpha = W_\alpha \setminus \bigcup \{ H_\beta m : m \leq \tau(\alpha) \text{ and } \beta \neq \alpha \}. \]
Then $H_\alpha \subset V_\alpha$, hence $\bigcup_{\alpha,n} V_\alpha = \bigcup_{\alpha} H_\alpha = \bigcup_{\alpha} U_\alpha$, cf. (3).

Given $x \in \bigcup_{\alpha} U_\alpha$, let us first pick $\beta$ with $x \in H_\beta$, and then $m$ with $x \in H_\beta m$. If $x \in V_\alpha$, then either $\beta = \alpha$ or else $\beta \neq \alpha$ and $\tau(\alpha) < m$.

Since for each fixed $\alpha$ there are only finitely many $n$ with $x \in V_\alpha$, we conclude that there are only finitely many pairs $(\alpha, n)$ with $x \in V_\alpha$.

Now, let us define $f : X \to \mathbb{N}$ by, cf. (4),
\[ f(x) = \begin{cases} \max \{ n(\alpha) : x \in V_\alpha \}, & \text{if } x \in \bigcup_{\alpha < \xi} U_\alpha, \\ 0, & \text{if } x \notin \bigcup_{\alpha < \xi} U_\alpha. \end{cases} \]
Then $f$ is lower-semicontinuous and we shall check that the epigraph of $f$ is disjoint from $\mathcal{A} \cap (\bigcup_{\alpha < \xi} U_\alpha \times \mathbb{N})$. Let $x \in \bigcup_{\alpha < \xi} U_\alpha$. Then, cf. (3), $x \in H_\alpha$ for some $\alpha < \xi$ and $n \in \mathbb{N}$, hence by (6), $f(x) \geq n(\alpha)$. By (2) and (4), if $(x,k) \in A$, then $k < n(\alpha)$, i.e., $k < f(x)$.

Having verified (5), we consider $F_\xi = X \setminus \bigcup_{\alpha < \xi} U_\alpha$ and we repeat the first step of the construction (note that, since $\infty \notin \mathcal{A}$, $F_\xi \neq \emptyset$). If all $Y_n \cap F_\xi$ are dense in $F_\xi$, cf. (2), $A$ is nowhere bounded on $F_\xi$ and we let $K = F_\xi$. Otherwise, we extend the induction, taking a nonempty, relatively open set $G$ in $F_\xi$ disjoint from some $Y_n(\xi)$, and we define $U_\xi = G \cup \bigcup_{\alpha < \xi} U_\alpha$.

The inductive process must terminate on some countable ordinal $\xi$, and then, as we have already noticed, $K = F_\xi$ is a required set. \hfill \Box

Remark 2.4. From Lemma 2.2 and Lemma 2.3 we infer immediately that for any separable metrizable space $X$ whose all closed subspaces are Baire, the Baire class one fan $S(X)$ has countable tightness.

3. Proof of Theorem 1.1

We shall prove the following proposition which is more general than Theorem 1.1.

Proposition 3.1. Let $S(X)$ be the Baire class one fan over the metrizable separable space $X$ all whose closed subspaces are Baire. Then the product $S(X) \times S(\mathbb{N})$ has countable tightness.
Proof. Since, by Remark 2.4, \( S(X) \) has countable tightness, it is enough to show that, whenever

\[
(7) \quad A \subset (X \times \mathbb{N}) \times (\mathbb{N} \times \mathbb{N}) \text{ and } (\infty, \infty) \in \overline{A}
\]

in \( S(X) \times S(\mathbb{N}) \), then \((\infty, \infty)\) is also in the closure of some countable subset of \( A \).

For each \( t \in \mathbb{N} \times \mathbb{N} \), let

\[
(8) \quad A(t) = \{ u \in X \times \mathbb{N} : (u, t) \in A \},
\]

and let

\[
(9) \quad T = \{ t \in \mathbb{N} \times \mathbb{N} : \infty \in \overline{A(t)} \}.
\]

We shall consider two possibilities.

Case (A). \( \infty \in \overline{T} \) in \( S(\mathbb{N}) \).

Then, since \( S(X) \) has countable tightness, for each \( t \in T \) one can choose a countable set \( C(t) \subset A(t) \) with \( \infty \in C(t) \), and then \((\infty, \infty) \in \bigcup \{ C(t) \times \{ t \} : t \in \mathbb{N} \times \mathbb{N} \} \).

Case (B). \( \infty \not\in \overline{T} \) in \( S(\mathbb{N}) \).

Let, for \( \varphi \in \mathbb{N}^{\mathbb{N}} \), \( \text{epi}(\varphi) = \{ (n, m) : m \geq \varphi(n) \} \).

Case (B) means that

\[
(10) \quad T \cap \text{epi}(\gamma) = \emptyset \text{ for some } \gamma \in \mathbb{N}^{\mathbb{N}}.
\]

Let, cf. (8), for \( \varphi \in \mathbb{N}^{\mathbb{N}} \),

\[
(11) \quad A(\varphi) = \bigcup \{ A(t) : t \in \text{epi}(\varphi) \} \subset X \times \mathbb{N}.
\]

Since \((\infty, \infty) \in \overline{A} \), for each \( \varphi \in \mathbb{N}^{\mathbb{N}} \), \( \infty \in \overline{A(\varphi)} \) in \( S(X) \), and by Lemma 2.3, there exists a closed nonempty set \( K(\varphi) \) in \( X \) such that

\[
(12) \quad A(\varphi) \text{ is nowhere bounded on } K(\varphi),
\]

\[
(13) \quad \infty \not\in \overline{A(\varphi) \cap ((X \setminus K(\varphi)) \times \mathbb{N})}.
\]

For \( \varphi, \psi \in \mathbb{N}^{\mathbb{N}} \), \( \varphi \leq^* \psi \) means that \( \varphi(n) \leq \psi(n) \) for all but finitely many \( n \), cf. [Kun], III.1.10. Let us check that, for the ordinal \( \gamma \) chosen in (10),

\[
(14) \quad \text{if } \varphi \leq^* \psi \text{ and } \gamma \leq \psi, \text{ then } K(\psi) \subset K(\varphi).
\]

Aiming at a contradiction, assume that \( L = K(\psi) \setminus K(\varphi) \neq \emptyset \). Then \( L \) is relatively open in \( K(\psi) \) and by (12), \( A(\psi) \) is nowhere bounded on \( L \), cf. Section 2, and by Lemma 2.2, \( \infty \in \overline{A(\psi) \cap (L \times \mathbb{N})} \). Since \( \text{epi}(\psi) \setminus \text{epi}(\varphi) \) is a finite subset of \( \text{epi}(\gamma) \), by (9) and (10), \( \infty \not\in \bigcup \{ A(t) : t \in \text{epi}(\psi) \setminus \text{epi}(\varphi) \} \). In effect, we conclude that \( \infty \in \overline{A(\varphi) \cap (L \times \mathbb{N})} \), cf. (11), which contradicts (13).

From (14) we infer that there is some \( \sigma \in \mathbb{N}^{\mathbb{N}} \) such that

\[
(15) \quad K(\sigma) = K(\varphi), \text{ whenever } \sigma \leq^* \varphi \text{ and } \gamma \leq \varphi.
\]

Indeed, if there were no such \( \sigma \), using (14) and the fact that each countable set in \( \mathbb{N}^{\mathbb{N}} \) is bounded with respect to \( \leq^* \), cf. [Kun], III.1.12, we could define a strictly decreasing sequence of type \( \omega_1 \) of closed sets \( K(\varphi_1) \supset \cdots \supset K(\varphi_{\xi}) \supset \cdots \) corresponding to \( \varphi_1 \leq^* \cdots \leq^* \varphi_{\xi} \leq^* \cdots, \gamma \leq \varphi_{\xi}, \xi < \omega_1 \). This, however, is impossible, \( X \) being separable metrizable, cf. [Kur], § 24.II.

With (15) at hand, we are ready to pick a countable subset of \( A \) whose closure contains \((\infty, \infty)\).
For each \( t \in \mathbb{N} \times \mathbb{N} \) and \( n \in \mathbb{N} \), let

\[
(16) \quad A(t, n) = A(t) \cap (K(\sigma) \times [n, +\infty)),
\]

cf. (8), and let us choose

\[
(17) \quad B(t, n) \subset A(t, n) \text{ countable, } \pi_X(A(t, n)) \subset \pi_X(B(t, n)).
\]

We check that

\[
(18) \quad (\infty, \infty) \in \bigcup \{B(t, n) \times \{t\} : t \in \mathbb{N} \times \mathbb{N}, n \in \mathbb{N}\}.
\]

To that end let us fix a neighbourhood \( \text{epi}(\varphi) \cup \{\infty\} \) of \( \infty \) in \( \mathbb{S}(\mathbb{N}) \) with \( \varphi \geq \sigma \) and \( \varphi \geq \gamma \) (such neighbourhoods form a basis of \( \infty \) in \( \mathbb{S}(\mathbb{N}) \)). We shall verify that

\[
B(\varphi) = \bigcup \{B(t, n) : t \in \text{epi}(\varphi), n \in \mathbb{N}\}
\]

is nowhere bounded on \( K(\sigma) \).

Indeed, let \( U \) be a nonempty relatively open set in \( K(\sigma) \) and \( n \in \mathbb{N} \). Since

\[
K(\sigma) = K(\sigma), \text{ cf. (15), } A(\varphi) \text{ is nowhere bounded on } K(\sigma), \text{ cf. (12), hence, for some } t \in \text{epi}(\varphi), \pi_X(A(t, n)) \cap U \neq \emptyset, \text{ cf. (16) and Definition 2.1, and by (17), } \pi_X(B(t, n)) \cap U \neq \emptyset.
\]

Now, since \( B(\varphi) \) is nowhere bounded on \( K(\sigma) \), \( \infty \in \overline{B(\varphi)} \), by Lemma 2.2. It follows that for any neighbourhood \( V \) of \( \infty \) in \( \mathbb{S}(X) \) and \( \varphi : \mathbb{N} \rightarrow \mathbb{N} \) with \( \varphi \geq \sigma, \varphi \geq \gamma \), some \( B(t, n) \times \{t\} \) intersects \( V \times \text{epi}(\varphi) \), which demonstrates (18) and ends the proof. \( \square \)

4. Comments

4.1. The fan \( \mathbb{S}(\mathbb{N}^\mathbb{N}) \) is not sequential. Let \( q_1, q_2, \ldots \) be a countable dense set in \( \mathbb{N}^\mathbb{N} \), and let \( a_n = (q_n, n), A = \{a_n : n = 1, 2, \ldots\} \subset \mathbb{S}(\mathbb{N}^\mathbb{N}) \). By Lemma 2.2, \( \infty \in \overline{A} \). However, if \( K \subset \mathbb{S}(\mathbb{N}^\mathbb{N}) \) is compact, then \( K \cap A \) is finite, and, in particular, no sequence of points of \( A \) converges to \( \infty \).

Indeed, if \( K \cap A \) is infinite, we can pick \( n_1 < n_2 < \ldots \) so that \( a_{n_k} \in K \) and \( \{q_{n_k} : k = 1, 2, \ldots\} \) has at most one accumulation point in \( \mathbb{N}^\mathbb{N} \). Then, there is a Baire class one function \( f : \mathbb{N}^\mathbb{N} \rightarrow \mathbb{N} \) such that \( f(q_{n_k}) > n_k \) and hence no \( a_{n_k} \) belongs to the epigraph of \( f \). It follows that the set \( \{a_{n_k} : k = 1, 2, \ldots\} \subset K \) is closed and discrete in \( \mathbb{S}(\mathbb{N}^\mathbb{N}) \), hence \( K \) is not compact.

This reasoning shows in fact that \( \mathbb{S}(X) \) is not sequential for any separable metrizable \( X \) whose all closed subspaces are Baire.

4.2. The fan \( \mathbb{S}(X) \) over \( \lambda \)-sets. Let us recall that \( X \subset \mathbb{R} \) is a \( \lambda \)-set if all countable subsets of \( X \) are \( G_\delta \)-sets in \( X \), cf. [Kur], §4,III, [Mi].

A result of A. Leiderman and the author [LP], combined with the Gruenhage theorem cited in Section 1, show that if \( X \) is a \( \lambda \)-set of cardinality \( \mathfrak{b} \) (such sets exist by a theorem of Rothberger, cf. [VL]), then \( \mathbb{S}(X) \times \mathbb{S}(\mathbb{N}) \) fails to have countable tightness.

4.3. The fan \( \mathbb{S}(X) \) over \( Q \)-sets. Let \( X \subset \mathbb{R} \) be a \( Q \)-set, i.e., all subsets of \( X \) are \( G_\delta \)-sets in \( X \), cf. [Mi]. Then every function \( f : X \rightarrow \mathbb{N} \) is of the first Baire class, cf. [Kur], §3, hence \( \mathbb{S}(X) \) is the sequential fan. In particular, by the result of Gruenhage cited in Section 1, for any \( Q \)-set \( X \) of cardinality less than \( \mathfrak{b} \), \( \mathbb{S}(X) \times \mathbb{S}(\mathbb{N}) \) has countable tightness.

4.4. The squares of fans. G. Gruenhage [Gr] proved that for any uncountable discrete \( X \), the square \( \mathbb{S}(X) \times \mathbb{S}(X) \) fails to have countable tightness. This is also the case for any uncountable \( \lambda \)-set in the real line, cf. [LP].

We do not know, however, if the square \( \mathbb{S}(\mathbb{N}^\mathbb{N}) \times \mathbb{S}(\mathbb{N}^\mathbb{N}) \) has countable tightness.
References


Mathematics Institute, University of Warsaw, ul Banacha 2, PL 02-097, Warsaw, Poland

E-mail address: R.Pol@mimuw.edu.pl