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POINTWISE ERGODIC THEOREMS FOR BOUNDED LAMPERTI REPRESENTATIONS OF AMENABLE GROUPS

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ABSTRACT. Dominated and Pointwise Ergodic Theorems for bounded representations of second countable locally compact amenable groups by Lamperti operators in $L^p(\Omega, \mathcal{F}, m)$, $p > 1$ fixed, are proved; we restrict ourselves to Cesàro averages in this paper. These theorems generalize or are closely related to well-known theorems for powers of power bounded Lamperti operators.

1. Introduction

The Birkhoff–Khinchin Pointwise Ergodic Theorem (PET) was extended to measure preserving actions of various kinds of locally compact amenable groups in [4, 5, 8, 10, 21, 27, 29–31, 35]. A. Shulman [29] proved the $(1, p)$ Maximal Ergodic Theorem (MET) and the PET for averages with weights, including as a special case Cesàro averages with respect to "tempered" sequences of sets, for actions of arbitrary amenable σ-compact groups and for functions $f \in L^p, 1 < p < \infty$ (see §5.6 and §6.1 in [33] where the general statements and, in the case $p = 2$, the proof were published). Lindenstrauss proved the (1,1) MET and the PET for Cesàro averages over "tempered" sequences of sets when $f \in L^1$.

In 1958, N. Dunford and J.T. Schwartz proved the PET for operators which are $L^1 - L^\infty$ contractions (see [7, Chapter VIII]). The first PET for operators acting in one space $L^p, 1 < p < \infty$, was obtained by A. Ionescu Tulcea [13] who proved it for positive invertible isometries. This was extended by M. A. Akcoglu [3] to positive contractions in $L^p, 1 < p < \infty$; Akcoglu's result stimulated research for analogous theorems for power-bounded positive operators in L^p and for bounded group representations by positive operators in L^p. A multi-parameter version of Akcoglu's theorem was obtained by S. A. McGrath [25]. F. J. Martin-Reyes and A. de la Torre [23, 24] proved the Dominated Ergodic Theorem (DET) and the PET for powers of invertible positive operators with uniformly bounded averages. C.-H. Kan [16] proved the DET and the PET for powers of, in general non-invertible, bounded operators with Lamperti conjugates (invertible bounded Lamperti operators possess this property).

The first DET and PET for representations of locally compact amenable groups by Lamperti operators in $L^p(\Omega, \mathcal{F}, m)$, $p > 1$, are due to M. Lin and R. Wittmann [20], who considered convolution powers of a symmetric probability measure as weights; for commutative groups, DETs for other kinds of weights had been proved earlier by N. Asmar, E. Berkson and T. A. Gillespie [2] (in his review of this paper

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in *Math. Reviews*, M. Cowling noted that the proof can be easily adapted to general amenable groups).

In this paper, Theorem 3.9 generalizes the part of the above results related to the PET for Cesàro averages for bounded Lamperti representations of $Z^d, d \geq 1$, in $L^p, 1 < p < \infty$, to arbitrary locally compact amenable groups with a countable base. On the other hand, this theorem generalizes the DET and the PET for measure preserving actions of amenable groups and functions in $L^p, p > 1$, for Cesàro averages with respect to ergodic tempered sequences of sets due to A. Shulman [29] (the complete generalization of Shulman's theorem will be presented in a forthcoming publication by the author where other methods are used). It is known (see [13]) that the PET may fail for some positive invertible isometries in $L^1(\Omega, \mathcal{F}, m)$; therefore our restriction to $p > 1$ seems to be quite natural. Counterexamples due to I. Assani [3] and M. Feder [9] show that the restriction to Lamperti representations is also essential even on $\mathbb{Z}$.

1.1. **Short review of the paper.** In Subsection 1.2 we introduce the notation used in the paper, and in Subsections 1.3-1.4 the main notions are defined. In §2 we present a generalized version of the Lin - Wittmann Transfer Principle (see Theorem 3.1 in [20]), which plays a crucial role in the proofs of the DETs in Subsections 3.1 and 4.1. §3 contains the main results of the paper: the Dominated and Pointwise Ergodic Theorems for Følner tempered sequences of compact sets. While averaging by sequences of sets is quite natural in countable groups, in continuous groups more general nets of sets may be preferable (for example, when averaging by concentric balls in $\mathbb{R}^d$, the net $\{B_r, r > 0\}$ is usually considered, i.e., the balls are indexed by their radii); therefore in §4 we generalize the results of §3 related to regular sequences to regular nets of sets. In §5 we prove a modulated PET that combines the PET for Lamperti representations and the Mean Value Theorem for Besicovitch almost periodic functions. In §6 we apply the results of §3 to prove PETs for bounded multiplicative cocycles over group actions. For the convenience of the reader, in the Appendix we provide some (well-known) examples of regular and Følner sequences and nets of sets in $\mathbb{R}^d, \mathbb{Z}^d (d \geq 1)$ and in groups of polynomial growth.

1.2. **Notation.** We use the following notation: G is a locally compact (l.c.) amenable group with a countable base; e is the identity in G; $\mathcal{B}$ is the Borel σ-algebra on G; μ (resp. ν) is the left (resp. right) Haar measure on $\mathcal{B}$; $A^{-1}B = \{z : z = a^{-1}b, a \in A, b \in B\}, A, B \subset G$.

":=" means "definition by the equation".

$(\Omega, \mathcal{F}, m)$ is a σ-finite measure space.

Let $p \geq 1$ (we always assume that p is a finite number). $L^p(\Omega, \mathcal{F}, m)$ or $L^p(\Omega)$ is the corresponding Lebesgue space and $L^p_+(\Omega, \mathcal{F}, m)$ or $L^p_+(\Omega)$ is the set of all non-negative functions in this space; $||\cdot||_p$ denotes the norm in $L^p(\Omega)$.

"Operator" means a *bounded linear operator in a space* $L^p(\Omega)$.

In §2-§6 "representation" means a *bounded measurable right representation of* G *in a space* $L^p(\Omega)$.

We denote by fT_x the image of a function f with respect to an operator T_x when $T : x \mapsto T_x$ is a right representation, i.e., when $T_{xy} = T_y T_x, x, y \in G$; for left representations we use the notation $T_x f$. $||T||_p := \sup_{x \in G} ||T_x||_p$ where $||T_x||_p$ is the norm of T_x.

$\mathbb{Z}$ and $\mathbb{R}$ are the groups of all integers and of all real numbers, respectively; $\mathbb{N}$ denotes the set of all natural numbers.

N is a linearly ordered set.

1.3. **Group representations and actions.** Let $1 \le p < \infty$. An operator T in L^p is called a *Lamperti operator* if for $f, g \in L^p$, $f \cdot g = 0$ m-a.e. implies $Tf \cdot Tg = 0$ m-a.e.

A mapping $T : x \mapsto T_x$ on G is said to be a *left (resp. right) L^p-representation of G* if, for each $x \in G$, T_x is an operator in $L^p(\Omega, \mathcal{F}, m)$, $T_{xy} = T_x T_y, x, y \in G$ (resp. $T_{xy} = T_y T_x, x, y \in G$) and T_e is the identity operator. We say that a representation T is *bounded* if $||T||_p =: \sup_{x\in G} ||T_x||_p < \infty$. For the T_x-image of $f \in L^p$ we use the notation $T_x f$ if T is a left representation and fT_x if T is a right one. We say that T is a *Lamperti* (resp. *positive*, resp. *isometric*) L^p-representation if all T_x have the corresponding property. Each isometric L^p-representation, when $p \ne 2$, and each positive L^p-representation are Lamperti ones (see [15]). With each Lamperti operator A in L^p we can associate its *modulus* $|A|$: a positive Lamperti operator in L^p such that $|Af| = |A|\,|f|$ for each $f \in L^p(\Omega)$; if $T : x \mapsto T_x$ is a Lamperti representation, then its *modulus* $|T| : x \mapsto |T_x|$ is a Lamperti representation too; moreover, $\| \, |T| \, \|_p = \|T\|_p$ (see [15]).

A mapping $\tau : x \mapsto \tau_x, x \in G$, is said to be a (left) *G-action* of Ω if all $\tau_x, x \in G$, are measurable point transformations of Ω, $\tau_x\tau_y = \tau_{xy}$ for all $x, y \in G$ and τ_e is the identity transformation. An action τ in Ω is said to be *measure preserving* if $m(\tau_x\Lambda) = m(\Lambda), \Lambda \in \mathcal{F}, x \in G$. We fix an action τ and consider the measures $\tilde{m}_x^- : \tilde{m}_x^-(\Lambda) = m(\tau_x^{-1}\Lambda)$; we say that τ is *non-singular* if $\tilde{m}_x^- \sim m, x \in G$, and it is *bounded* if $||\tau||_m := \sup_{x\in G} \operatorname{vrai\,sup}_\omega \frac{d\tilde{m}_x^-}{dm}(\omega) < \infty$. Let $p, 1 \le p < \infty$; with any left action $\tau = \{\tau_x, x \in G\}$ of G we associate a positive right Lamperti representation T in $L^p(\Omega, \mathcal{F}, m)$ defined by $x \mapsto T_x : fT_x(\omega) = f(\tau_x\omega)$; this representation is bounded if and only if the action τ is bounded; moreover $||T||_p^p = ||\tau||_m$.

We say that a right representation T in a space $L^p(\Omega) = L^p(\Omega, \mathcal{F}, m)(p \ge 1)$ is *measurable* if for each $f \in L^p$ the function $(\omega, x) \mapsto fT_x(\omega)$ is $\mathcal{F} \times \mathcal{B}$-measurable. Let us note that, for each $x \in G$, $\|fT_x\|_p \le ||T||_p||f||_p$; therefore, by the Fubini theorem, for each finite Borel measure κ on G and for each $f \in L^p(\Omega)$ we have: for m-a.e. $\omega \in \Omega$ the function $x \mapsto fT_x(\omega)$ belongs to $L^p(G, \mathcal{B}, \kappa)$ and hence it is integrable with respect to κ.

A left action τ is *measurable* if $\{(\omega, x) : \tau_x\omega \in \Lambda\} \in \mathcal{F} \times \mathcal{B}$ for each $\Lambda \in \mathcal{F}$. This is equivalent to measurability of the associated right representation in each $L^p(\Omega), p \ge 1$.

The definitions of measurability of a left representation and of a right action are similar.

If T is a measurable Lamperti right representation in a space

$$L^p(\Omega) = L^p(\Omega, \mathcal{F}, m)(p \ge 1),$$

then its modulus $|T|$ is measurable too; this follows from the construction of the operators $|T_x|$ (see [15]).

1.4. **Følner nets, regular nets and tempered sequences of sets.** Let N be a linearly ordered set (we use this notation throughout the paper).

A net $\{A_n, n \in N\}$ of sets in $\mathcal{B}$ with $0 < \mu(A_n) < \infty$ is called a *Følner net* if

$$\lim_{n\in N} \frac{\mu(A_n \triangle xA_n)}{\mu(A_n)} = 0, \ x \in G. \tag{1.1}$$

Each amenable σ-compact group G possesses Følner sequences of compact sets (see, e.g., [26]).

Let $\mathcal{A} = \{A_n, n \in N\}$ be a net of sets on G with $0 < \mu(A_n) < \infty$. For each $n \in N$ denote: $\tilde{A}_n = \bigcup_{k\le n}^{n} A_k$. $\mathcal{A}$ is said to be (left) *regular* if

$$r_l(\mathcal{A}) := \sup_{n\in N} \frac{\mu(\tilde{A}_n^{-1} A_n)}{\mu(A_n)} < \infty \tag{1.2}$$

(see [30], [31]). Regular Følner sequences exist in all compactly generated l.c. groups of polynomial growth (see §7) while they may not exist in solvable groups.

A sequence $\mathcal{A} = \{A_n\}$ of compact sets in G is said to be (left) *tempered* if

$$t_l(\mathcal{A}) := \sup_{n} \frac{\mu(\tilde{A}_{n-1}^{-1} A_n)}{\mu(A_n)} < \infty; \tag{1.3}$$

such sequences were introduced (under a different name)[1] by A. Shulman [29] (see §5.6 and §6.1 in [33]). E. Lindenstrauss [21, 22] showed that tempered Følner sequences exist in any σ-compact l.c. amenable group. $r_l(\mathcal{A})$ and $t_l(\mathcal{A})$ are the *left regularity* and the *left tempering indexes*; it is clear that $t_l(\mathcal{A}) \le r_l(\mathcal{A})$.

The "right" versions of the above notions are, respectively,

$$r_r(\mathcal{A}) := \sup_{n\in N} \frac{\nu(A_n A_n^{-1})}{\nu(A_n)} < \infty$$

and

$$t_r(\mathcal{A}) := \sup_{n} \frac{\nu(A_n A_n^{-1})}{\nu(A_n)} < \infty.$$

1.5. **Separable nets of measures, densities and sets.** A set of probability measures Γ is said to be *separable* if it is a separable space with respect to the topology of weak convergence. Let N be an infinite ordered set. A net of probability measures $\Gamma = \{\gamma_n, n \in N\}$ is *cofinally separable* if it is separable and there is a countable subset $N_0 \subset N$ such that set $\{\gamma_k, k \in N_0\}$ is dense in $\{\gamma_n \in N\}$ and for each $n \in N$ the set $\{\gamma_k, k > n, k \in N_0\}$ is dense in $\{\gamma_k, k > n\}$; such a set N_0 is called a *c-separability* set. Of course each countable net is cofinally separable. The above definitions can be readily modified for to a net $\Phi = \{\varphi_n, n \in N\}$ of probability densities with respect to μ and to a net of integrable sets $\mathcal{A} = \{A_n, n \in N\}$. To be consistent with the above definitions, we consider these sets as subsets of two separable metric spaces: of $L^1(G, \mathcal{B}, \mu)$ and of the space $\mathcal{S}$ of integrable sets in $\mathcal{B}$ with the metric $\rho(A, B) = \mu(A \triangle B)$, resp. Hence such nets are separable. An increasing net of integrable sets $\mathcal{A} = \{A_n \in N\}$ is cofinally separable (see (3.C) in [33]).

The following statement will be useful when we consider non-countable nets (see (13.D) in [33]).

[1]The term "tempered" was used by E. Lindenstrauss in [21, 22].

Proposition 1.1. *Let T be a measurable right representation of G in some $L^p(\Omega)$, $p \geq 1, f \in L^p(\Omega)$.*

1. Let Γ be a net of absolutely continuous probability measures. Then the function $\omega \mapsto \sup_{\gamma\in\Gamma} \int_G fT_x(\omega)\gamma(dx)$ is measurable.

2. Let $\Gamma = \{\gamma_n, \in N\}$ be a cofinally separable net and N_0 be a c-separability set. If the limit $\lim_{n\in N_0} \int_G fT_x(\omega)\gamma(dx)$ exists, then the limit $\lim_{n\in N} \int_G fT_x(\omega)\gamma(dx)$ also exists and both limits are equal to m-a.e.

1.6. **Assumptions.** *In what follows we assume that G is a second countable l.c. amenable group, T is a measurable bounded right representation of G in some $L^p(\Omega, m), 1 < p < \infty$.*

We consider "left" objects: left actions, right representations, the left Haar measure μ, left tempered sequences of sets, left Følner and left regular nets of sets without special references to the "side". Translation of our "left" statements to the "right" ones is straightforward: if $x \mapsto T_x$ is a left representation, then $x \mapsto T_{x^{-1}}$ is a right one; if $\{A_n\}$ is a right tempered and right ergodic sequence, then $\{A_n^{-1}\}$ is left tempered and left ergodic, and $\frac{1}{\nu(A_n)} \int_{A_n} T_x f\nu(dx) = \frac{1}{\mu(A_n^{-1})} \int_{A_n^{-1}} fT_{x^{-1}}\mu(dx)$.

2. Transfer theorem

We provide a generalized version of the Lin - Wittmann Transfer Principle (Theorem 3.1 in [20]).

Theorem 2.1. *Let $1 < p < \infty$. Let Γ be a set of Borel probability measures on G. Assume that for each $u \in L^p_+(G, \mathcal{B}, \mu)$ the function $y \mapsto \sup_{\gamma\in\Gamma} \int_G u(xy)\gamma(dx)$ is $\mathcal{B}$-measurable and that there is a constant $C > 0$ such that*

$$\int_G \left(\sup_{\gamma\in\Gamma} \int_G u(xy)\gamma(dx)\right)^p \mu(dy) \leq C||u||^p_{L^p(G)}. \tag{2.1}$$

Let T be a Lamperti representation in $L^p(\Omega)$, $f \in L^p(\Omega, \mathcal{F}, m)$ and let the function $f^*(\omega) = \sup_{\gamma\in\Gamma} \int_G |fT_x(\omega)|\gamma(dx)$ be $\mathcal{F}$-measurable.[2] *Then*

$$\int_\Omega [\sup_{\gamma\in\Gamma} \int_G |fT_x(\omega)|\gamma(dx)]^p m(d\omega) \leq C||T||^{2p}_p \|f\|^p_p.$$

Proof. Let us consider the modulus $|T|$ of the representation T (see Subsection 1.3). Since $|T| : x \mapsto |T_x|$ is a positive measurable Lamperti L^p-representation and $|fT(\omega)| = |f|\,|T_x|(\omega)$ m-a.e., we may assume that T_x is a positive representation and $f \in L^p_+(\Omega)$. Let K and A be compact sets in G of positive measure μ. Put $u_\omega(x) = fT_x(\omega)I_{KA}(x)$ m-a.e.; it is clear that this function is measurable. Since $I_K(x)I_A(y) \leq I_{KA}(xy)$, by (2.1)

$$\begin{aligned} &\int_G [\sup_{\gamma\in\Gamma} \int_G fT_{xy}(\omega)I_K(x)I_A(y)\gamma(dx)]^p \mu(dy) \\ &\leq \int_G [\sup_{\gamma\in\Gamma} \int_G u_\omega(xy)\gamma(dx)]^p \mu(dy) \leq C\|u_\omega\|^p_{L^p(G)}. \end{aligned} \tag{2.2}$$

[2] Both measurability conditions are fulfilled if Γ is separable (see Subsection 1.5); in particular, this is true if the measures in Γ are absolutely continuous with respect to μ.

By the Fubini theorem,

$$\int_\Omega \|u_\omega\|^p_{L^p(G)} m(d\omega) = \int_\Omega \left[\int_G (fT_x(\omega)I_{KA}(x))^p \gamma(dx)\right] m(d\omega)$$
$$\leq \int_G I_{KA}(x) \int_\Omega (fT_x(\omega))^p m(d\omega)\mu(dx) \leq \|T\|_p^p \|f\|_p^p [\mu(KA)].$$

This shows that $u_\omega \in L^p(G)$ for m-a.e. ω. We integrate the extreme parts of (2.2) over Ω with respect to the measure m and use the previous relation:

$$\int_\Omega \int_G [\sup_{\gamma\in\Gamma} \int_G fT_{xy} I_K(x) I_A(y)\gamma(dx)]^p \mu(dy)m(d\omega) \leq C\|T\|_p^p \|f\|_p^p \mu(KA),$$

hence

$$\int_\Omega [\sup_{\gamma\in\Gamma} \int_G fT_x I_K(x)\gamma(dx)]^p m(d\omega)$$
$$= \frac{1}{\mu(A)} \int_G \int_\Omega [\sup_{\gamma\in\Gamma} \int_G fT_x I_K(x) I_A(y)\gamma(dx)]^p m(d\omega)\mu(dy)$$
$$\leq \frac{1}{\mu(A)} \int_G \int_\Omega [\sup_{\gamma\in\Gamma} \int_G fT_{xy}T_{y^{-1}} I_K(x) I_A(y)\gamma(dx)]^p m(d\omega)\mu(dy)$$
$$\leq \|T\|_p^p \frac{1}{\mu(A)} \int_\Omega \int_G [\sup_{\gamma\in\Gamma} \int_G fT_{xy} I_K(x) I_A(y)\gamma(dx)]^p \mu(dy)m(d\omega)$$
$$\leq C\|T\|_p^{2p} \|f\|_p^p \frac{\mu(KA)}{\mu(A)}.$$

By Leptin's theorem (see, e.g., §2.7 in [26]), for each compact set K there is a sequence of compact sets A_n such that $\frac{\mu(KA_n)}{\mu(A_n)} \to 1$. Therefore,

$$\int_\Omega [\sup_{\gamma\in\Gamma} \int_G fT_x I_K(x)\gamma(dx)]^p m(d\omega) \leq C\|T\|_p^{2p} \|f\|_p^p.$$

It remains to put $K_l \uparrow G$ for K and to apply the B. Levi theorem. □

3. Ergodic theorems for tempered sequences

3.1. **Maximal and Dominated Ergodic Theorems.** We start with the well-known Maximal Ergodic Theorem due to Lindenstrauss (see Theorem 3.1 in [22]).[3]

Lemma 3.1. *If $\tau : x \mapsto \tau_x$ is a measure preserving action in $(\Omega, \mathcal{F}, m)$, $f \in L^1(\Omega, \mathcal{F}, m)$ and $\mathcal{A} = \{A_n\}$ is a tempered sequence of sets on G, then for each $\varepsilon > 0$*

$$m\{\sup_n \frac{1}{\mu(A_n)} \int_{A_n} |f(\tau_x\omega)|\mu(dx) > \varepsilon\} \leq \varepsilon^{-1} 2(1 + t_l(\mathcal{A}))\|f\|_1.$$

Lemma 3.2. *Under the conditions of Lemma 3.1, for each $p > 1$*

$$||\sup_n \frac{1}{\mu(A_n)} \int_{A_n} |f(\tau_x\omega)|\mu(dx)||_p^p \leq \frac{2^{p+1}p(1 + t_l(\mathcal{A}))}{p-1} ||f||_p^p.$$

Proof. This statement follows from Lemma 3.1 and from Proposition 5.1.5 in [33]. □

[3]Although in [22] this theorem is stated under the condition $m(\Omega) = 1$, it is valid for any σ-finite measure m.

Theorem 3.3. *Let T be a Lamperti representation of G in $L^p(\Omega, \mathcal{F}, m)$ $(1 < p < \infty)$ and let $\mathcal{A} = \{A_n\}$ be a tempered sequence of sets on G. Then for each $f \in L^p(\Omega, \mathcal{F}, m)$*

$$\|\sup_{n\geq 1} \frac{1}{\mu(A_n)} \int_{A_n} |fT_x(\omega)|\mu(dx)\|_p^p \leq \frac{2^{p+1}p(1+t_l(\mathcal{A}))}{p-1}\|T\|_p^{2p}\|f\|_p^p.$$

Proof. We apply Lemma 3.2 to the case when $(\Omega, \mathcal{F}, m) = (G, \mathcal{B}, \mu)$ and $\tau_x y = xy$. We obtain: for each $u \in L_+^p(G, \mathcal{B}, \mu)$

$$||\sup_n \frac{1}{\mu(A_n)} \int_{A_n} u(xy)\mu(dx)||_p^p \leq \frac{2^{p+1}p(1+t_l(\mathcal{A}))}{p-1}||g||_p^p.$$

It remains to apply Theorem 2.1. $\square$

Lemma 3.4. *Let $\gamma \geq 1, b = \frac{p}{\gamma}$ and let T be a Lamperti L^p-representation. We consider mappings $T_x^{(b)} : f^b \mapsto f^b T_x^{(b)}$ defined as follows:*

$$f^b T_x^{(b)}(\omega) := (fT_x(\omega))^b$$

where $f \in L^p(\Omega, \mathcal{F}, m)$.[4]

a) The mapping $T^{(b)} : x \mapsto T_x^{(b)}$ is a Lamperti representation of G in $L^\gamma(\Omega)$.

b) $||f^b||_\gamma^\gamma = \|f||_p^p, \quad ||T_x^{(b)}||_\gamma^\gamma = ||T_x||_p^p$.

Proof. All features of a Lamperti representation for the mapping $x \mapsto T_x^{(b)}$ can be readily deduced from the above definition and from the corresponding properties of the representation T. It is easy to verify that $\|f^b\|_\gamma^\gamma = \|f||_p^p$ for each $f \in L^p$ and that $f \mapsto f^b$ is a one-to-one mapping from L^p onto L^γ; this implies $||f^b T_x^{(b)}||_\gamma^\gamma = ||fT_x||_p^p$ so each $T_x^{(b)}$ maps L^γ onto L^γ. According to Theorem 4.1 in [15], each Lamperti operator T_x is of the following form: $fT_x(\omega) = h_x(\omega)f\Psi_x(\omega)$ where Ψ_x is a linear transformation in the space of all measurable functions $M(\Omega, \mathcal{F}, m)$ with the following property: $(f\Psi_x)^b = f^b\Psi_x, f \in M(\Omega, \mathcal{F}, m), b > 0$; therefore $f^b T_x^{(b)}(\omega) = (h_x(\omega))^b f^b \Psi_x(\omega)$ for each $f^b \in L^\gamma$; this shows that the transformations $T_x^{(b)}$ in L^γ are linear. Moreover,

$$\sup_{||f^b||_\gamma \leq 1} ||f^b T_x^{(b)}||_\gamma^\gamma = \sup_{||f||_p \leq 1} ||fT_x)||_p^p,$$

therefore $||T_x^{(b)}||_\gamma^\gamma = ||T_x||_p^p \leq ||T||_p^p$. Each operator $T_x^{(b)}$ is Lamperti: if $f, g \in L^p, f(\omega)g(\omega) = 0$ m-a.e., then $f^b T_x^{(b)} \cdot g^b T_x^{(b)} = (fT_x \cdot gT_x)^b = 0$. It is clear that, for each $f^b \in L^\gamma$, the mapping $(\omega, x) \mapsto f^b T_x^{(b)}(\omega)$ is $\mathcal{F} \times \mathcal{B}$-measurable. At last, for each $f \in L^p$ and $x, y \in G$, we have: $f^b T_{xy}^{(b)} = (fT_{xy})^b = (fT_x T_y)^b = (fT_x)^b T_y^{(b)} = f^b T_x^{(b)} T_y^{(b)}$, hence $x \mapsto T_x^{(b)}$ is a Lamperti representation. $\square$

Let us apply Theorem 3.3 with $\gamma > 1$ instead of p to the representation $T^{(\frac{p}{\gamma})}$ defined in Lemma 3.4 and to each function $f^{\frac{p}{\gamma}} \in L^\gamma$. We obtain the following generalized version of Theorem 3.3.

[4]For each complex number z we consider the principal value of z^b; in particular, $z^b w^b$ is identified with the principal value of $(zw)^b$.

Theorem 3.5. *Let $p > 1, \gamma > 1$, and let $\mathcal{A} = \{A_n\}$ be a tempered sequence of sets on G. Assume that $T : x \mapsto T_x$ is a Lamperti L^p-representation. Then for each $f \in L^p(\Omega, \mathcal{F}, m)$*

$$\int_\Omega \left(\sup_{n\geq 1} \frac{1}{\mu(A_n)} \int_{A_n} |fT_x(\omega)|^{\frac{p}{\gamma}} \mu(dx) \right)^\gamma m(d\omega) \leq \frac{2^{\gamma+1}\gamma(1+t_l(\mathcal{A}))}{\gamma - 1} \|T\|_p^{2p} ||f||_p^p.$$

Corollary 3.6. *Under the conditions of Theorem 3.5, for each $f \in L^p(\Omega, \mathcal{F}, m)$*

$$\sup_{n\geq 1} [\frac{1}{\mu(A_n)} \int_{A_n} |fT_x(\omega)|^{\frac{p}{\gamma}} \mu(dx)] < \infty \quad m-\text{a.e.}$$

3.2. **Pointwise Ergodic Theorems.** Let us fix a representation T in

$$L^p = L^p(\Omega, \mathcal{F}, m), p > 1.$$

For any $f \in L^p$ and $x \in G$ we denote: $f_x = fT_{x^{-1}} - f$. Let D_0^p be the linear manifold generated by such functions and D^p be its closure in L^p. We denote by I^p the set of all T-invariant functions in L^p. By reflexivity of L^p, the "ergodic decomposition" holds:

$$L^p = D^p \oplus I^p \tag{3.1}$$

(see [33, Theorem 1.5.9 and Proposition 1.3.7]). The projection of f onto I^p (along D^p), denoted by $f\mathbf{M}_T^{(p)}$ (or simply by $f\mathbf{M}^{(p)}$ when T is fixed), is called the *mean value* of the orbit $\{fT_x,\ x \in G\}$. It is T_x-invariant: $f\mathbf{M}^{(p)}T_x = f\mathbf{M}^{(p)}$ for all $x \in G, f \in L^p$. $f \mapsto f\mathbf{M}^{(p)}$ is a projection in L^p and $||\mathbf{M}_T^{(p)}||_p \leq ||T||_p$.

Let $\{\varphi_n\}$ be a sequence of probability densities with respect to μ; we denote $f\mathbf{M}_n := \int_G fT_x\varphi_n(x)\mu(dx)$. It is clear that M_n are operators in $L^p(\Omega, \mathcal{F}, m)$ and, moreover, by virtue of the Hölder inequality and of the Fubini theorem, $||\mathbf{M}_n||_p \leq ||T||_p$.

Definition 3.7. We say that a net of probability densities $\{\varphi_n, n \in N\}$ is *ergodic* if $\lim_{n\in N} \int_G |\varphi_n(xy) - \varphi_n(x)|\mu(dx) = 0$ for each $y \in G$.

By the Mean Ergodic Theorem, if $\{\varphi_n, n \in N\}$ is ergodic, then for each $f \in L^p$ $f\mathbf{M}^{(p)} = (L^p) \lim_{n\to\infty} f\mathbf{M}_n$.

Lemma 3.8. *Let T be a bounded measurable representation of G in $L^p(\Omega), p \geq 1$. Assume that $\{\varphi_n, n \in N\}$ is a cofinally separable ergodic net of probability densities on G with respect to μ such that there is a number $c > 1$ such that for each $f \in L^p(\Omega, \mathcal{F}, m)$*

$$\sup_{n\in N} [\int_G |fT_x(\omega)|^c \varphi_n(x)\mu(dx)] < \infty \quad m-\text{a.e.} \tag{3.2}$$

Then for each $f \in L^p(\Omega, \mathcal{F}, m)$

$$\lim_{n\in N} \int_G fT_x\varphi_n(x)\mu(dx) = f\mathbf{M}^{(p)} \quad m\text{-a.e.}$$

Proof. First let us note that, by the Hölder's inequality, condition (3.2) implies that for each $f \in L^p(\Omega, \mathcal{F}, m)$

$$\sup_{n\in N} [\int_G |fT_x(\omega)|^b \varphi_n(x)\mu(dx)] < \infty \quad m-\text{a.e.} \quad \text{if} \quad 1 \leq b \leq c. \tag{3.3}$$

For any $g \in L^P(\Omega)$ and any $y \in G$ consider $g_y = gT_{y^{-1}} - g$. We will prove that for each such y and g

$$\lim_{n\in N} g_y\mathbf{M}_n = 0 \ m\text{-a.e.} \tag{3.4}$$

Since

$$gT_{y^{-1}}\mathbf{M}_n = \int_G gT_{y^{-1}}T_x\varphi_n(x)\mu(dx) = \int_G gT_{y^{-1}x}\varphi_n(x)\mu(dx) = \int_G gT_z\varphi_n(yz)\mu(dz),$$

we have: $g_y\mathbf{M}_n = \int_G gT_x(\omega)(\varphi_n(yx) - \varphi_n(x))\mu(dx)$. Choose some $b, 1 < b \le c$. Using Hölder's inequality we obtain:

$$|g_y\mathbf{M}_n(\omega)|^b \le (\int_G |gT_x(\omega)|\,|\varphi_n(yx) - \varphi_n(x)|\mu(dx))^b$$
$$\le \int_G |gT_x(\omega)|^b|\varphi_n(yx) - \varphi_n(x)|\mu(dx)(\int_G |\varphi_n(yx) - \varphi_n(x)|\mu(dx))^{b-1}$$
$$\le \int_G |gT_x(\omega)|^b(\varphi_n(yx) + \varphi_n(x))\mu(dx)(\int_G |\varphi_n(yx) - \varphi_n(x)|\mu(dx))^{b-1}.$$

Property (3.3) (applied to the functions g and $gT_{y^{-1}}$) shows that, in the extreme right term of above relation, the first factors form an m-a.e. bounded sequence, while the second factors tend to 0 since the net $\{\varphi_n\}$ is ergodic. This implies (3.4). Now, by (3.3) with $b = 1$, $\sup_{n\in N} |f\mathbf{M}_n(\omega)| < \infty$ m-a.e. for each $f \in L^p$. If N is countable, by virtue of the Banach Convergence Principle (see, e.g., [7]) we obtain: $\lim_{n\in N} f\mathbf{M}_n = 0$ for any $f \in D^p$; it remains to employ the ergodic decomposition (3.1). For arbitrary cofinally separable nets apply Proposition 1.1. ☐

Theorem 3.9. *Let $\mathcal{A} = \{A_n\}$ be a tempered Følner sequence in G. If $1 < p < \infty$ and $T : x \mapsto T_x$ is a Lamperti L^p-representation, then for each $f \in L^p(\Omega, \mathcal{F}, m)$*

$$\lim_{n\to\infty} \frac{1}{\mu(A_n)} \int_{A_n} fT_x(\omega)\mu(dx) = f\mathbf{M}^{(p)}(\omega) \ \ m\text{-a.e.}$$

Proof. Our statement follows from Lemma 3.8 with $N = \mathbb{N}$, $\varphi_n(x) = \frac{1}{\mu(A_n)}\mathbf{1}_{A_n}(x)$, $1 < c < p$. Indeed, the Følner condition implies that this sequence is ergodic, and we have $\frac{1}{\mu(A_n)}\int_{A_n} fT_x\mu(dx) = \int_G fT_x\varphi_n(x)\mu(dx)$; condition (3.2) is guaranteed by Corollary 3.6. ☐

Corollary 3.10. *Let $\mathcal{A} = \{A_n\}$ be tempered and Følner. If $\tau : x \mapsto \tau_x$ is a bounded G-action, then for each $f \in L^p(\Omega, \mathcal{F}, m)$ with $p > 1$*

$$\lim_{n\to\infty} \frac{1}{\mu(A_n)} \int_{A_n} f(\tau_x\omega)\mu(dx) = f\mathbf{M}^{(p)}(\omega) \ \ m\text{-a.e.}$$

Remark 3.11. If $m(\Omega) < \infty$, then for each bounded non-singular left action $x \mapsto \tau_x$ of G in $(\Omega, \mathcal{F}, m)$ there exists an equivalent to m finite τ-invariant probability measure m' (see [11]); moreover, the proof of Theorem 1 in [11] implies that τ is m-bounded if and only if $c_1m' \le m \le c_2m'$ for some constants $c_1, c_2 > 0$. Therefore if $m(\Omega) < \infty$ Corollary 3.10 follows from Lindenstrauss' theorem (stated above as Lemma 3.1) which gives also the m-a.e. convergence for $f \in L^1(\Omega)$.

Remark 3.12. A class of, generally speaking, non-regular averaging sequences $\{A_n\}$ in connected amenable groups was considered by F. Greenleaf and W. Emerson [10] (we call them *GE sequences*). They have proved that each connected amenable group with trivial largest compact normal subgroup K is a multiple semidirect product of a compact subgroup and subgroups isomorphic to $\mathbb{R}$; the sets A_n are simply the "semidirect parallelepipeds" with their "edges" in the component subgroups; in the general case the inverse images of the A_ns with respect to the natural map $G \mapsto G/K$ are considered (see [10] for interesting examples of such sequences of sets); it has been proved therein that each such sequence contains a Følner subsequence. In [10] the PET for measure preserving actions with Cesàro averages over any GE sequence has been proved. Theorem 3.1 in [10] shows that, if $\{A_n\}$ is a GE sequence, the DET is valid for measure preserving actions. Theorem 2.1 implies that an analog of our Theorem 3.3 is valid for such sequences, too, and, as above, this implies the analog of Theorem 3.5 for GE sequences. By reasoning as in the proof of Theorem 3.9, we come to the analog of Theorem 3.9 for Følner GE sequences; following the proof of Theorem 3.1 in [10], one can remove the Følner condition by application of the "multi-parameter" Dunford-Zygmund type generalization of Theorem 3.9.

4. Ergodic theorems for regular nets

4.1. **Dominated inequalities.** In this subsection N means an ordered set and $\mathcal{A} = \{A_n, n \in N\}$ is a regular separable net of Borel sets with $0 < \mu(A_n) < \infty$. The following statement is contained in Theorem 5.3.2 in [33].

Lemma 4.1. *Let $u \in L^1_+(G, \mathcal{B}, \mu), \varepsilon > 0$. Then*

$$\mu(\{y : \sup_{n\in N} \frac{1}{\mu(A_n)} \int_{A_n} u(xy)(dx) > \varepsilon\}) \leq \varepsilon^{-1} r_l(\mathcal{A}) \int_G u(x)\mu(dx).$$

Lemma 4.2. *If $p > 1$ and $u \in L^p(G)$, then*

$$|| \sup_{n\in N} \frac{1}{\mu(A_n)} \int_{A_n} |u(xy)|\mu(dx)||^p_p \leq r_l(\mathcal{A}) \frac{2^p p}{p-1} \int_G |u(x)|^p \mu(dx)||g||^p_p.$$

Proof. Apply Lemma 4.1 and Proposition 5.1.5 in [33]. □

Lemma 4.2 and Theorem 2.1 bring us to the following statement.

Theorem 4.3. *Let T be a Lamperti representation of G in $L^p(\Omega, \mathcal{F}, m)$ $(p > 1)$. Then for each $f \in L^p(\Omega, \mathcal{F}, m)$*

$$\| \sup_{n\in N} \frac{1}{\mu(A_n)} \int_{A_n} fT_x(\omega)\mu(dx)\|^p_p \leq r_l(\mathcal{A}) \frac{2^p p}{p-1} \|T\|^{2p}_p \|f\|^p_p.$$

Let us apply Theorem 4.3 with $\gamma > 1$ instead of p to the Lamperti L^γ-representation $T^{(\frac{p}{\gamma})}$, defined in Lemma 3.4, and to every function $f^{\frac{p}{\gamma}} \in L^\gamma(\Omega)$. We obtain the following generalized version of this theorem.

Theorem 4.4. *Let $p > 1, \gamma > 1$. Assume that $T : x \mapsto T_x$ is a Lamperti L^p-representation. Then for each $f \in L^p(\Omega, \mathcal{F}, m)$*

$$\int_\Omega \left(\sup_{n\in N} \frac{1}{\mu(A_n)} \int_{A_n} |fT_x(\omega)|^{\frac{p}{\gamma}} \mu(dx) \right)^\gamma m(d\omega) \leq r_l(\mathcal{A}) \frac{2^\gamma \gamma}{\gamma - 1} \|T\|^{2p}_p ||f||^p_p.$$

Hence m-a.e.

$$\sup_{n\in N} \left(\frac{1}{\mu(A_n)} \int_{A_n} |fT_x(\omega)|^{\frac{p}{\gamma}} \mu(dx) \right)^\gamma m(d\omega) < \infty.$$

4.2. **Pointwise Ergodic Theorem.** Lemma 3.8 and Theorem 4.4 imply the following statement.

Theorem 4.5. *Let $p > 1$, N be an ordered set, and let the net $\mathcal{A} = \{A_n, n \in N\}$ be cofinally separable,*[5] *Følner and regular. If $T : x \mapsto T_x$ is a Lamperti L^p-representation, then for each $f \in L^p(\Omega, \mathcal{F}, m)$*

$$\lim_{n \in N} \frac{1}{\mu(A_n)} \int_{A_n} fT_x(\omega)\mu(dx) = f\mathbf{M}^{(p)}(\omega) \ \ m\text{-a.e.}$$

5. Modulated ergodic theorems

We present here a general version of "modulated" ergodic theorems (see [32], [28], [19], [18] and the references therein); for simplicity, *in this section we assume that G is commutative.* Let $\hat{G}$ denote its dual group of multiplicative characters. Let us fix a Følner sequence of sets $\mathcal{A} = \{A_n\}$ in G and let $1 \le q < \infty$. We consider the space $W^q(\mathcal{A})$ of all measurable functions u on G with finite semi-norm

$$||u||_q = \limsup_{n \to \infty} \left[\frac{1}{\mu(A_n)} \int_{A_n} |u(x)|^q \mu(dx) \right]^{1/q}$$

and the space $B^q(\mathcal{A})$ of q-Besicovitch almost periodic functions (with respect to the sequence $\mathcal{A}$), i.e., the closure in $W^q(\mathcal{A})$ of the set AP of all uniformly almost periodic functions. It is easy to check that $B^q \subset B^r$ if $r < q$. Each $u \in B^q(\mathcal{A})$ has the Fourier series expansion: $u = \sum_{\chi \in \hat{G}} c_u(\chi)\chi$ (the series converges in $B^q(\mathcal{A})$).

Since $\mathcal{A}$ is Følner, for each $u \in B^q(\mathcal{A}), q \ge 1$, the *mean value*

$$M(u) = \lim_{n \to \infty} \frac{1}{\mu(A_n)} \int_{A_n} u(x)\mu(dx)$$

exists (see Theorem B in [32]).

In this section we prove a statement incorporating both the PET (Theorem 3.9) and, for $q > 1$, the stated mean value theorem for Besicovitch functions.

We assume $1 < p < \infty$. Let $\mathcal{A}$ be Følner and tempered. For each Lamperti L^p-representation of G and for each $\chi \in \hat{G}$ the mapping $x \mapsto \chi(x)T_x$ is also a Lamperti L^p-representation. Therefore $c_{f,T}(\chi) := \lim_{n \to \infty} \frac{1}{\mu(A_n)} \int_{A_n} \chi^{-1}(x) T_x f(\omega)\mu(dx)$ is defined m-a.e. for every $f \in L^p(\Omega, \mathcal{F}, m)$, by Theorem 3.9. We denote $\sigma(u) := \{\chi \in \hat{G} : c_u(\chi) \neq 0\}; \sigma(f, T) := \{\chi \in \hat{G} : c_{f,T}(\chi) \neq 0\}$.

Theorem 5.1. *Let T be a Lamperti L^p-representation of G, and let $\mathcal{A} = \{A_n\}$ be a Følner tempered sequence.*

(i) Let $1 < p < \infty, q = \frac{p}{p-1}$. If $f \in L^p(\Omega)$, $u \in AP$ and the set $\sigma(u)$ is finite, then

$$M_T(f; u) := \lim_{n \to \infty} \frac{1}{\mu(A_n)} \int_{A_n} T_x f(\omega) u(x) \mu(dx) \ \text{exists } m\text{-a.e.} \tag{5.1}$$

and

$$M_T(f, \bar{u}) = \lim_{n \to \infty} \frac{1}{\mu(A_n)} \int_{A_n} T_x f(\omega)\overline{u(x)}\mu(dx) = \sum_{\chi \in \sigma(u) \cap \sigma(f,T)} c_{f,T}(\chi)\overline{c_u(\chi)} \ \ m\text{-a.e.} \tag{5.2}$$

[5] Recall that this is the case when N is countable or when the net $\mathcal{A}$ is increasing.

(ii) Assume that $1 < p < \infty, q = \frac{p}{p-1}, f \in L^p(\Omega)$ and $u \in B^{q_1}(\mathcal{A})$ where $q_1 > q$. Then assertion (5.1) *holds and*

$$||M_T(f;u)||_p \leq ||T||_p||f||_p||u||_{q_1}. \tag{5.3}$$

(iii) Assume that T is a unitary Lamperti L^2-representation (e.g. T is generated by a measure preserving action τ in $(\Omega, \mathcal{F}, m)$), $f \in L^2(\Omega), u \in B^{q_1}(G)$ where $q_1 > 2$; then (5.1) *and* (5.2) *hold (the series converges in $L^2(\Omega)$).*

Proof. (i) Assume $\sigma(u) \cap \sigma(f,T) = \{\chi_1, ..., \chi_k\}$ and consider the AP-function $u(x) = \sum_{l=1}^k c_u(\chi_l)\chi_l(x)$. We have:

$$M_T(f;\bar{u}) = \lim_{n\to\infty} \frac{1}{\mu(A_n)} \int_{A_n} \overline{u(x)} T_x f(\omega)\mu(dx)$$
$$= \sum_{l=1}^k \overline{c_u(\chi_l)} \lim_{n\to\infty} \frac{1}{\mu(A_n)} \int_{A_n} \overline{\chi_l(x)} T_x f \mu(dx) = \sum_{l=1}^k \overline{c_u(\chi_l)} c_{f,T}(\chi_l)$$
$$= \sum_{\chi \in \sigma(u)\cap\sigma(f,T)} \overline{c_u(\chi)} c_{f,T}(\chi) \quad m\text{-a.e.}$$

Thus statement (i) is proved.

(ii) We have to consider the case when the set $\sigma(u)$ is infinite. Let $p_1 = \frac{q_1}{q_1-1}$, $\sigma(u) = \{\chi_1, \chi_2, ...\}$. Denote:

$$u_k(x) = \sum_{l=1}^k c_l(u)\chi_l(x);$$
$$a_n(\omega) = \frac{1}{\mu(A_n)} \int_{A_n} T_x f(\omega) u(x)\mu(dx);$$
$$a_{k,n}(\omega) = \frac{1}{\mu(A_n)} \int_{A_n} T_x f(\omega) u_k(x)\mu(dx);$$
$$\hat{a}_k = M_T(f;u_k).$$

By virtue of statement (i), $\lim_{n\to\infty} a_{k,n} = \hat{a}_k, k = 1, 2, ...$ By the Hölder inequality,

$$|a_n - a_{k,n}| \leq [\frac{1}{\mu(A_n)} \int_{A_n} |T_x f(\omega)|^{p_1} \mu(dx)]^{\frac{1}{p_1}} [\frac{1}{\mu(A_n)} \int_{A_n} |(u-u_k)|^{q_1} \mu(dx)]^{\frac{1}{q_1}},$$

hence, by Corollary 3.6, $\lim_{k\to\infty} \limsup_{n\to\infty} |a_n - a_{k,n}| = 0$. The triangle inequality implies: $\lim_{k\to\infty} \limsup_{n\to\infty} |a_n - \hat{a}_k| = 0$ and $\limsup_{m,n\to\infty} |a_m - a_n| \leq 2\lim_{k\to\infty} \limsup_{n\to\infty} |a_n - \hat{a}_k| = 0$. So $M_T(f;u) = \lim_{n\to\infty} a_n(\omega)$ exists m-a.e. By the Hölder inequality, $\int_\Omega |\frac{1}{\mu(A_n)} \int_{A_n} T_x f(\omega)u(x)\mu(dx)|^p m(d\omega) \leq ||T||_p^p ||f||_p^p ||u||_{q_1}^p$ and by the Modulated Mean ET (see Theorem 2.6(iii) in [19]),

$$\lim_{n\to\infty} ||\frac{1}{\mu(A_n)} \int_{A_n} T_x f(\omega)u(x)\mu(dx)||_p = ||M_T(f;u)||_p.$$

These two relations imply (5.3).

(iii) For each $\chi_0 \in \hat{G}$ the subspace $I^{(\chi_0)} \subset L^2(\Omega)$ of functions invariant with respect to the representation $T^{(\chi_0)} : x \mapsto \overline{\chi_0(x)} T_x$ coincides with the space of all $\chi_0(x)$-eigenfunctions with respect to each operator T_x. It is easy to check that all such (non-trivial) subspaces are pairwise orthogonal, that the mean value operator $\mathbf{M}^{(\chi_0)}$ is the orthogonal projector onto $I^{(\chi_0)}$ and that $c_{f,T}(\chi_0) = \mathbf{M}^{(\chi_0)} f$. Therefore statement (iii) follows from Proposition 5.1 in [19]. □

Remark 5.2. If the representation T is generated by a measure preserving action, then Theorem 5.1 is valid also when $p \geq 1, q_1 = q$ (see [17], [32] and §3 in [19] for more results).

Remark 5.3. Let $\mathcal{A}$ be a Følner sequence, $p \geq 1$. For any bounded representation T in $L^p(\Omega)$, the limit (5.1) exists in the sense of L^p-convergence for each $f \in L^p(\Omega)$ and each $u \in B^1(\mathcal{A})$ (see [19, Theorem 2.6]) and statement (iii) is valid for each unitary representation in $L^2(\Omega)$ (see [19, Proposition 5.1]).

6. PETs for multiplicative cocycles over group actions

Let τ be a non-singular left G-action in $(\Omega, \mathcal{F}, m)$. Recall our definition of the measure $\tilde{m}_x^- : \tilde{m}_x^-(\Lambda) = m(\tau_x^{-1}\Lambda), \Lambda \in \mathcal{F}$; we will also consider the measure $\tilde{m}_x^+ : \tilde{m}_x^+(\Lambda) = m(\tau_x \Lambda)$. We say that a $\mathcal{B} \times \mathcal{F}$-measurable function $g : (x, \omega) \mapsto g(x, \omega)$ is a right *cocycle over the action* τ if for each $x \in G \quad g(x, \omega) \neq 0$ for m-almost all $\omega \in \Omega$ and for each $x_1, x_2 \in G$, $g(x_1 x_2, \omega) = g(x_1, \tau_{x_2}\omega) g(x_2, \omega)$ m-a.e. and $g(e, \omega) = 1$ m-a.e. Let $p \geq 1$. A cocycle g over τ is L^p-*bounded* if

$$||g||_p := \sup_{x \in G} \{ \operatorname{vrai} \sup_{\omega \in \Omega} |g(x, \tau_x^{-1}\omega)| [\frac{d\tilde{m}_x^-}{dm}(\omega)]^{\frac{1}{p}} \} < \infty.$$

Example 6.1. The function $g_D(x, \omega) := [\frac{d\tilde{m}_x^+}{dm}(\omega)]^{\frac{1}{p}}$ is a right cocycle: for each pair $x_1, x_2 \in G$ we have m-a.e.

$$\frac{d\tilde{m}_{x_1 x_2}^+}{dm}(\omega) = \frac{d\tilde{m}_{x_1 x_2}^+}{d\tilde{m}_{x_2}^+}(\omega) \frac{d\tilde{m}_{x_2}^+}{dm}(\omega) = \frac{d\tilde{m}_{x_1}^+}{dm}(\tau_{x_2}\omega) \frac{d\tilde{m}_{x_2}^+}{dm}(\omega),$$

$g_D(e, \omega) = 1$ a.e. and $||g_D||_p = 1$ for each $p \geq 1$ since $\frac{d\tilde{m}_x^+}{dm}(\tau_x^{-1}\omega) = [\frac{d\tilde{m}_x^-}{dm}(\omega)]^{-1}$.

Example 6.2. For each $\mathcal{F}$-measurable function $\omega \mapsto h(\omega)$, such that $h(\omega) \neq 0$ m-a.e., the associated multiplicative *coboundary over* $\tau \quad g_h(x, \omega) := \frac{h(\tau_x \omega)}{h(\omega)}$ is a right cocycle over τ.

Proposition 6.3. *If $p \geq 1$ and g is an L^p-bounded right cocycle over τ, then $T^{(g)} : x \mapsto f T_x^{(g)}(\omega) := g(x, \omega) f(\tau_x \omega)$ is a Lamperti right representation of G in $L^p(\Omega)$; moreover, $||T^{(g)}||_p = ||g||_p$.*

Proof. First $T_e = I$. We have m-a.e.:

$$\begin{aligned} f T_{x_1}^{(g)} T_{x_2}^{(g)}(\omega) &= (g(x_1, \omega) f(\tau_{x_1}\omega)) T_{x_2}^{(g)} = g(x_2, \omega) g(x_1, \tau_{x_2}\omega) f(\tau_{x_1}\tau_{x_2}\omega) \\ &= g(x_1 x_2, \omega) f(\tau_{x_1 x_2}\omega) = f T_{x_1 x_2}^{(g)}(\omega). \end{aligned}$$

Thus $T^{(g)}$ is a right representation of G; it is clear that it is Lamperti and that it is measurable as well. At last,

$$\int_\Omega |f T_x^{(g)}|^p dm = \int_\Omega |g(x, \omega)|^p |f(\tau_x \omega)|^p m(d\omega) = \int_\Omega |g(x, \tau_x^{-1}\omega)|^p \frac{dm_x^-}{dm} |f(\omega)|^p m(d\omega)$$

for each $f \in L^p(\Omega)$; since f^p runs over $L^1(\Omega)$ as f runs over $L^p(\Omega)$, $||T_x^{(g)}||_p = [\operatorname{vrai} \sup_\omega |g(x, \tau_x^{-1}\omega)|^p \frac{d\tilde{m}_x^-}{dm}(\omega)]^{\frac{1}{p}} = ||g||_p$. □

Theorem 3.9 and Proposition 6.3 imply the following statement.

Theorem 6.4. *Let $\{A_n\}$ be a Følner tempered sequence and let g be a bounded cocycle over τ. Then for each $f \in L^p(\Omega)\,, p > 1$, the following limit exists m-a.e.:*

$$\lim_{n\to\infty} \frac{1}{\mu(A_n)} \int_{A_n} g(x,\omega) f(\tau_x \omega)\mu(dx) = f\mathbf{M}^{(p)}_{T^{(g)}}(\omega).$$

In particular, if $m(\Omega) < \infty$, we put $f = 1$ m-a.e. and obtain: the mean value

$$M(g)(\omega) := \lim_{n\to\infty} \frac{1}{\mu(A_n)} \int_{A_n} g(x,\omega)\mu(dx) = \mathbf{1}\mathbf{M}^{(p)}_{T^{(g)}}(\omega)$$

exists m-a.e.

Remark 6.5. By the properties of the operator $\mathbf{M}^{(p)}_{T^{(g)}}$ (see Subsection 3.2), $||M(g)||_p \leq ||g||_p (m(\Omega))^{\frac{1}{p}}$.

We conclude with the modulated PET for cocycles which is a simple corollary of Proposition 6.3 and Theorem 5.1.

Theorem 6.6. *Assume that the group G is commutative, $\mathcal{A} = \{A_n\}$ is a Følner tempered sequence in G, g is a bounded cocycle over τ and that $u \in B^{q_1}(\mathcal{A})$ where $q_1 > q$. Then m-a.e. the following limit exists:*

$$\lim_{n\to\infty} \frac{1}{\mu(A_n)} \int_{A_n} g(x,\omega)u(x) f(\tau_x \omega)\mu(dx).$$

Remark 6.7. It is easy to verify that the results of §5 and in §6 are valid for Følner regular cofinally separable nets.

7. Appendix

Examples: Følner and regular sequences and nets

We state here some well-known examples (see [31]–[33] for proofs of the statements in Examples 7.1-7.3).

Example 7.1. Each increasing net $\mathcal{A} = \{A_n, n \in N\}$ of bounded convex bodies in $\mathbb{R}^d$ is regular; $r(\mathcal{A}) \leq \binom{2d}{d}; r(\mathcal{A}) = 2^d$ if all A_n are symmetric. If the intrinsic diameters $D_n \to \infty$, $\mathcal{A}$ is also Følner (see [6]). If $\{A_n, n \in N\}$ is an increasing net of bounded convex bodies in $\mathbb{R}^d$ with $D_n \to \infty$, then the sets $A_n \cap \mathbb{Z}^d$ form a Følner regular net in $\mathbb{Z}^d$.

Example 7.2. If A is a compact star-shaped set in $\mathbb{R}^d, \mu(A) > 0$, and $\{t_n, n \in N\}$ is an increasing net of positive numbers converging to ∞, then the net of homothetic sets $\mathcal{A} = \{t_n A, n \in N\}$ is Følner and $r(\mathcal{A}) = \frac{\mu(t_n A - t_n A)}{\mu(t_n A)} = \frac{\mu(t_n(A-A))}{\mu(t_n A)} = \frac{\mu(A-A)}{\mu(A)} < \infty$, so $\{t_n A, n \in N\}$ is regular, too.

Example 7.3. Each compactly generated commutative l.c. group possesses regular Følner sequences. However such sequences are absent in countable groups with $\sup\{n : \mathbb{Z}^n$ is isomorphic to a subgroup of $G\} = \infty$ (see [12]).

Example 7.4. R. Tessera [34] has proved that, if U is a symmetric compact generating set in an l.c. group of polynomial growth, then the sequence $\{U^n\}$ is regular and Følner.

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