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BOUNDS ON VOLUME GROWTH OF GEODESIC BALLS FOR EINSTEIN WARPED PRODUCTS

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Abstract. The purpose of this note is to provide some volume estimates for Einstein warped products similar to a classical result due to Calabi and Yau for complete Riemannian manifolds with nonnegative Ricci curvature. To do so, we make use of the approach of quasi-Einstein manifolds which is directly related to Einstein warped products. In particular, we present an obstruction for the existence of such a class of manifolds.

1. Introduction

It has long been a goal of mathematicians to understand the geometry of Einstein manifolds as well as Einstein-type manifolds, for instance, Ricci solitons and m-quasi-Einstein manifolds. Surely, this is a fruitful problem in Riemannian Geometry. Ricci solitons model the formation of singularities in the Ricci flow and they correspond to self-similar solutions for this flow; for more details on this subject we recommend the survey by Cao [7]. On the other hand, one of the motivations to study m-quasi-Einstein metrics on a Riemannian manifold is its direct relation to Einstein warped products. For comprehensive references on such a theory, see [2], [3], [5], [9], [10], [11] and [20].

One fundamental ingredient to understand the behavior of Einstein warped products is the m-Bakry-Emery Ricci tensor which appeared previously in [1] and [17] as a modification of the classical Bakry-Emery tensor $Ric_f = Ric + \nabla^2 f$. More exactly, the m-Bakry-Emery Ricci tensor is given by

$$Ric_f^m = Ric + \nabla^2 f - \frac{1}{m} df \otimes df, \tag{1.1}$$

where f is a smooth function on M^n and $\nabla^2 f$ stands for the Hessian form.

A Riemannian manifold (M^n, g), $n \geq 2$, will be called *m-quasi-Einstein manifold*, or simply *quasi-Einstein manifold*, if there exist a smooth potential function f on M^n and a constant λ satisfying the following fundamental equation:

$$Ric_f^m = Ric + \nabla^2 f - \frac{1}{m} df \otimes df = \lambda g. \tag{1.2}$$

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When m goes to infinity, equation (1.2) reduces to the one associated with a Ricci soliton. Furthermore, when m is a positive integer it corresponds to a warped product Einstein metric; for more details see, for instance, [5]. Following the terminology of Ricci solitons, an m-quasi-Einstein metric g on a manifold M^n will be called *expanding*, *steady* or *shrinking*, respectively, if $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$. Moreover, an m-quasi-Einstein manifold will be called *trivial* if f is constant, otherwise it will be *nontrivial*. We notice that the triviality implies that M^n is an Einstein manifold.

In order to proceed we recall a classical result which gives a characterization for Einstein warped products.

Theorem 1 ([3], [11]). *Let $M^n \times_u F^m$ be an Einstein warped product with Einstein constant λ, warping function $u = e^{-\frac{f}{m}}$ and Einstein fiber F^m. Then the weighted manifold $(M^n, g_M, e^{-f}dM)$ satisfies the m-quasi-Einstein equation* (1.2)*. Furthermore the Einstein constant μ of the fiber satisfies*

$$\Delta f - |\nabla f|^2 = m\lambda - m\mu e^{\frac{2f}{m}}. \tag{1.3}$$

Conversely, if the weighted manifold $(M^n, g_M, e^{-f}dM)$ satisfies (1.2)*, then f satisfies* (1.3) *for some constant $\mu \in \mathbb{R}$. Considering the warped product $N^{n+m} = M^n \times_u F^m$, with $u = e^{-\frac{f}{m}}$ and Einstein fiber F with $Ric_F = \mu g_F$, then N is also Einstein with $Ric_N = \lambda g_N$, where $g_N = g_M + u^2 g_F$.*

Clearly, Theorem 1 shows that an m-quasi-Einstein structure provides a structure of Einstein warped product; for more details we recommend [11]. Therefore, classifying m-quasi-Einstein manifolds or understanding their geometry is definitely an important issue.

One should point out that some examples of expanding m-quasi-Einstein manifolds with arbitrary μ as well as steady quasi-Einstein manifolds with $\mu > 0$ were constructed in [3]. While Case [6] showed that steady m-quasi-Einstein manifolds with $\mu \leq 0$ are trivial. In [17], Qian proved that shrinking m-quasi-Einstein manifolds must be compact. Moreover, as already noticed in [11] the converse result is also true. From that it follows that an m-quasi-Einstein manifold is compact if and only if $\lambda > 0$. See also [20] for further discussion.

Based on the above results and inspired by works by Calabi and Yau, we shall investigate bounds on volume growth of geodesic balls for noncompact m-quasi-Einstein manifolds, in particular λ must be nonpositive. For our purposes we recall that Calabi [4] and Yau [21] proved that every metric with nonnegative Ricci tensor on a noncompact smooth manifold satisfies

$$Vol(B_p(r)) \geq cr \tag{1.4}$$

for any $r > r_0$ where r_0 is a positive constant and $B_p(r)$ is the geodesic ball of radius r centered at $p \in M^n$, and c is a constant that does not depend on r. In a similar way Munteanu and Sesum [14] obtained the same type of growth for a steady gradient Ricci soliton.

Now we state our first result concerning the growth of volume of geodesic balls for noncompact m-quasi-Einstein manifolds which is similar to the Calabi-Yau estimate.

Theorem 2. *Let $\big(M^n,\, g,\, f\big)$ be a noncompact steady m-quasi-Einstein manifold with $m \in (1, \infty]$. Then there exist constants c and $r_0 > 0$ such that for any $r > r_0$*

$$Vol(B_p(r)) \geq cr. \tag{1.5}$$

This immediately yields the following corollary.

Corollary 1. *Let $N = M^n \times_u F^m$ be a Ricci flat warped product. Then there exist constants c and $r_0 > 0$ such that for any $r > r_0$*

$$Vol(B_p(r)) \geq cr, \tag{1.6}$$

for geodesic balls of the base.

One question that naturally arises from the above results is to know what occurs on expanding m-quasi-Einstein manifolds. In this case, we obtain the following volume growth of geodesic balls.

Theorem 3. *Let $\big(M^n,\, g,\, f, \lambda\big)$ be a noncompact expanding m-quasi-Einstein manifold with $m \in (1, \infty)$ and $\mu \leq 0$. Supposing that $f \geq -k$ for some positive constant k, then there exist constants c and $r_0 > 0$ such that for any $r > r_0$*

$$Vol(B_p(r)) \geq cr. \tag{1.7}$$

In the sequel, inspired by ideas developed in [8], [12] and [13] we shall prove an f-volume estimate of geodesic balls on expanding m-quasi-Einstein manifolds. More precisely, we have the following result.

Theorem 4. *Let $\big(M^n,\, g,\, f, \lambda\big)$ be a noncompact expanding m-quasi-Einstein manifold with $m \in [1, \infty)$ and $\mu = 0$. Then there exists a constant c such that for any $r > 1$*

$$Vol_f(B_p(r)) \geq ce^{\sqrt{-\lambda m}r}. \tag{1.8}$$

It should be emphasized that there are several further interesting obstructions to the existence of Einstein metrics. Based on this, in [3] (cf. page 265) the following question was posed:

> *"Does there exist a compact Einstein warped product with nonconstant warping function?"*

In 2003 Kim and Kim [11] gave a partial answer for this question by means of the quasi-Einstein approach, while Case [6] studied this problem without compactness assumption for m-quasi-Einstein manifolds with $\lambda = 0$ and $\mu \leq 0$. Here, we use the weak Maximum Principle at infinity for the f-Laplacian to obtain a triviality result for Einstein warped products with negative scalar curvature. More precisely, we have the following result.

Theorem 5. *Let $N = M^n \times_u F^m$ be a complete Einstein warped product with Einstein constant $\lambda < 0$, warping function u and Einstein constant of the fiber F satisfying $\mu < 0$. If the warping function satisfies*

$$u \leq \sqrt{\frac{2\mu}{\lambda}},$$

then u is a constant function and N is a Riemannian product.

2. Proof of the results

In order to set the stage for the proofs to follow, let us recall some classic equations. First, considering the function $u = e^{-\frac{f}{m}}$ on M^n, we immediately have $\nabla u = -\frac{u}{m}\nabla f$ as well as

$$\nabla^2 f - \frac{1}{m} df \otimes df = -\frac{m}{u}\nabla^2 u. \tag{2.1}$$

Next, a straightforward computation involving (1.2) and (1.3) gives

$$\frac{u^2}{m}(R - \lambda n) + (m-1)|\nabla u|^2 = -\lambda u^2 + \mu. \tag{2.2}$$

We also recall that Wang [19] proved that if $\lambda \leq 0$, then $R \geq \lambda n$, from which it follows that

$$(m-1)|\nabla u|^2 \leq -\lambda u^2 + \mu. \tag{2.3}$$

Now we are ready to prove the results.

2.1. **Proof of Theorem 2.**

Proof. To begin with, we notice that when $m = \infty$ we have a gradient Ricci soliton and in this case the result follows from [14]. From now on we can assume that $m \in (1, \infty)$. From this, since $\lambda = 0$ we get

$$|\nabla u|^2 \leq \frac{\mu}{m-1}. \tag{2.4}$$

Taking into account that $R \geq 0$ and $u > 0$, we deduce

$$\int_{B_p(r)} uR d\sigma \geq 0$$

for each $r > 0$, where $d\sigma$ denotes the Riemannian volume form. Consequently, if for all $r > 0$ we have

$$\int_{B_p(r)} uR d\sigma = 0,$$

then $R = 0$ on M^n.

On the other hand, Wang [19] (see also [5]) proved that every m-quasi-Einstein manifold satisfies

$$\begin{aligned} \frac{1}{2}\Delta R - \frac{m+2}{2m}\langle \nabla f, \nabla R\rangle &= -\frac{m-1}{m}\Big|Ric - \frac{R}{n}g\Big|^2 \\ &\quad -\frac{n+m-1}{mn}(R - n\lambda)\Big(R - \frac{n(n-1)}{n+m-1}\lambda\Big). \end{aligned} \tag{2.5}$$

In particular, in the steady case we have $Ric = 0$, provided that $R = 0$, whence relation (1.5) remounts to Calabi [4] and Yau [21].

It is well-known that such a metric is real analytic (cf. Proposition 2.8 in [10]), hence the zeroes of the scalar curvature R are isolated. Therefore, if $R \geq 0$, but $R \neq 0$, we choose $p \in M^n$ such that $R(p) > 0$ and a ball $B_p(r_0)$ with radius $r_0 > 0$ such that

$$\int_{B_p(r_0)} uR d\sigma = mC_0$$

is a positive constant. Then we use the trace of (2.1) to conclude that for all $r \geq r_0$

$$mC_0 = \int_{B_p(r_0)} uRd\sigma \leq \int_{B_p(r)} uRd\sigma = m\int_{B_p(r)} \Delta u d\sigma.$$

Next we invoke Stokes formula and (2.4) to deduce

$$\begin{aligned} mC_0 &\leq m\int_{\partial B_p(r)} \frac{\partial u}{\partial \eta} ds \leq m\int_{\partial B_p(r)} |\nabla u| ds \\ &\leq m\sqrt{\frac{\mu}{m-1}} \cdot Area(\partial B_p(r)). \end{aligned}$$

This implies that for $r \geq r_0$ we have

$$Area(\partial B_p(r)) \geq c > 0 \tag{2.6}$$

for a uniform constant c.

Finally, on integrating (2.6) from r_0 to r we arrive at

$$Vol(B_p(r)) \geq c(r - r_0) \geq c_0 \cdot r$$

for all $r \geq 2r_0$. This finishes the proof of the theorem. □

2.2. **Proof of Theorem 3.**

Proof. Firstly we notice that using the hypotheses on f and μ in (1.8) we deduce

$$|\nabla u|^2 \leq \frac{-\lambda e^{2k/m}}{m-1}. \tag{2.7}$$

Next, taking into account that $R \geq \lambda n$ and $u > 0$ we infer

$$\int_{B_p(r)} u(R - \lambda n) d\sigma \geq 0$$

for each $r > 0$. Moreover, if for all $r > 0$ we have

$$\int_{B_p(r)} u(R - \lambda n) d\sigma = 0,$$

then $R = \lambda n$ on M^n. Hence from (2.5) we deduce that M^n is Einstein, but this gives a contradiction. In fact, according to [6] an Einstein manifold with nontrivial expanding quasi-Einstein structure and potential function bounded from below does not exist.

From now on the proof looks like the one from the previous theorem. In particular, there is $p \in M^n$ such that $R(p) > \lambda n$. Since u is positive there exists $r_0 > 0$ such that

$$\int_{B_p(r_0)} u(R - \lambda n) d\sigma = mC_0$$

is a positive constant. Moreover, from the analyticity of R it follows that for all $r \geq r_0$

$$\begin{aligned} mC_0 &\leq \int_{B_p(r)} u(R - \lambda n) d\sigma = m\int_{Bp(r)} \Delta u d\sigma \\ &= m\int_{\partial B_p(r)} \frac{\partial u}{\partial \eta} ds \leq m\int_{\partial B_p(r)} |\nabla u| ds \\ &\leq m\sqrt{\frac{-\lambda e^{2k/m}}{m-1}} \cdot Area(\partial B_p(r)), \end{aligned} \tag{2.8}$$

where we have used Stokes formula and (2.7). Therefore, (2.8) allows us to deduce that for $r \geq r_0$

$$Area(\partial B_p(r)) \geq c > 0 \tag{2.9}$$

for an uniform constant c.

In order to conclude it suffices to integrate (2.9) from r_0 to r to arrive at

$$Vol(B_p(r)) \geq c(r - r_0) \geq c_0 \cdot r$$

for all $r \geq 2r_0$, which gives the requested result. □

2.3. **Proof of Theorem 4.**

Proof. First, since $\mu = 0$ we use (1.3) to infer

$$\Delta e^{-f} = (-\Delta f + |\nabla f|^2)e^{-f} = -\lambda m e^{-f}. \tag{2.10}$$

Now, upon integrating of (2.10) over $B_p(r)$ we deduce

$$\begin{aligned} -\lambda m \int_{B_p(r)} e^{-f} d\sigma &= \int_{B_p(r)} \Delta e^{-f} d\sigma \\ &= \int_{\partial B_p(r)} \frac{\partial}{\partial \eta}(e^{-f}) ds. \end{aligned} \tag{2.11}$$

On the other hand, from [19] we have $|\frac{\partial f}{\partial \eta}| \leq |\nabla f| \leq \sqrt{-\lambda m}$. This enables us to use (2.11) to arrive at

$$-\lambda m \int_{B_p(r)} e^{-f} d\sigma \leq \sqrt{-\lambda m} \int_{\partial B_p(r)} e^{-f} ds. \tag{2.12}$$

Next, denoting

$$\xi(r) := Vol_f(B_p(r)) = \int_{B_p(r)} e^{-f} d\sigma,$$

we can use (2.12) to get

$$\xi'(r) \geq \sqrt{-\lambda m}\,\xi(r).$$

Whence, on integrating this inequality from 1 to r we conclude that

$$\xi(r) = \int_{B_p(r)} e^{-f} d\sigma \geq c e^{\sqrt{-\lambda m} r}$$

for any $c > 0$. From here we conclude the proof of the theorem. □

2.4. **Proof of Theorem 5.**

Proof. We start invoking Theorem 1 to deduce that $(M^n, g, \nabla f, \lambda)$ is an expanding m-quasi-Einstein manifold with potential function $f = -m \ln u$ and $\mu < 0$.

Next, we combine f-volume estimates obtained by Qian [17] and Theorem 9 in [16] to conclude that the weak Maximum Principle at infinity is valid for the f-Laplacian on $(M^n, g, \nabla f, \lambda)$. We also highlight that the potential function f satisfies $|\nabla f|^2 \leq -\lambda m$ (cf. [19], see also [18]). From this setting, we apply the weak Maximum Principle at infinity for $|\nabla f|^2$ to conclude that there exists a sequence $\{p_k\} \subset M^n$, such that

$$|\nabla f|^2(p_k) \geq \overline{|\nabla f|}^2 - \frac{1}{k} \qquad \text{and} \qquad \Delta_f |\nabla f|^2(p_k) \leq \frac{1}{k},$$

where $\overline{|\nabla f|}^2 = \sup_M |\nabla f|^2$.

We now recall the weighted Bochner formula:

$$\frac{1}{2}\Delta_f|\nabla f|^2 = |Hessf|^2 + Ric(\nabla f, \nabla f) + Hessf(\nabla f, \nabla f) + \langle \nabla f, \nabla \Delta_f f\rangle. \tag{2.13}$$

Therefore, (1.2) and (1.3) substituted in (2.13) gives

$$\begin{aligned} \frac{1}{2}\Delta_f|\nabla f|^2 &\geq \lambda|\nabla f|^2 + \frac{1}{m}|\nabla f|^4 - 2\mu e^{2f/m}|\nabla f|^2 \\ &\geq \big(\lambda - 2\mu u^{-2} + \frac{1}{m}|\nabla f|^2\big)|\nabla f|^2. \end{aligned} \tag{2.14}$$

Now, since $u \leq \sqrt{\frac{2\mu}{\lambda}}$ we immediately deduce

$$\frac{1}{2}\Delta_f|\nabla f|^2 \geq \frac{1}{m}|\nabla f|^4. \tag{2.15}$$

From this, over $\{p_k\}$ we get

$$\frac{1}{2k} \geq \frac{1}{m}\Big(\overline{|\nabla f|}^2 - \frac{1}{k}\Big)^2.$$

So, when k goes to infinity we conclude that $\overline{|\nabla f|}^2 = 0$ and this forces f to be constant which concludes the proof of the theorem. □

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