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THE SMITH NORMAL FORM OF A MATRIX ASSOCIATED WITH YOUNG'S LATTICE

TOMMY WUXING CAI AND RICHARD P. STANLEY

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Abstract. We prove a conjecture of Miller and Reiner on the Smith normal form of the operator DU associated with a differential poset for the special case of Young's lattice. Equivalently, this operator can be described as $\frac{\partial}{\partial p_1} p_1$ acting on homogeneous symmetric functions of degree n.

1. Introduction

Let R be a commutative ring with 1 and M an $m \times m$ matrix over R. We say that M has a *Smith normal form* (SNF) over R if there exist matrices $P, Q \in \mathrm{SL}(m, R)$ (so $\det P = \det Q = 1$) such that PMQ is a diagonal matrix $\mathrm{diag}(d_1, d_2, \ldots, d_m)$ with $d_i \mid d_{i+1}$ $(1 \le i \le m-1)$ in R. It is well-known that if R is an integral domain and if the SNF of M exists, then it is unique up to multiplication of each d_i by a unit u_i (with $u_1 \cdots u_m = 1$).

We see that Smith normal form is a refinement of the determinant, since $\det M = d_1 d_2 \cdots d_m$. In the case that R is a principal ideal domain (PID), it is well known that all matrices in R have a Smith normal form. Not very much is known in general.

In this work we are interested in the ring $\mathbb{Z}[x]$ of integer polynomials in one variable. Since $\mathbb{Z}[x]$ is not a PID, a matrix over this ring need not have an SNF. For instance, it can be shown that the matrix $\begin{bmatrix} x & 0 \\ 0 & x+2 \end{bmatrix}$ does not have a Smith normal form over $\mathbb{Z}[x]$.

We will use symmetric function notation and terminology from [5]. The ring Λ of symmetric functions has several standard $\mathbb{Z}$-bases: monomial symmetric functions m_λ, elementary symmetric functions e_λ, complete symmetric functions h_λ and Schur functions s_λ. The power sum symmetric functions p_λ form a $\mathbb{Q}$-basis of $\Lambda_{\mathbb{Q}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$. The ring $\Lambda_{\mathbb{Q}}$ is a graded $\mathbb{Q}$-algebra: $\Lambda_{\mathbb{Q}} = \bigoplus_{n=0}^{\infty} \Lambda_{\mathbb{Q}}^n$, where $\Lambda_{\mathbb{Q}}^n$ is the vector space spanned over $\mathbb{Q}$ by $\{s_\lambda : \lambda \vdash n\}$. Similarly, Λ is a graded ring.

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Regard elements of $\Lambda^n_{\mathbb{Q}}$ as polynomials in the p_j's. Define a linear map T_1 on $\Lambda^n_{\mathbb{Q}}$ by

$$T_1(v) = \frac{\partial}{\partial p_1}(p_1 v), \quad v \in \Lambda^n_{\mathbb{Q}}. \tag{1.1}$$

Remark 1.1. Note that the power sum p_λ is an eigenvector of $T_1(v)$ with corresponding eigenvalue $1 + m_1(\lambda)$, where $m_1(\lambda)$ denotes the number of 1's in λ.

Denote by $M = M_1^{(n)}$ the matrix of T_1 with respect to the basis $\{s_\lambda : \lambda \vdash n\}$ (arranged in, say, lexicographic order). It is known and easy to see that M is an integer symmetric matrix of order $p(n)$, the number of partitions of n. Let $\lambda(n) = (p(n)-p(n-1), \ldots, p(2)-p(1), p(1))$, so $\lambda(n)$ is a partition of $p(n)$. Let the conjugate of $\lambda(n)$ be $\lambda(n)' = (j_{p(n)-p(n-1)}, \ldots, j_2, j_1)$ (so $j_{p(n)-p(n-1)} = n$). We prove the following result.

Theorem 1.2. *Let $\alpha_k(x) = a_1(x)a_2(x)\cdots a_k(x)$ with $a_i(x) = i+x$ ($i = 1, 2, \ldots, n-1$) and $a_n(x) = n+1+x$. There exist $P(x), Q(x) \in \mathrm{SL}(p(n), \mathbb{Z}[x])$ such that $P(x)(M + xI_{p(n)})Q(x)$ is the following diagonal matrix:*

$$\mathrm{diag}(1, 1, \ldots, 1, \alpha_{j_1}(x), \alpha_{j_2}(x), \ldots, \alpha_{j_{p(n)-p(n-1)}}(x)). \tag{1.2}$$

As an example of Theorem 1.2, let us consider the case $n = 6$. First $\lambda(6) = (4, 2, 2, 1, 1, 1)$ and $\lambda(6)' = (6, 3, 1, 1)$. Hence the diagonal entries of an SNF of $M + xI$ are seven 1's followed by

$$1+x, 1+x, (1+x)(2+x)(3+x), (1+x)(2+x)(3+x)(4+x)(5+x)(7+x).$$

In general, the number of diagonal entries equal to 1 is $p(n-1)$.

The origin of Theorem 1.2 is as follows. Let P be a differential poset, as defined in [3] or [4, §3.21], with levels $P_0, P_1, \ldots$. Let $\mathbb{Z}[x]P_n$ denote the free $\mathbb{Z}[x]$-module with basis P_n. Let U, D be the up and down operators associated with P. Miller and Reiner [2] conjectured a certain Smith normal form of the operator $DU + xI\colon \mathbb{Z}[x]P_n \to \mathbb{Z}[x]P_n$. Our result is equivalent to the conjecture of Miller and Reiner for the special case of Young's lattice Y. Our result also specializes to a proof of Miller's Conjecture 14 in [1]. After proving the theorem, we state a conjecture which generalizes it. It seems natural to try to generalize our work to the differential poset Y^r for $r \geq 2$, but we have been unable to do this.

2. The proof of the theorem

Instead of Schur functions, we consider the matrix with respect to the complete symmetric functions $\{h_\lambda\colon \lambda \vdash n\}$. Since $\{s_\lambda\colon \lambda \vdash n\}$ and $\{h_\lambda\colon \lambda \vdash n\}$ are both $\mathbb{Z}$-bases for Λ^n, the Smith normal form does not change when we switch to the h_λ basis. We introduce a new ordering on the set $\mathcal{P}_n$ of all partitions of n. The matrix A with respect to this new ordering turns out to be much easier to manipulate than the original matrix. In fact, we show that $A + xI_{p(n)}$ can be turned into an upper triangular matrix after some simple row operations. Then we use more row/column operations to cancel the nondiagonal elements. The resulting diagonal matrix is the SNF that we are looking for.

From now on, we fix a positive integer n. The case $n = 1$ is trivial, so we assume $n \geq 2$.

2.1. **A new ordering on partitions.**

Definition 2.1. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_i, 1)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_i \geq 1$ $(i \geq 1)$. We define $\lambda^+ = (\lambda_1 + 1, \lambda_2, \ldots, \lambda_i)$ and write $\lambda^+ \nwarrow \lambda$. We call a partition λ *initial* if there is no μ such that $\mu^+ = \lambda$, i.e., $\lambda_1 = \lambda_2$. We call a partition μ *terminal* if μ^+ is not well defined, i.e., $m_1(\mu) = 0$, where $m_i(\lambda)$ denotes the number of parts of λ equal to i.

For a sequence $\lambda^0, \lambda^1, \lambda^2, \ldots, \lambda^t$ of partitions, we write

$$\lambda^t \nwarrow \lambda^{t-1} \nwarrow \cdots \nwarrow \lambda^0 \tag{2.1}$$

and call it a *rising string* of length t if $\lambda^{i+1} \nwarrow \lambda^i$, i.e., $\lambda^{i+1} = (\lambda^i)^+$, for $0 \leq i \leq t-1$.

In (2.1), if λ^0 is initial and λ^t is terminal, then we call (2.1) a *full* (rising) string. Moreover, we say that λ^0 is the initial element of the string and λ^t is the terminal element of the string. One cannot add a partition to a full string to make it longer.

A partition $\mu = (\mu_1, \mu_2, \ldots, \mu_i)$ with $\mu_1 = \mu_2 \geq \cdots \geq \mu_i \geq 2$ is not in any rising string of nonzero length. We also call this μ a (point) string of length 0; it is both initial and terminal.

Note that every partition μ is in exactly one full string; we denote the terminal element of this string by $T(\mu)$. Thus $T(\lambda) = T(\mu)$ if and only if λ, μ are in the same string.

Definition 2.2. We define a total ordering $\preceq$ on the set of partitions of n as follows:

(1) For two partitions in the rising string (2.1), we define $\lambda^i \preceq \lambda^j$ for $i \leq j$.
(2) For λ, μ in different full strings, we write $\lambda \preceq \mu$ if $T(\lambda) \leq_L T(\mu)$, i.e., $T(\lambda)$ is lexicographically less than $T(\mu)$.

We arrange the partitions in $\mathcal{P}_n$ from largest to smallest according to the following order:

$$\lambda^{11}, \lambda^{12}, \ldots, \lambda^{1i_1}; \lambda^{21}, \lambda^{22}, \ldots, \lambda^{2i_2}; \ldots; \lambda^{t1}, \lambda^{t2}, \ldots, \lambda^{ti_t}, \tag{2.2}$$

where $\lambda^{j1} \nwarrow \lambda^{j2} \nwarrow \cdots \nwarrow \lambda^{ji_j}$ is the jth full string of partitions of n, and we use semicolons to separate neighboring strings. It's easy to see that $\lambda^{11} = (n)$, $i_1 = n$, $\lambda^{1i_1} = (1^n)$, and t is the number of terminal elements of n, which is equal to the number of partitions of n with no part equal to 1, viz., $p(n) - p(n-1)$. In fact, these cardinalities $i_1, i_2, \ldots, i_t$ of strings can be expressed explicitly. We will see this point after the following example.

Example 2.3. The following is the list of partitions of 6:

$$6,\ 51,\ 41^2,\ 31^3,\ 21^4,\ 1^6;\ 42,\ 321,\ 2^21^2;\ 3^2;\ 2^3.$$

The eigenvalues of M arranged in accordance with this order are by Remark 1.1 as follows:

$$1, 2, 3, 4, 5, 7; 1, 2, 3; 1; 1.$$

On the other hand, we can arrange the eigenvalues of M in the following form:

$$\begin{aligned} p(6)-p(5) &= 4:1\ \ 1\ \ 1\ \ 1, \\ p(5)-p(4) &= 2:2\ \ 2, \\ p(4)-p(3) &= 2:3\ \ 3, \\ p(3)-p(2) &= 1:4, \\ p(2)-p(1) &= 1:5, \\ p(1) &= 1:7. \end{aligned}$$

We see that the eigenvalues (to the right side of the colons) form a tableau (with constant rows and increasing columns) of shape $\lambda(6) = (4,2,2,1,1,1)$.

Notice that the eigenvalues associated with the first string are 1,2,3,4,5, and 7 and that they form the first column of the above tableau. In fact, the eigenvalues corresponding to every string form a column of the tableau. This is not a coincidence, but rather because the eigenvalues corresponding to a string form a sequence of consecutive integers starting from 1: $1,2,3,\dots,i$, except there is a gap for those corresponding to the first string. Therefore the cardinality of a string is the length of a column of the tableau.

We can easily formalize this argument to prove the following.

Lemma 2.4. *Rearrange the cardinalities of the strings $i_1, i_2, \dots, i_t$ in (2.2) in weakly decreasing order: $J = (j_t, \dots, j_2, j_1)$. Then J is exactly the conjugate of the partition $\lambda(n) = (p(n)-p(n-1), \dots, p(2)-p(1), p(1))$ as defined in the introduction.*

Note that $i_1, i_2, \dots, i_t$ are not necessarily in weakly decreasing order.

2.2. **The transition matrix with respect to the new ordering.** Let $A = (a_{\mu\lambda})$ be the $p(n) \times p(n)$ matrix of the action of $\frac{\partial}{\partial p_1} p_1$ on the basis h_λ, i.e., $\frac{\partial}{\partial p_1} p_1 \cdot h_\lambda = \sum_\mu a_{\mu\lambda} h_\mu$. Here we use the total ordering $\preceq$, with the greatest partition (n) corresponding to the first row and column. Recall that the notation $\frac{\partial}{\partial p_1} v$ means that we write v as a polynomial in the power sums $p_1, p_2, \dots$ and regard each p_i, $i \geq 2$, as a constant when we differentiate.

Lemma 2.5. *The matrix $A = (a_{\mu\lambda})$ has the following properties:*

(1) *$a_{\mu\lambda} \neq 0$ only if $\lambda \nwarrow \mu$, $\mu = \lambda$ or $T(\mu) > T(\lambda)$ (greater but not equal in dominance order).*
(2) *$a_{\mu\lambda} = 1$ if $\lambda \nwarrow \mu$;*
$a_{\lambda\lambda} = m_1(\lambda) + 1$;
$m_1(\mu)$ equals $m_1(\lambda) + 1$ or $m_1(\lambda) + 2$ if $a_{\mu\lambda} \neq 0$ and $T(\mu) > T(\lambda)$.

Proof. Let $h_\lambda = h_{\lambda_1} \cdots h_{\lambda_k} \cdots h_{\lambda_i} h_1^j$, with $\lambda_1 \geq \cdots \geq \lambda_i \geq 2$. From e.g. the basic identity

$$\sum_{m \geq 0} h_m t^m = \exp \sum_{i \geq 1} p_i \frac{t^i}{i}$$

it follows that $\frac{\partial h_m}{\partial p_1} = h_{m-1}$. Then (since $h_1 = p_1$)

$$\begin{aligned}\frac{\partial}{\partial p_1} p_1 \cdot h_\lambda &= \frac{\partial}{\partial p_1} h_{\lambda_1} \cdots h_{\lambda_k} \cdots h_{\lambda_i} h_1^{j+1} \\ &= (j+1) h_\lambda + \sum_{k=1}^{i} h_{\lambda_1} \cdots h_{\lambda_k - 1} \cdots h_{\lambda_i} h_1^{j+1}.\end{aligned}$$

We see that $a_{\lambda\lambda} = m_1(\lambda) + 1$. Let μ be the partition such that

$$h_{\lambda_1} \cdots h_{\lambda_k - 1} \cdots h_{\lambda_i} h_1^{j+1} = h_\mu, \quad 1 \le k \le i.$$

In the case that $k = 1$ and λ is not an initial element, i.e., $\lambda_1 > \lambda_2$, then $\mu = (\lambda_1 - 1, \lambda_2, \ldots, \lambda_i, 1^{j+1})$, and thus $\lambda \nwarrow \mu$.

In the other cases, μ is of the form $\mu = (\lambda_1, \lambda_2, \ldots, \lambda_r - 1, \ldots, \lambda_i, 1^{j+1})$ for some $1 < r \le i$. Hence $T(\mu) = (\lambda_1 + j + 1, \lambda_2, \ldots, \lambda_r - 1, \ldots, \lambda_i)$ or $T(\mu) = (\lambda_1 + j + 2, \lambda_2, \ldots, \lambda_{i-1})$ (when $i = r$ and $\lambda_r = 2$). Note that $T(\lambda) = (\lambda_1 + j, \lambda_2, \ldots, \lambda_i)$. We see that $T(\mu) > T(\lambda)$ (in dominance order). This proves (1) and (2). $\square$

In the following we set $a'_i = i$ for $i = 1, 2, \ldots, n-1$ and $a'_n = n+1$. We separate rows and columns corresponding to different strings and write A in the block matrix form $A = (A_{kl})_{t \times t}$. We have the following properties of these A_{kl}'s by Lemma 2.5:

(1) For $k > l$, $A_{kl} = 0$.
(2) A_{kk} is an $i_k \times i_k$ lower triangular matrix. Its diagonal entries are $a'_1, a'_2, \ldots, a'_{i_k}$; the entries on the line right below and parallel to the diagonal are all 1, and all the other entries are 0.

It looks as follows:

$$A = \begin{bmatrix}
a'_1 & & & & b^{12}_{11} & b^{12}_{12} & \cdots & b^{12}_{1i_2} & \cdots & b^{1t}_{11} & b^{1t}_{12} & \cdots & b^{1t}_{1i_t} \\
1 & a'_2 & & & b^{12}_{21} & b^{12}_{22} & \cdots & b^{12}_{2i_2} & \cdots & b^{1t}_{21} & b^{1t}_{22} & \cdots & b^{1t}_{2i_t} \\
 & \ddots & \ddots & & \cdots & & & & & \cdots & & & \\
 & & 1 & a'_{i_1} & b^{12}_{i_1 1} & b^{12}_{i_1 2} & \cdots & b^{12}_{i_1 i_2} & \cdots & b^{1t}_{i_1 1} & b^{1t}_{i_1 2} & \cdots & b^{1t}_{i_1 i_t} \\
 & & & & a'_1 & & & & \cdots & b^{2t}_{11} & b^{2t}_{12} & \cdots & b^{2t}_{1i_t} \\
 & & & & 1 & a'_2 & & & \cdots & b^{2t}_{21} & b^{2t}_{22} & \cdots & b^{2t}_{2i_t} \\
 & & & & & \ddots & \ddots & & & \cdots & & & \\
 & & & & & & 1 & a'_{i_2} & \cdots & b^{2t}_{i_2 1} & b^{2t}_{i_2 2} & \cdots & b^{2t}_{i_2 i_t} \\
 & & & & & & & & \ddots & & & & \\
 & & & & & & & & & a'_1 & & & \\
 & & & & & & & & & 1 & a'_2 & & \\
 & & & & & & & & & & \ddots & \ddots & \\
 & & & & & & & & & & & 1 & a'_{i_t}
\end{bmatrix}.$$

Furthermore, we have

$$b^{kl}_{ij} = 0 \text{ if } i \le j, \tag{2.3}$$

i.e., A_{kl} is a strict lower triangular matrix if $k < l$. The reason is that $b^{kl}_{ij} = a_{\lambda^{ki}\lambda^{lj}}$, and if it is nonzero, then we have the following by Lemma 2.5:

$$i - 1 = m_1(\lambda^{ki}) \ge m_1(\lambda^{lj}) + 1 = j - 1 + 1 = j.$$

(Here we use that $m_1(\lambda^{pr}) = r - 1$ for $\lambda^{pr} \neq (1^n)$. It is possible that $\lambda^{ki} = (1^n)$, but then again $i = n \geq i_l + 1 \geq j + 1$.)

Now we replace the a_i' on the diagonal of A with an arbitrary $f_i = f_i(x) \in \mathbb{Z}[x]$ and change A into a matrix $A(x)$ with entries in $\mathbb{Z}[x]$. We will apply some row/column operations to $A(x)$ and transform it into an SNF in $\mathbb{Z}[x]$. (Some of these operations depend on the f_i's.) This is the same as saying that there are $P_1(x), Q_1(x) \in \mathrm{SL}(p(n), \mathbb{Z}[x])$ such that $P_1(x)A(x)Q_1(x)$ is an SNF in $\mathbb{Z}[x]$.

Notice that the original matrix M is equal to PAP^{-1} for some $P \in \mathrm{SL}(p(n), \mathbb{Z})$. If we take $f_i = a_i' + x$ (which is $a_i(x)$ in our theorem) in the beginning, then $A(x) = A + xI_{p(n)}$, and thus the SNF $P_1(x)A(x)Q_1(x)$ is equal to

$$P_1(x)(P^{-1}MP + xI_{p(n)})Q_1(x) = P_1(x)P^{-1}(M + xI_{p(n)})PQ_1(x),$$

as desired.

2.3. **Transformation into an upper triangular matrix.** If we use horizontal lines to separate rows of $A(x)$ corresponding to different full strings, then $A(x)$ is partitioned into t submatrices. We see that we should consider a matrix of the following form:

$$B = \begin{bmatrix} f_1 & & & & b_{11} & b_{12} & \cdots & b_{1m} \\ 1 & f_2 & & & b_{21} & b_{22} & \cdots & b_{2m} \\ & \ddots & \ddots & & \cdots & & & \\ & & 1 & f_s & b_{s1} & b_{s2} & \cdots & b_{sm} \end{bmatrix}.$$

We can apply row operations to B and transform it first into

$$B_1 = \begin{bmatrix} 1 & f_2 & & & b_{21} & b_{22} & \cdots & b_{2m} \\ & \ddots & \ddots & & \cdots & & & \\ & & 1 & f_s & b_{s1} & b_{s2} & \cdots & b_{sm} \\ f_1 & & & & b_{11} & b_{12} & \cdots & b_{1m} \end{bmatrix},$$

and then into

$$B_2 = \begin{bmatrix} 1 & f_2 & & & b_{21} & b_{22} & \cdots & b_{2m} \\ & \ddots & \ddots & & \cdots & & & \\ & & 1 & f_s & b_{s1} & b_{s2} & \cdots & b_{sm} \\ 0 & \cdots & 0 & \alpha & \beta_1 & \beta_2 & \cdots & \beta_m \end{bmatrix},$$

with $\alpha = f_1 \cdots f_s$ and

$$(-1)^{s-1}\beta_j = b_{1j} - f_1 b_{2j} + f_1 f_2 b_{3j} + \cdots + (-1)^{s-1} f_1 \cdots f_{s-1} b_{sj}.$$

Apply this process to the t submatrices of $A(x)$. We turn $A(x)$ into the matrix

$$A_1(x) = \begin{bmatrix}
1 & f_2 & & & b_{21}^{12} & b_{22}^{12} & \cdots & b_{2i_2}^{12} & \cdots & b_{21}^{1t} & b_{22}^{1t} & \cdots & b_{2i_t}^{1t} \\
 & \ddots & \ddots & & \cdots & & & & & \cdots & & & \\
 & & 1 & f_{i_1} & b_{i_11}^{12} & b_{i_12}^{12} & \cdots & b_{i_1i_2}^{12} & \cdots & b_{i_11}^{1t} & b_{i_12}^{1t} & \cdots & b_{i_1i_t}^{1t} \\
 & & & \alpha_{i_1} & \beta_1^{12} & \beta_2^{12} & \cdots & \beta_{i_2}^{12} & \cdots & \beta_1^{1t} & \beta_2^{1t} & \cdots & \beta_{i_t}^{1t} \\
 & & & & 1 & f_2 & & & \cdots & b_{21}^{2t} & b_{22}^{2t} & \cdots & b_{2i_t}^{2t} \\
 & & & & & \ddots & \ddots & & & \cdots & & & \\
 & & & & & & 1 & f_{i_2} & \cdots & b_{i_21}^{2t} & b_{i_22}^{2t} & \cdots & b_{i_2i_t}^{2t} \\
 & & & & & & & \alpha_{i_2} & \cdots & \beta_1^{2t} & \beta_2^{2t} & \cdots & \beta_{i_t}^{2t} \\
 & & & & & & & & \ddots & & & & \\
 & & & & & & & & & 1 & f_2 & & \\
 & & & & & & & & & & \ddots & \ddots & \\
 & & & & & & & & & & & 1 & f_{i_t} \\
 & & & & & & & & & & & & \alpha_{i_t}
\end{bmatrix},$$

where $\alpha_k = f_1 f_2 \cdots f_k$ and

$$(-1)^{i_k-1}\beta_j^{kl} = b_{1j}^{kl} - f_1 b_{2j}^{kl} + f_1 f_2 b_{3j}^{kl} + \cdots + (-1)^{i_k-1} f_1 f_2 \cdots f_{i_k-1} b_{i_kj}^{kl}.$$

Recalling that $b_{ij}^{kl} = 0$ for $i \le j$ (see (2.3)), we find that

$$f_1 f_2 \cdots f_j \mid \beta_j^{kl}, \text{ and as a special case, } \alpha_{i_l} \mid \beta_{i_l}^{kl}. \tag{2.4}$$

This property is crucial for later cancellation.

Remark 2.6. Next we will show that we can cancel the nondiagonal entries without altering the diagonal. Then by the definition of α_k, we see that the matrix $A(x)$ has the Smith normal form

$$\operatorname{diag}(1, 1, \ldots, 1, \alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_t}),$$

where $(j_1, \ldots, j_t)$ is the rearrangement of $i_1, \ldots, i_t$ in weakly increasing order. Combining Lemma 2.4, everything in our theorem is now clear.

2.4. **The cancellation of the nondiagonal entries.** Now we want to cancel the nondiagonal elements, completing the proof. For those nonzero elements above the diagonal, we can do the following:

(C1) First apply column operations to cancel the entries on the rows with diagonal elements equal to 1 (starting from the first row).
(C2) Then apply row operations to cancel the entries on the columns with diagonal elements equal to 1.

The matrix turns into the following:

$$A_2(x) = \begin{bmatrix}
1 & & & & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
 & \ddots & \ddots & & \cdots & & & & & \cdots & & & \\
 & & 1 & & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
 & & & \alpha_{i_1} & 0 & 0 & \cdots & \beta^{12} & \cdots & 0 & 0 & \cdots & \beta^{1t} \\
 & & & & 1 & & & & & 0 & 0 & \cdots & 0 \\
 & & & & & \ddots & \ddots & & & \cdots & & & \\
 & & & & & & 1 & & & 0 & 0 & \cdots & 0 \\
 & & & & & & & \alpha_{i_2} & & 0 & 0 & \cdots & \beta^{2t} \\
 & & & & & & & & \ddots & & & & \\
 & & & & & & & & & 1 & & & \\
 & & & & & & & & & & 1 & & \\
 & & & & & & & & & & & \ddots & \\
 & & & & & & & & & & & & \alpha_{i_t}
\end{bmatrix}.$$

The only entries we cannot cancel in (C1) and (C2) are those on the intersection of rows and columns with α_{i_l}'s, i.e., the β^{kl}'s in $A_2(x)$. If we can prove that each β^{kl} is a multiple of α_{i_l}, then we can apply row operations to cancel all those β^{kl}'s, and we are done.

To see this, let us first go back to $A_1(x)$. We know that at the beginning $\beta^{kl}_{i_l}$ was a multiple of α_{i_l} by (2.4). Then this entry was changed to β^{kl} after we applied (C1) and (C2). More precisely, (C1) changed it but (C2) did not. If we look more closely, we find that terms were added to $\beta^{kl}_{i_l}$ only when we were doing the column operations to cancel the nonzero entries below this entry $\beta^{kl}_{i_l}$. We claim that each term which was added to this entry was actually a multiple of α_{i_l}. Thus the new term β^{kl} in $A_2(x)$ is still a multiple of α_{i_l}.

Now let us prove our claim. For simplicity we consider only the entry $\beta^{1t}_{i_t}$. The general case can be treated similarly. The entries which were below this $\beta^{1t}_{i_t}$ and which were canceled in (C1) were $b^{kt}_{ci_t}$ ($2 \le k \le t-1$, $2 \le c \le i_k$), together with f_{i_t} which was right above α_{i_t}.

(a) If $b^{kt}_{ci_t}$ was nonzero, then $c \ge i_t + 1$ by (2.3). To cancel this $b^{kt}_{ci_t}$, we added $-b^{kt}_{ci_t}$ times the $\lambda^{k,(c-1)}$ column (i.e., the column indexed by $\lambda^{k,(c-1)}$) to the λ^{ti_t} column. Thus $-b^{kt}_{ci_t}\beta^{1k}_{c-1}$ was added to $\beta^{1t}_{i_t}$. By the fact (2.4), $f_1 \cdots f_{c-1} \mid \beta^{1k}_{c-1}$. But $c-1 \ge i_t$, so this term added to $\beta^{1t}_{i_t}$ did have $f_1 \cdots f_{i_t} = \alpha_{i_t}$ as a factor.

(b) To cancel f_{i_t}, we added $-f_{i_t}$ times the λ^{t,i_t-1} column to the λ^{ti_t} column. This added $-f_{i_t}\beta^{1t}_{i_t-1}$ to $\beta^{1t}_{i_t}$. But again $f_1 \cdots f_{i_t-1} \mid \beta^{1t}_{i_t-1}$ by (2.4); we see that this term added is a multiple of α_{i_t}. □

3. A conjecture

We conjecture that our theorem can be generalized to the action $k\frac{\partial}{\partial p_k}p_k$ for $k \ge 1$.

Conjecture 3.1. *Let $M_k^{(n)}$ be the matrix of the map $k\frac{\partial}{\partial p_k}p_k$ with respect to an integral basis for homogeneous symmetric functions of degree n. Then there exists $P(x), Q(x) \in \mathrm{SL}(p(n), \mathbb{Z}[x])$ such that $P(x)(M_k^{(n)} + xI_{p(n)})Q(x)$ is the diagonal*

matrix $\operatorname{diag}(f_1(x), \ldots, f_{p(n)}(x))$*, where* $f_i(x)$ *may be described as follows. Let* $\mathcal{M}$ *be the multiset of all numbers* $m_k(\lambda)$ *for* $\lambda \vdash n$*. First,* $f_{p(n)}(x)$ *is a product of factors* $x + k(a_i + 1)$ *where the* a_i*'s are the* distinct *elements of* $\mathcal{M}$*. Then* $f_{p(n)-1}(x)$ *is a product of factors* $x + k(b_i + 1)$ *where the* b_i*'s are the remaining* distinct *elements of* $\mathcal{M}$*, etc. (After a while we will have exhausted all the elements of* $\mathcal{M}$*. The remaining diagonal elements are the empty product* 1*.)*

We can prove the following special case of the above conjecture. The proof is based on the result that for a partition $\lambda \vdash n$ there is at most one k-border strip if and only if $k > n/2$, though we omit the details here.

Proposition 3.2. *If* $k > n/2$*, then an SNF of* $M_k^{(n)} + xI_{p(n)}$ *over* $\mathbb{Z}[x]$ *is given by*

$$\operatorname{diag}(1, \ldots, 1, x + k, \ldots, x + k, (x + k)(x + 2k), \ldots, (x + k)(x + 2k)),$$

where there are $p(n - k)$ 1*'s and* $p(n - k)$ $(x + k)(x + 2k)$*'s.*

Thus it is known that Conjecture 3.1 is true for $k = 1$ or $k > n/2$.

Note added in proof

Zipei Nie has extended the authors' work to Y^r and has proved Conjecture 3.1

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School of Sciences, South China University of Technology, Guangzhou 510640, People's Republic of China

E-mail address: `caiwx@scut.edu.cn`

Department of mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

E-mail address: `rstan@math.mit.edu`