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THE ALPERIN-MCKAY CONJECTURE FOR METACYCLIC, MINIMAL NON-ABELIAN DEFECT GROUPS

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Abstract. We prove the Alperin-McKay Conjecture for all p-blocks of finite groups with metacyclic, minimal non-abelian defect groups. These are precisely the metacyclic groups whose derived subgroup have order p. In the special case $p = 3$, we also verify Alperin's Weight Conjecture for these defect groups. Moreover, in case $p = 5$ we do the same for the non-abelian defect groups $C_{25} \rtimes C_{5^n}$. The proofs do *not* rely on the classification of the finite simple groups.

1. Introduction

Let B be a p-block of a finite group G with respect to an algebraically closed field of characteristic p. Suppose that B has a metacyclic defect group D. We are interested in the number $k(B)$ (respectively $k_i(B)$) of irreducible characters of B (of height $i \geq 0$), and the number $l(B)$ of irreducible Brauer characters of B. If $p = 2$, these invariants are well understood and the major conjectures are known to be true by the work of several authors (see [4, 9, 11, 31, 35, 37]). Thus we will focus on the case $p > 2$ in the present work. Here at least Brauer's $k(B)$-Conjecture, Olsson's Conjecture and Brauer's Height Zero Conjecture are satisfied for B (see [14, 38, 43]). By a result of Stancu [40], B is a controlled block. Moreover, if D is a non-split extension of two cyclic groups, it is known that B is nilpotent (see [7]). Then a result by Puig [33] describes the source algebra of B in full detail. Thus we may assume in the following that D is a split extension of two cyclic groups. A famous theorem by Dade [6] handles the case where D itself is cyclic by making use of Brauer trees. The general situation is much harder – even the case $D \cong C_3 \times C_3$ is still open (see [24–26, 42]). Now consider the subcase where D is non-abelian. Then a work by An [1] shows that G is not a quasisimple group. On the other hand, the algebra structure of B in the p-solvable case can be obtained from Külshammer [27]. If B has maximal defect (i.e. $D \in \mathrm{Syl}_p(G)$), the block invariants of B were determined in [15]. If B is the principal block, Horimoto and Watanabe [20] constructed a perfect isometry between B and its Brauer correspondent in $\mathrm{N}_G(D)$.

Let us suppose further that D is a split extension of a cyclic group and a group of order p (i.e. D is the unique non-abelian group with a cyclic subgroup of index p). Here the difference $k(B) - l(B)$ is known from [16]. Moreover, under additional assumptions on G, Holloway, Koshitani and Kunugi [19] obtained the block invariants precisely. In the special case where D has order p^3, incomplete information is

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given by Hendren [17]. Finally, one has full information in case $|D| = 27$ by work of the present author [38, Theorem 4.5].

In the present work we consider the following class of non-abelian split metacyclic groups

$$D = \langle x, y \mid x^{p^m} = y^{p^n} = 1,\ yxy^{-1} = x^{1+p^{m-1}} \rangle \cong C_{p^m} \rtimes C_{p^n} \tag{1.1}$$

where $m \geq 2$ and $n \geq 1$. By a result of Rédei (see [21, Aufgabe III.7.22]) these are precisely the metacyclic, minimal non-abelian groups. A result by Knoche (see [21, Aufgabe III.7.24]) implies further that these are exactly the metacyclic groups with derived subgroup of order p. In particular the family includes the non-abelian group with a cyclic subgroup of index p mentioned above. The main theorem of the present paper states that $k_0(B)$ is locally determined. In particular the Alperin-McKay Conjecture holds for B. Recall that the Alperin-McKay Conjecture asserts that $k_0(B) = k_0(b)$ where b is the Brauer correspondent of B in $\mathrm{N}_G(D)$. This improves some of the results mentioned above. We also prove that every irreducible character of B has height 0 or 1. This is in accordance with the situation in $\mathrm{Irr}(D)$. In the second part of the paper we investigate the special case $p = 3$. Here we are able to determine $k(B)$, $k_i(B)$ and $l(B)$. This gives an example of Alperin's Weight Conjecture and the Ordinary Weight Conjecture. Finally, we determine the block invariants for $p = 5$ and $D \cong C_{25} \rtimes C_{5^n}$ where $n \geq 1$.

As a new ingredient (compared to [38]) we make use of the focal subgroup of B.

2. The Alperin-McKay Conjecture

Let p be an odd prime, and let B be a p-block with split metacyclic, non-abelian defect group D. Then D has a presentation of the form

$$D = \langle x, y \mid x^{p^m} = y^{p^n} = 1,\ yxy^{-1} = x^{1+p^l} \rangle$$

where $0 < l < m$ and $m - l \leq n$. Elementary properties of D are stated in the following lemma.

Lemma 2.1.
(i) $D' = \langle x^{p^l} \rangle \cong C_{p^{m-l}}$.
(ii) $\mathrm{Z}(D) = \langle x^{p^{m-l}} \rangle \times \langle y^{p^{m-l}} \rangle \cong C_{p^l} \times C_{p^{n-m+l}}$.

Proof. Omitted. □

We fix a Sylow subpair (D, b_D) of B. Then the conjugation of subpairs $(Q, b_Q) \leq (D, b_D)$ forms a saturated fusion system $\mathcal{F}$ on D (see [2, Proposition IV.3.14]). Here $Q \leq D$ and b_Q is a uniquely determined block of $\mathrm{C}_G(Q)$. We also have subsections (u, b_u) where $u \in D$ and $b_u := b_{\langle u \rangle}$. By Proposition 5.4 in [40], $\mathcal{F}$ is controlled. Moreover by Theorem 2.5 in [14] we may assume that the inertial group of B has the form $\mathrm{N}_G(D, b_D)/\,\mathrm{C}_G(D) = \mathrm{Aut}_{\mathcal{F}}(D) = \langle \mathrm{Inn}(D), \alpha \rangle$ where $\alpha \in \mathrm{Aut}(D)$ such that $\alpha(x) \in \langle x \rangle$ and $\alpha(y) = y$. By a slight abuse of notation we often write $\mathrm{Out}_{\mathcal{F}}(D) = \langle \alpha \rangle$. In particular the inertial index $e(B) := |\mathrm{Out}_{\mathcal{F}}(D)|$ is a divisor of $p - 1$. Let

$$\mathfrak{foc}(B) := \langle f(a)a^{-1} : a \in Q \leq D,\ f \in \mathrm{Aut}_{\mathcal{F}}(Q) \rangle$$

be the *focal subgroup* of B (or of $\mathcal{F}$). Then it is easy to see that $\mathfrak{foc}(B) \subseteq \langle x \rangle$. In case $e(B) = 1$, B is nilpotent and $\mathfrak{foc}(B) = D'$. Otherwise $\mathfrak{foc}(B) = \langle x \rangle$.

For the convenience of the reader we collect some estimates on the block invariants of B.

Proposition 2.2. *Let B be as above. Then*

$$\left(\frac{p^l+p^{l-1}-p^{2l-m-1}-1}{e(B)}+e(B)\right)p^n \le k(B) \le \left(\frac{p^l-1}{e(B)}+e(B)\right)(p^{n+m-l-2}+p^n-p^{n-2}),$$
$$2p^n \le k_0(B) \le \left(\frac{p^l-1}{e(B)}+e(B)\right)p^n,$$
$$\sum_{i=0}^{\infty} p^{2i}k_i(B) \le \left(\frac{p^l-1}{e(B)}+e(B)\right)p^{n+m-l},$$
$$l(B) \ge e(B) \mid p-1,$$
$$p^n \mid k_0(B), \qquad p^{n-m+l} \mid k_i(B) \qquad \text{for } i \ge 1,$$
$$k_i(B) = 0 \qquad \text{for } i > 2(m-l).$$

Proof. Most of the inequalities are contained in Proposition 2.1 to Corollary 2.5 in [38]. By Theorem 1 in [36] we have $p^n \mid |D : \mathfrak{foc}(B)| \mid k_0(B)$. In particular $p^n \le k_0(B)$. In case $k_0(B) = p^n$ it follows from [23] that B is nilpotent. However then we would have $k_0(B) = |D : D'| = p^{n+l} > p^n$. Therefore $2p^n \le k_0(B)$. Theorem 2 in [36] implies $p^{n-m+l} \mid |\mathrm{Z}(D) : \mathrm{Z}(D) \cap \mathfrak{foc}(B)| \mid k_i(B)$ for $i \ge 1$. □

Now we consider the special case where $m = l + 1$. As mentioned in the introduction these are precisely the metacyclic, minimal non-abelian groups. We prove the main theorem of this section.

Theorem 2.3. *Let B be a p-block of a finite group with metacyclic, minimal non-abelian defect groups for an odd prime p. Then*

$$k_0(B) = \left(\frac{p^{m-1}-1}{e(B)}+e(B)\right)p^n$$

with the notation from (1.1)*. In particular the Alperin-McKay Conjecture holds for B.*

Proof. By Proposition 2.2 we have

$$p^n \mid k_0(B) \le \left(\frac{p^{m-1}-1}{e(B)}+e(B)\right)p^n.$$

Thus, by way of contradiction we may assume that

$$k_0(B) \le \left(\frac{p^{m-1}-1}{e(B)}+e(B)-1\right)p^n.$$

We also have

$$k(B) \ge \left(\frac{p^{m-1}+p^{m-2}-p^{m-3}-1}{e(B)}+e(B)\right)p^n$$

from Proposition 2.2. Hence the sum $\sum_{i=0}^{\infty} p^{2i}k_i(B)$ will be small if $k_0(B)$ is large and $k_1(B) = k(B) - k_0(B)$. This implies the following contradiction:

$$\begin{aligned}\Big(\frac{p^m-1}{e(B)}+p^2+e(B)-1\Big)p^n = &\Big(\frac{p^{m-1}-1}{e(B)}+e(B)-1\Big)p^n \\ &+\Big(\frac{p^{m-2}-p^{m-3}}{e(B)}+1\Big)p^{n+2} \\ \le &\sum_{i=0}^{\infty} p^{2i}k_i(B) \le \Big(\frac{p^m-p}{e(B)}+pe(B)\Big)p^n \\ < &\Big(\frac{p^m-1}{e(B)}+p^2\Big)p^n.\end{aligned}$$

Since the Brauer correspondent of B in $\mathrm{N}_G(D)$ has the same fusion system, the Alperin-McKay Conjecture follows. □

Isaacs and Navarro [22, Conjecture D] proposed a refinement of the Alperin-McKay Conjecture by invoking Galois automorphisms. We show (as an improvement of Theorem 4.3 in [38]) that this conjecture holds in the special case $|D| = p^3$ of Theorem 2.3. We will denote the subset of $\mathrm{Irr}(B)$ of characters of height 0 by $\mathrm{Irr}_0(B)$.

Corollary 2.4. *Let B be a p-block of a finite group G with non-abelian, metacyclic defect group of order p^3. Then Conjecture D in [22] holds for B.*

Proof. Let D be a defect group of B. For $k \in \mathbb{N}$, let $\mathbb{Q}_k$ be the cyclotomic field of degree k. Let $|G|_{p'}$ be the p'-part of the order of G. It is well known that the Galois group $\mathcal{G} := \mathrm{Gal}(\mathbb{Q}_{|G|}|\mathbb{Q}_{|G|_{p'}})$ acts canonically on $\mathrm{Irr}(B)$. Let $\gamma \in \mathcal{G}$ be a p-element. Then it suffices to show that γ acts trivially on $\mathrm{Irr}_0(B)$. By Lemma IV.6.10 in [12] it is enough to prove that γ acts trivially on the $\mathcal{F}$-conjugacy classes of subsections of B via ${}^{\gamma}(u, b_u) := (u^{\overline{\gamma}}, b_u)$ where $u \in D$ and $\overline{\gamma} \in \mathbb{Z}$. Since γ is a p-element, this action is certainly trivial unless $|\langle u\rangle| = p^2$. Here however, the action of γ on $\langle u\rangle$ is just the D-conjugation. The result follows. □

In the situation of Corollary 2.4 one can say a bit more: By Proposition 3.3 in [38], $\mathrm{Irr}(B)$ splits into the following families of p-conjugate characters:

- $(p-1)/e(B) + e(B)$ orbits of length $p-1$,
- two orbits of length $(p-1)/e(B)$,
- at least $e(B)$ p-rational characters.

Without loss of generality, let $e(B) > 1$. By Theorem 4.1 in [38] we have $k_1(B) \le (p-1)/e(B) + e(B) - 1$. Moreover, Proposition 4.1 of the same paper implies $k_1(B) < p - 1$. In particular, all orbits of length $p-1$ of p-conjugate characters must lie in $\mathrm{Irr}_0(B)$. In case $e(B) = p-1$ the remaining $(p-1)/e(B) + e(B)$ characters in $\mathrm{Irr}_0(B)$ must be p-rational. Now let $e(B) < \sqrt{p-1}$. Then it is easy to see that $\mathrm{Irr}_0(B)$ contains just one orbit of length $(p-1)/e(B)$ of p-conjugate characters. Unfortunately, it is not clear if this also holds for $e(B) \ge \sqrt{p-1}$.

Next we improve the bound coming from Proposition 2.2 on the heights of characters.

Proposition 2.5. *Let B be a p-block of a finite group with metacyclic, minimal non-abelian defect groups. Then $k_1(B) = k(B) - k_0(B)$. In particular, B satisfies the following conjectures.*

- *Eaton's Conjecture* [8],
- *Eaton-Moretó Conjecture* [10],
- *Robinson's Conjecture* [28, Conjecture 4.14.7],
- *Malle-Navarro Conjecture* [29].

Proof. By Theorem 2 in [37] we may assume $p > 2$ as before. By way of contradiction suppose that $k_i(B) > 0$ for some $i \geq 2$. Since

$$k(B) \geq \left(\frac{p^{m-1} + p^{m-2} - p^{m-3} - 1}{e(B)} + e(B)\right)p^n,$$

we have $k(B) - k_0(B) \geq (p^{m-1} - p^{m-2})p^{n-1}/e(B)$ by Theorem 2.3. By Proposition 2.2, $k_1(B)$ and $k_i(B)$ are divisible by p^{n-1}. This shows

$$\left(\frac{p^{m-1} - 1}{e(B)} + e(B)\right)p^n + \left(\frac{p^{m-1} - p^{m-2}}{e(B)} - 1\right)p^{n+1} + p^{n+3} \leq \sum_{i=0}^{\infty} p^{2i}k_i(B) \leq \left(\frac{p^{m-1} - 1}{e(B)} + e(B)\right)p^{n+1}.$$

Hence, we derive the following contradiction:

$$p^{n+3} - p^{n+1} \leq \left(\frac{1-p}{e(B)} + e(B)(p-1)\right)p^n \leq p^{n+2}.$$

This shows $k_1(B) = k(B) - k_0(B)$. Now Eaton's Conjecture is equivalent to Brauer's $k(B)$-Conjecture and Olsson's Conjecture. Both are known to hold for all metacyclic defect groups. Also, the Eaton-Moretó Conjecture and Robinson's Conjecture are trivially satisfied for B. The Malle-Navarro Conjecture asserts that $k(B)/k_0(B) \leq k(D') = p$ and $k(B)/l(B) \leq k(D)$. By Theorem 2.3 and Proposition 2.2, the first inequality reduces to $p^{n-1} + p^n - p^{n-2} \leq p^{n+1}$ which is true. For the second inequality we observe that every conjugacy class of D has at most p elements, since $|D : \mathrm{Z}(D)| = p^2$. Hence, $k(D) = |\mathrm{Z}(D)| + \frac{|D| - |\mathrm{Z}(D)|}{p} = p^{n+m-1} + p^{n+m-2} - p^{n+m-3}$. Now Proposition 2.2 gives

$$\begin{aligned}\frac{k(B)}{l(B)} \leq k(B) &\leq \left(\frac{p^{m-1} - 1}{e(B)} + e(B)\right)(p^{n-1} + p^n - p^{n-2}) \\ &\leq p^{n+m-1} + p^{n+m-2} - p^{n+m-3} = k(D).\end{aligned}$$

□

We use the opportunity to present a result for $p = 3$ and a different class of metacyclic defect groups (where $l = 1$ with the notation above).

Theorem 2.6. *Let B be a 3-block of a finite group G with defect group*

$$D = \langle x, y \mid x^{3^m} = y^{3^n} = 1,\ yxy^{-1} = x^4 \rangle$$

where $2 \leq m \leq n + 1$. Then $k_0(B) = 3^{n+1}$. In particular, the Alperin-McKay Conjecture holds for B.

Proof. We may assume that B is non-nilpotent. By Proposition 2.2 we have $k_0(B) \in \{2 \cdot 3^n, 3^{n+1}\}$. By way of contradiction, suppose that $k_0(B) = 2 \cdot 3^n$. Let $P \in \mathrm{Syl}_p(G)$. Since $D/\mathfrak{foc}(B)$ acts freely on $\mathrm{Irr}_0(B)$, there are 3^n characters of degree

$a|P:D|$, and 3^n characters of degree $b|P:D|$ in B for some $a,b \geq 1$ such that $3 \nmid a, b$. Hence,

$$\left|\sum_{\chi \in \mathrm{Irr}_0(B)} \chi(1)^2\right|_3 = 3^n|P:D|^2(a^2+b^2)_3 = |P:D|^2|D:\mathfrak{foc}(B)|.$$

Now Theorem 1.1 in [23] gives a contradiction. □

A generalization of the argument in the proof shows that in the situation of Proposition 2.2, $k_0(B) = 2p^n$ can only occur if $p \equiv 1 \pmod 4$.

3. Lower defect groups

In the following we use the theory of lower defect groups in order to estimate $l(B)$. We cite a few results from the literature. Let B be a p-block of a finite group G with defect group D and Cartan matrix C. We denote the multiplicity of an integer a as elementary divisor of C by $m(a)$. Then $m(a) = 0$ unless a is a p-power. It is well known that $m(|D|) = 1$. Brauer [3] expressed $m(p^n)$ $(n \geq 0)$ in terms of 1-*multiplicities* of lower defect groups (see also Corollary V.10.12 in [12]):

$$m(p^n) = \sum_{R \in \mathcal{R}} m_B^{(1)}(R) \tag{3.1}$$

where $\mathcal{R}$ is a set of representatives for the G-conjugacy classes of subgroups $R \leq D$ of order p^n. Later (3.1) was refined by Broué and Olsson by invoking the fusion system $\mathcal{F}$ of B.

Proposition 3.1 (Broué-Olsson [5])**.** *For $n \geq 0$ we have*

$$m(p^n) = \sum_{R \in \mathcal{R}} m_B^{(1)}(R, b_R)$$

where $\mathcal{R}$ is a set of representatives for the $\mathcal{F}$-conjugacy classes of subgroups $R \leq D$ of order p^n.

Proof. This is (2S) of [5]. □

In the present paper we do not need the precise (and complicated) definition of the non-negative numbers $m_B^{(1)}(R)$ and $m_B^{(1)}(R, b_R)$. We say that R is a *lower defect group* for B if $m_B^{(1)}(R, b_R) > 0$. In particular, $m_B^{(1)}(D, b_D) = m_B^{(1)}(D) = m(|D|) = 1$. A crucial property of lower defect groups is that their multiplicities can usually be determined locally. In the next lemma, $b_R^{\mathrm{N}_G(R,b_R)}$ denotes the (unique) Brauer correspondent of b_R in $\mathrm{N}_G(R, b_R)$.

Lemma 3.2. *For $R \leq D$ and $B_R := b_R^{\mathrm{N}_G(R,b_R)}$ we have $m_B^{(1)}(R, b_R) = m_{B_R}^{(1)}(R)$. If R is fully $\mathcal{F}$-normalized, then B_R has defect group $\mathrm{N}_D(R)$ and fusion system $\mathrm{N}_{\mathcal{F}}(R)$.*

Proof. The first claim follows from (2Q) in [5]. For the second claim we refer to Theorem IV.3.19 in [2]. □

Another important reduction is given by the following lemma.

Lemma 3.3. *For $R \leq D$ we have $\sum_{Q \in \mathcal{R}} m_{B_R}^{(1)}(Q) \leq l(b_R)$ where $\mathcal{R}$ is a set of representatives for the $\mathrm{N}_G(R, b_R)$-conjugacy classes of subgroups Q such that $R \leq Q \leq \mathrm{N}_D(R)$.*

Proof. This is implied by Theorem 5.11 in [32] and the remark following it. Notice that in Theorem 5.11 it should read $B \in \mathrm{Bl}(G)$ instead of $B \in \mathrm{Bl}(Q)$. ☐

In the local situation for B_R the next lemma is also useful.

Lemma 3.4. *If* $\mathrm{O}_p(\mathrm{Z}(G)) \nsubseteq R$, *then* $m_B^{(1)}(R) = 0$.

Proof. See Corollary 3.7 in [32]. ☐

Now we apply these results.

Lemma 3.5. *Let B be a p-block of a finite group with metacyclic, minimal non-abelian defect group D for an odd prime p. Then every lower defect group of B is D-conjugate either to $\langle y\rangle$, $\langle y^p\rangle$, or to D.*

Proof. Let $R < D$ be a lower defect group of B. Then $m(|R|) > 0$ by Proposition 3.1. Corollary 5 in [36] shows that $p^{n-1} \mid |R|$. Since $\mathcal{F}$ is controlled, the subgroup R is fully $\mathcal{F}$-centralized and fully $\mathcal{F}$-normalized. The fusion system of b_R (on $\mathrm{C}_D(R)$) is given by $\mathrm{C}_{\mathcal{F}}(R)$ (see Theorem IV.3.19 in [2]). Suppose for the moment that $\mathrm{C}_{\mathcal{F}}(R)$ is trivial. Then b_R is nilpotent and $l(b_R) = 1$. Let $B_R := b_R^{\mathrm{N}_G(R,b_R)}$. Then B_R has defect group $\mathrm{N}_D(R)$ and $m_{B_R}^{(1)}(\mathrm{N}_D(R)) = 1$. Hence, Lemmas 3.2 and 3.3 imply $m_B^{(1)}(R, b_R) = m_{B_R}^{(1)}(R) = 0$. This contradiction shows that $\mathrm{C}_{\mathcal{F}}(R)$ is non-trivial. In particular R is centralized by a non-trivial p'-automorphism $\beta \in \mathrm{Aut}_{\mathcal{F}}(D)$. By the Schur-Zassenhaus Theorem, β is $\mathrm{Inn}(D)$-conjugate to a power of α. Thus, R is D-conjugate to a subgroup of $\langle y\rangle$. The result follows. ☐

Proposition 3.6. *Let B be a p-block of a finite group with metacyclic, minimal non-abelian defect groups for an odd prime p. Then $e(B) \le l(B) \le 2e(B) - 1$.*

Proof. Let

$$D = \langle x, y \mid x^{p^m} = y^{p^n} = 1,\ yxy^{-1} = x^{1+p^{m-1}}\rangle$$

be a defect group of B. We argue by induction on n. Let $n = 1$. By Proposition 2.2 we have $e(B) \le l(B)$ and

$$k(B) \le \left(\frac{p^{m-1} - 1}{e(B)} + e(B)\right)(1 + p - p^{-1}).$$

Moreover, Theorem 3.2 in [38] gives

$$k(B) - l(B) = \frac{p^m + p^{m-1} - p^{m-2} - p}{e(B)} + e(B)(p - 1).$$

Hence,

$$\begin{aligned} l(B) = k(B) - (k(B) - l(B)) &\le \left(\frac{p^{m-1} - 1}{e(B)} + e(B)\right)(1 + p - p^{-1}) \\ &\quad - \frac{p^m + p^{m-1} - p^{m-2} - p}{e(B)} - e(B)(p - 1) \\ &= 2e(B) - \frac{1}{p}\left(e(B) - \frac{1}{e(B)}\right) - \frac{1}{e(B)}, \end{aligned}$$

and the claim follows in this case.

Now suppose $n \ge 2$. We determine the multiplicities of the lower defect groups by using Lemma 3.5. As usual $m(|D|) = 1$. Consider the subpair $(\langle y\rangle, b_y)$. By

Lemmas 3.1 and 3.2 we have $m(p^n) = m_B^{(1)}(\langle y\rangle, b_y) = m_{B_y}^{(1)}(\langle y\rangle)$ where $B_y := b_y^{\mathrm{N}_G(\langle y\rangle, b_y)}$. Since $\mathrm{N}_D(\langle y\rangle) = \mathrm{C}_D(y)$, it follows easily that $\mathrm{N}_G(\langle y\rangle, b_y) = \mathrm{C}_G(y)$ and $B_y = b_y$. By Theorem IV.3.19 in [2] the block b_y has defect group $\mathrm{C}_D(y)$ and fusion system $\mathrm{C}_{\mathcal{F}}(\langle y\rangle)$. In particular $e(b_y) = e(B)$. It is well known that b_y dominates a block $\overline{b_y}$ of $\mathrm{C}_G(y)/\langle y\rangle$ with cyclic defect group $\mathrm{C}_D(y)/\langle y\rangle$ and $e(\overline{b_y}) = e(b_y) = e(B)$ (see [30, Theorem 5.8.11]). By Dade's Theorem [6] on blocks with cyclic defect groups we obtain $l(b_y) = e(B)$. Moreover, the Cartan matrix of $\overline{b_y}$ has elementary divisors 1 and $|\mathrm{C}_D(y)/\langle y\rangle|$ where 1 occurs with multiplicity $e(B) - 1$ (this follows for example from [13]). Therefore, the Cartan matrix of b_y has elementary divisors p^n and $|\mathrm{C}_D(y)|$ where p^n occurs with multiplicity $e(B) - 1$. Since $\langle y\rangle \subseteq \mathrm{Z}(\mathrm{C}_G(y))$, Lemma 3.4 implies $m(p^n) = m_{b_y}^{(1)}(\langle y\rangle) = e(B) - 1$.

Now consider $(\langle u\rangle, b_u)$ where $u := y^p \in \mathrm{Z}(D)$. Here b_u has defect group D. By the first part of the proof (the case $n = 1$) we obtain $l(b_u) = l(\overline{b_u}) \le 2e(B) - 1$. As above we have $m(p^{n-1}) = m_B^{(1)}(\langle u\rangle, b_u) = m_{b_u}^{(1)}(\langle u\rangle)$. Since p^n occurs as elementary divisor of the Cartan matrix of b_u with multiplicity $e(B) - 1$ (see above), it follows that $m(p^{n-1}) = m_{b_u}^{(1)}(\langle u\rangle) \le e(B) - 1$. Now $l(B)$ is the sum over the multiplicities of elementary divisors of the Cartan matrix of B which is at most $m(|D|) + m(\langle y\rangle) + m(\langle u\rangle) \le 1 + e(B) - 1 + e(B) - 1 = 2e(B) - 1$. □

The next proposition gives a reduction method.

Proposition 3.7. *Let $p > 2$, $m \ge 2$ and $e \mid p - 1$ be fixed. Suppose that $l(B) = e$ holds for every block B with defect group*

$$D = \langle x, y \mid x^{p^m} = y^p = 1,\ yxy^{-1} = x^{1+p^{m-1}}\rangle$$

and $e(B) = e$. Then every block B with $e(B) = e$ and defect group

$$D = \langle x, y \mid x^{p^m} = y^{p^n} = 1,\ yxy^{-1} = x^{1+p^{m-1}}\rangle$$

where $n \ge 1$ satisfies the following:

$$k_0(B) = \left(\frac{p^{m-1} - 1}{e(B)} + e(B)\right)p^n, \qquad k_1(B) = \frac{p^{m-1} - p^{m-2}}{e(B)}p^{n-1},$$

$$k(B) = \left(\frac{p^m + p^{m-1} - p^{m-2} - p}{e(B)} + e(B)p\right)p^{n-1}, \qquad l(B) = e(B).$$

Proof. We use induction on n. In case $n = 1$ the result follows from Theorem 3.2 in [38], Theorem 2.3 and Proposition 2.5.

Now let $n \ge 2$. Let $\mathcal{R}$ be a set of representatives for the $\mathcal{F}$-conjugacy classes of elements of D. We are going to use Theorem 5.9.4 in [30]. For $1 \ne u \in \mathcal{R}$, b_u has metacyclic defect group $\mathrm{C}_D(u)$ and fusion system $\mathrm{C}_{\mathcal{F}}(\langle u\rangle)$. If $\mathrm{C}_{\mathcal{F}}(\langle u\rangle)$ is non-trivial, $\alpha \in \mathrm{Aut}_{\mathcal{F}}(D)$ centralizes a D-conjugate of u. Hence, we may assume that $u \in \langle y\rangle$ in this case. If $\langle u\rangle = \langle y\rangle$, then b_u dominates a block $\overline{b_u}$ of $\mathrm{C}_G(u)/\langle u\rangle$ with cyclic defect group $\mathrm{C}_D(u)/\langle u\rangle$. Hence, $l(b_u) = l(\overline{b_u}) = e(B)$. Now suppose that $\langle u\rangle < \langle y\rangle$. Then by induction we obtain $l(b_u) = l(\overline{b_u}) = e(B)$. Finally assume that $\mathrm{C}_{\mathcal{F}}(\langle u\rangle)$ is trivial. Then b_u is nilpotent and $l(b_u) = 1$. It remains to determine $\mathcal{R}$. The powers of y are pairwise non-conjugate in $\mathcal{F}$. As in the proof of Proposition 2.5, D has precisely $p^{n+m-3}(p^2 + p - 1)$ conjugacy classes. Let C be one of these classes which do not intersect $\langle y\rangle$. Assume $\alpha^i(C) = C$ for some $i \in \mathbb{Z}$ such that $\alpha^i \ne 1$. Then there are elements $u \in C$ and $w \in D$ such that $\alpha^i(u) = wuw^{-1}$. Hence

$\gamma := w^{-1}\alpha^i \in \mathrm{N}_G(D, b_D) \cap \mathrm{C}_G(u)$. Since γ is not a p-element, we conclude that u is conjugate to a power of y which was excluded. This shows that no non-trivial power of α can fix C as a set. Thus, all these conjugacy classes split in

$$\frac{p^2 + p - p^{3-m} - 1}{e(B)} p^{n+m-3}$$

orbits of length $e(B)$ under the action of $\mathrm{Out}_{\mathcal{F}}(D)$. Now Theorem 5.9.4 in [30] implies

$$k(B) - l(B) = \left(\frac{p^{m-1} + p^{m-2} - p^{m-3} - 1}{e(B)} + e(B)\right)p^n - e(B).$$

By Proposition 3.6 it follows that

$$k(B) \leq \left(\frac{p^m + p^{m-1} - p^{m-2} - p}{e(B)} + e(B)p\right)p^{n-1} + e(B) - 1. \tag{3.2}$$

By Proposition 2.2 the left-hand side of (3.2) is divisible by p^{n-1}. Since $e(B) - 1 < p^{n-1}$, we obtain the exact value of $k(B)$. It follows that $l(B) = e(B)$. Finally, Theorem 2.3 and Proposition 2.5 give $k_i(B)$. □

For $p = 3$, Proposition 3.6 implies $l(B) \leq 3$. Here we are able to determine all block invariants.

Theorem 3.8. *Let B be a non-nilpotent 3-block of a finite group with metacyclic, minimal non-abelian defect groups. Then*

$$k_0(B) = \frac{3^{m-2} + 1}{2} 3^{n+1}, \qquad k_1(B) = 3^{m+n-3},$$
$$k(B) = \frac{11 \cdot 3^{m-2} + 9}{2} 3^{n-1}, \qquad l(B) = e(B) = 2$$

with the notation from (1.1).

Proof. By Proposition 3.7 it suffices to settle the case $n = 1$. Here the claim holds for $m \leq 3$ by Theorem 3.7 in [38]. We will extend the proof of this result in order to handle the remaining $m \geq 4$. Since B is non-nilpotent, we have $e(B) = 2$. By Theorem 2.3 we know $k_0(B) = (3^m + 9)/2$. By way of contradiction, we may assume that $l(B) = 3$ and $k_1(B) = 3^{m-2} + 1$ (see Theorem 3.4 in [38]).

We consider the generalized decomposition numbers $d^z_{\chi\varphi_z}$ where $z := x^3 \in \mathrm{Z}(D)$ and φ_z is the unique irreducible Brauer character of b_z. Let $d^z := (d^z_{\chi\varphi_z} : \chi \in \mathrm{Irr}(B))$. By the orthogonality relations we have $(d^z, d^z) = 3^{m+1}$. As in [18, Section 4] we can write

$$d^z = \sum_{i=0}^{2\cdot 3^{m-2}-1} a_i \zeta_{3^{m-1}}^i$$

for integral vectors a_i and a primitive 3^{m-1}-th root of unity $\zeta_{3^{m-1}} \in \mathbb{C}$. Since z is $\mathcal{F}$-conjugate to z^{-1}, the vector d^z is real. Hence, the vectors a_i are linearly dependent. More precisely, it turns out that the vectors a_i are spanned by $\{a_j : j \in J\}$ for a subset $J \subseteq \{0, \ldots, 2 \cdot 3^{m-2} - 1\}$ such that $0 \in J$ and $|J| = 3^{m-2}$.

Let q be the quadratic form corresponding to the Dynkin diagram of type $A_{3^{m-2}}$. We set $a(\chi) := (a_j(\chi) : j \in J)$ for $\chi \in \mathrm{Irr}(B)$. Since the subsection (z, b_z) gives equality in Theorem 4.10 in [18], we have

$$k_0(B) + 9k_1(B) = \sum_{\chi \in \mathrm{Irr}(B)} q(a(\chi))$$

for a suitable ordering of J. This implies $q(a(\chi)) = 3^{2h(\chi)}$ for $\chi \in \mathrm{Irr}(B)$ where $h(\chi)$ is the height of χ. Moreover, if $a_0(\chi) \neq 0$, then $a_0(\chi) = \pm 3^{h(\chi)}$ by Lemma 3.6 in [38]. By Lemma 4.7 in [18] we have $(a_0, a_0) = 27$.

In the next step we determine the number β of 3-rational characters of height 1. Since $(a_0, a_0) = 27$, we have $\beta < 4$. On the other hand, the Galois group $\mathcal{G}$ of $\mathbb{Q}(\zeta_{\zeta_{3^{m-1}}}) \cap \mathbb{R}$ over $\mathbb{Q}$ acts on d^z and the length of every non-trivial orbit is divisible by 3 (because $\mathcal{G}$ is a 3-group). This implies $\beta = 1$, since $k_1(B) = 3^{m-2} + 1$.

In order to derive a contradiction, we repeat the argument with the subsection (x, b_x). Again we get equality in Theorem 4.10 in [18], but this time for $k_0(B)$ instead of $k_0(B) + 9k_1(B)$. Hence, $d^x(\chi) = 0$ for characters $\chi \in \mathrm{Irr}(B)$ of height 1. Again we can write $d^x = \sum_{i=0}^{2\cdot 3^{m-1}-1} \overline{a_i}\zeta_{3^m}^i$ where $\overline{a_i}$ are integral vectors. Lemma 4.7 in [18] implies $(\overline{a}_0, \overline{a}_0) = 9$. Using Lemma 3.6 in [38] we also have $\overline{a}_0(\chi) \in \{0, \pm 1\}$. By Proposition 3.3 in [38] we have precisely three 3-rational characters $\chi_1, \chi_2, \chi_3 \in \mathrm{Irr}(B)$ of height 0 (note that altogether we have four 3-rational characters). Then $a_0(\chi_i) = \pm\overline{a}_0(\chi_i) = \pm 1$ for $i = 1, 2, 3$. By [36, Section 1] we have $\lambda * \chi_i \in \mathrm{Irr}_0(B)$ and $(\lambda * \chi_i)(u) = \chi_i(u)$ for $\lambda \in \mathrm{Irr}(D/\mathfrak{foc}(B)) \cong C_3$ and $u \in \{x, z\}$. Since this action on $\mathrm{Irr}_0(B)$ is free, we have nine characters $\psi \in \mathrm{Irr}(B)$ such that $a_0(\psi) = \pm\overline{a}_0(\psi) = \pm 1$. In particular $(a_0, \overline{a_0}) \equiv 1 \pmod 2$. By the orthogonality relations we have $(d^z, d^{x^j}) = 0$ for all $j \in \mathbb{Z}$ such that $3 \nmid j$. Using Galois theory we get the final contradiction $0 = (d^z, \overline{a}_0) = (a_0, \overline{a}_0) \equiv 1 \pmod 2$. □

In the smallest case $D \cong C_9 \rtimes C_3$ of Theorem 3.8, even more information on B was given in Theorem 4.5 in [38].

Corollary 3.9. *Alperin's Weight Conjecture and the Ordinary Weight Conjecture are satisfied for every* 3*-block with metacyclic, minimal non-abelian defect groups.*

Proof. Let D be a defect group of B. Since B is controlled, Alperin's Weight Conjecture asserts that $l(B) = l(B_D)$ where B_D is a Brauer correspondent of B in $\mathrm{N}_G(D)$. Since both numbers equal $e(B)$, the conjecture holds.

Now we prove the Ordinary Weight Conjecture in the form of [2, Conjecture IV.5.49]. Since $\mathrm{Out}_{\mathcal{F}}(D)$ is cyclic, all 2-cocycles appearing in this version are trivial. Therefore the conjecture asserts that $k_i(B)$ only depends on $\mathcal{F}$ and thus on $e(B)$. Since the conjecture is known to hold for the principal block of the solvable group $G = D \rtimes C_{e(B)}$, the claim follows. □

We remark that Alperin's Weight Conjecture is also true for the abelian defect groups $D \cong C_{3^n} \times C_{3^m}$ where $n \neq m$ (see [34, 41]).

We observe another consequence for arbitrary defect groups.

Corollary 3.10. *Let B be a* 3*-block of a finite group with defect group D. Suppose that $D/\langle z \rangle$ is metacyclic, minimal non-abelian for some $z \in \mathrm{Z}(D)$. Then Brauer's $k(B)$-Conjecture holds for B, i.e. $k(B) \leq |D|$.*

Proof. Let (z, b_z) be a major subsection of B. Then b_z dominates a block $\overline{b_z}$ of $C_G(z)/\langle z\rangle$ with metacyclic, minimal non-abelian defect group $D/\langle z\rangle$. Hence, Theorem 3.8 implies $l(b_z) = l(\overline{b_z}) \le 2$. Now the claim follows from Theorem 2.1 in [39]. □

In the situation of Theorem 3.8 it is straight-forward to distribute $\mathrm{Irr}(B)$ into families of 3-conjugate and 3-rational characters (cf. Proposition 3.3 in [38]). However, it is not so easy to see which of these families lie in $\mathrm{Irr}_0(B)$.

Now we turn to $p = 5$.

Theorem 3.11. *Let B be a 5-block of a finite group with non-abelian defect group $C_{25} \rtimes C_{5^n}$ where $n \ge 1$. Then*

$$k_0(B) = \left(\frac{4}{e(B)} + e(B)\right)5^n, \qquad k_1(B) = \frac{4}{e(B)}5^{n-1},$$
$$k(B) = \left(\frac{24}{e(B)} + 5e(B)\right)5^{n-1}, \qquad l(B) = e(B).$$

Proof. By Proposition 3.7 it suffices to settle the case $n = 1$. Moreover by Theorem 4.4 in [38] we may assume that $e(B) = 4$. Then by Theorem 2.3 above and Proposition 4.2 in [38] we have $k_0(B) = 25$, $1 \le k_1(B) \le 3$, $26 \le k(B) \le 28$ and $4 \le l(B) \le 6$. We consider the generalized decomposition numbers $d^z_{\chi\varphi_z}$ where $z := x^5 \in \mathrm{Z}(D)$ and φ_z is the unique irreducible Brauer character of b_z. Since all non-trivial powers of z are $\mathcal{F}$-conjugate, the numbers $d^z_{\chi\varphi_z}$ are integral. Also, these numbers are non-zero, because (z, b_z) is a major subsection. Moreover, $d^z_{\chi\varphi_z} \equiv 0 \pmod p$ for characters $\chi \in \mathrm{Irr}(B)$ of height 1 (see Theorem V.9.4 in [12]). Let $d^z := (d^z_{\chi\varphi_z} : \chi \in \mathrm{Irr}(B))$. By the orthogonality relations we have $(d^z, d^z) = 125$. Suppose by way of contradiction that $k_1(B) > 1$. Then it is easy to see that $d^z_{\chi\varphi_z} = \pm 5$ for characters $\chi \in \mathrm{Irr}(B)$ of height 1. By [36, Section 1], the numbers $d^z_{\chi\varphi_z}$ ($\chi \in \mathrm{Irr}_0(B)$) split in five orbits of length 5 each. Let α (respectively β, γ) be the number of orbits of entries ± 1 (respectively ± 2, ± 3) in d^z. Then the orthogonality relations read

$$\alpha + 4\beta + 9\gamma + 5k_1(B) = 25.$$

Since $\alpha + \beta + \gamma = 5$, we obtain

$$3\beta + 8\gamma = 20 - 5k_1(B) \in \{5, 10\}.$$

However, this equation cannot hold for any choice of α, β, γ. Therefore we have proved that $k_1(B) = 1$. Now Theorem 4.1 in [38] implies $l(B) = 4$. □

Corollary 3.12. *Alperin's Weight Conjecture and the Ordinary Weight Conjecture are satisfied for every 5-block with non-abelian defect group $C_{25} \rtimes C_{5^n}$.*

Proof. See Corollary 3.9. □

Unfortunately, the proof of Theorem 3.11 does not work for $p = 7$ and $e(B) = 6$ (even by invoking the other generalized decomposition numbers). However, we have the following partial result.

Proposition 3.13. *Let $p \in \{7, 11, 13, 17, 23, 29\}$ and let B be a p-block of a finite group with defect group $C_{p^2} \rtimes C_{p^n}$ where $n \geq 1$. If $e(B) = 2$, then*

$$k_0(B) = \frac{p+3}{2}p^n, \qquad k_1(B) = \frac{p-1}{2}p^{n-1},$$
$$k(B) = \frac{p^2+4p-1}{2}p^{n-1}, \qquad l(B) = 2.$$

Proof. We follow the proof of Theorem 4.4 in [38] in order to handle the case $n = 1$. After that the result follows from Proposition 3.7.

In fact the first part of the proof of Theorem 4.4 in [38] applies to any prime $p \geq 7$. Hence, we know that the generalized decomposition numbers $d^z_{\chi\varphi_z} = a_0(\chi)$ for $z := x^p$ and $\chi \in \operatorname{Irr}_0(B)$ are integral. Moreover,

$$\sum_{\chi \in \operatorname{Irr}_0(B)} a_0(\chi)^2 = p^2.$$

The action of $D/\mathfrak{foc}(B)$ on $\operatorname{Irr}_0(B)$ shows that the values $a(\chi)$ distribute in $(p+3)/2$ parts of p equal numbers each. Therefore, (4.1) in [38] becomes

$$\sum_{i=2}^{\infty} r_i(i^2 - 1) = \frac{p-3}{2}$$

for some $r_i \geq 0$. This gives a contradiction. □

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