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CHAINS OF THEORIES AND COMPANIONABILITY

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Abstract. The theory of fields that are equipped with a countably infinite family of commuting derivations is not companionable, but if the axiom is added whereby the characteristic of the fields is zero, then the resulting theory is companionable. Each of these two theories is the union of a chain of companionable theories. In the case of characteristic 0, the model-companions of the theories in the chain form another chain, whose union is therefore the model-companion of the union of the original chain. However, in a signature with predicates, in all finite numbers of arguments, for linear dependence of vectors, the two-sorted theory of vector-spaces with their scalar-fields is companionable, and it is the union of a chain of companionable theories, but the model-companions of the theories in the chain are mutually inconsistent. Finally, the union of a chain of non-companionable theories may be companionable.

A **theory** in a given signature is a set of sentences, in the first-order logic of that signature, that is closed under logical implication. We shall consider chains $(T_m : m \in \omega)$ of theories: this means

$$(*) \qquad T_0 \subseteq T_1 \subseteq T_2 \subseteq \cdots .$$

The signature of T_m will be $\mathscr{S}_m$, so automatically $\mathscr{S}_0 \subseteq \mathscr{S}_1 \subseteq \mathscr{S}_2 \subseteq \cdots$.

In one motivating example, $\mathscr{S}_m$ is $\{0, 1, -, +, \cdot, \partial_0, \dots, \partial_{m-1}\}$, the signature of fields with m additional singulary operation-symbols, and T_m is m-DF, the theory of fields (of any characteristic) with m commuting derivations. In this example, each T_{m+1} is a **conservative extension** of T_m; that is, $T_{m+1} \supseteq T_m$ and every sentence in T_{m+1} of signature $\mathscr{S}_m$ is already in T_m. We establish this by showing that every model of T_m expands to a model of T_{m+1}. (This condition is sufficient, but not necessary [3, §2.6, exer. 8, p. 66].) If $(K, \partial_0, \dots, \partial_{m-1}) \models m$-DF, then $(K, \partial_0, \dots, \partial_m) \models (m+1)$-DF, where ∂_m is the 0-derivation.

The union of the theories m-DF can be denoted by ω-DF: it is the theory of fields with ω-many commuting derivations. Each of the theories m-DF has a *model-companion* called m-DCF [16], but we shall show (as Theorem 3 below) that ω-DF has no model-companion. Let us recall that a **model-companion** of a theory T is a theory T^* in the same signature such that (1) $T_\forall = {T^*}_\forall$, that is, every model of one of the theories embeds in a model of the other, and (2) T^* is **model-complete**, that is, $T^* \cup \operatorname{diag}(\mathfrak{M})$ axiomatizes a complete theory for all models $\mathfrak{M}$ of T^*. Here $\operatorname{diag}(\mathfrak{M})$ is the quantifier-free theory of $\mathfrak{M}$ with parameters; equivalently, $\operatorname{diag}(\mathfrak{M})$ is the theory of all structures in which $\mathfrak{M}_M$ embeds. (These notions with historical

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references are reviewed further in [16].) A theory has at most one model-companion, by an argument with interwoven elementary chains.

Let m-DF_0 be m-DF with the additional requirement that the field have characteristic 0. Then m-DF_0 has a model-companion, called m-DCF_0 [10]. It is known (and will be proved anew below) that m-$\mathrm{DCF}_0 \subseteq (m+1)$-$\mathrm{DCF}_0$. It will follow then that the union ω-DF_0 of the m-DF_0 has a model-companion, which is the union of the m-DCF_0. This is, by the following general result, a version of which has been observed also by Alice Medvedev [11,12]. Again, the theories T_k are as in $(*)$ above.

Theorem 1. *Suppose each theory T_k has a model-companion ${T_k}^*$, and*

$$(\dagger) \qquad {T_0}^* \subseteq {T_1}^* \subseteq {T_2}^* \subseteq \cdots .$$

Then the theory $\bigcup_{k\in\omega} T_k$ has a model-companion, namely $\bigcup_{k\in\omega} {T_k}^*$.

Proof. Write U for $\bigcup_{k\in\omega} T_k$, and write U^* for $\bigcup_{k\in\omega} {T_k}^*$. Suppose $\mathfrak{A} \models U$ and Γ is a finite subset of $U^* \cup \operatorname{diag}(\mathfrak{A})$. Then Γ is a subset of ${T_k}^* \cup \operatorname{diag}(\mathfrak{A} \restriction \mathscr{S}_k)$ for some k in ω and also $\mathfrak{A} \restriction \mathscr{S}_k \models T_k$. Since $({T_k}^*)_\forall \subseteq T_k$, the structure $\mathfrak{A} \restriction \mathscr{S}_k$ must embed in a model of ${T_k}^*$; this model will be a model of Γ. We conclude that Γ is consistent. Therefore, $U^* \cup \operatorname{diag}(\mathfrak{A})$ is consistent. Thus ${U^*}_\forall \subseteq U$. By symmetry $U_\forall \subseteq U^*$.

Similarly, if $\mathfrak{B} \models U^*$, then ${T_k}^* \cup \operatorname{diag}(\mathfrak{B} \restriction \mathscr{S}_k)$ axiomatizes a complete theory in each case, and therefore $U^* \cup \operatorname{diag}(\mathfrak{B})$ is complete. □

The foregoing proof does not require that the signatures $\mathscr{S}_k$ form a chain, but needs only that every finite subset of $\bigcup_{k\in\omega} \mathscr{S}_k$ be included in some $\mathscr{S}_k$. This is the setting for Medvedev's preprint [12, Prop. 2.4, p. 6], which then has the same proof as the foregoing. Also in Medvedev's setting, each ${T_{k+1}}^*$ is a conservative extension of ${T_k}^*$, but only the weaker assumption ${T_k}^* \subseteq {T_{k+1}}^*$ is needed in the proof.

Medvedev notes that many properties that the theories T_k might have are "local" and are therefore preserved in $\bigcup_{k\in\omega} T_k$; examples are completeness, elimination of quantifiers, stability, and simplicity. In her main application, $\mathscr{S}_n$ is the signature of fields with singulary operation-symbols $\sigma_{m/n!}$, where $m \in \mathbb{Z}$, and T_n is the theory of fields on which the $\sigma_{m/n!}$ are automorphisms such that

$$\sigma_{k/n!} \circ \sigma_{m/n!} = \sigma_{(k+m)/n!}.$$

Then T_n includes the theory S_n of fields with the single automorphism $\sigma_{1/n!}$. Using [15, §1] (which is based on [3, ch. 5]), we may observe at this point that reduction of models of T_n to models of S_n is actually an equivalence of the categories $\mathrm{Mod}^{\subseteq}(T_n)$ and $\mathrm{Mod}^{\subseteq}(S_n)$, whose objects are models of the indicated theories and whose morphisms are embeddings. We thus have at hand a (rather simple) instance of the hypothesis of the following theorem.

Theorem 2. *Suppose (I, J) is a bi-interpretation of theories S and T such that I is an equivalence of the categories $\mathrm{Mod}^{\subseteq}(S)$ and $\mathrm{Mod}^{\subseteq}(T)$. If S has the model-companion S^*, and $S \subseteq S^*$, then T also has a model-companion, which is the theory of those models $\mathfrak{B}$ of T such that $J(\mathfrak{B}) \models S^*$.*

Proof. The class of models $\mathfrak{B}$ of T such that $J(\mathfrak{B}) \models S^*$ is elementary. Let T^* be its theory. Then $T \subseteq T^*$. Suppose $\mathfrak{B} \models T$. Then $J(\mathfrak{B}) \models S$, so $J(\mathfrak{B})$ embeds in a model $\mathfrak{A}$ of S^*. Consequently, $I(J(\mathfrak{B}))$ embeds in $I(\mathfrak{A})$. Also $I(\mathfrak{A}) \models T^*$

since $\mathfrak{A} \cong J(I(\mathfrak{A}))$. Since also $\mathfrak{B} \cong I(J(\mathfrak{B}))$, we conclude that $\mathfrak{B}$ embeds in a model of T^*. Finally, T^* is model-complete. Indeed, suppose now $\mathfrak{B}$ and $\mathfrak{C}$ are models of T^* such that $\mathfrak{B} \subseteq \mathfrak{C}$. An embedding of $J(\mathfrak{B})$ in $J(\mathfrak{C})$ is induced, and these structures are models of S^*, so the embedding is elementary. Therefore, the induced embedding of $I(J(\mathfrak{B}))$ in $I(J(\mathfrak{C}))$ is also elementary. By the equivalence of the categories, $\mathfrak{B} \preccurlyeq \mathfrak{C}$. □

In the present situation, the theory S_n has a model-companion [1, 9]; let us denote this by ACFA_n. By the theorem then, T_n has a model-companion ${T_n}^*$, which is axiomatized by $T_n \cup \mathrm{ACFA}_n$. We have $\mathrm{ACFA}_n \subseteq {T_{n+1}}^*$ by [1, 1.12, Cor. 1, p. 3013]. By Theorem 1 then, $\bigcup_{n\in\omega} T_n$ has a model-companion, which is the union of the ${T_n}^*$. Medvedev calls this union $\mathbb{Q}$ACFA; she shows, for example, that it preserves the simplicity of the ACFA_n, as noted above, though it does not preserve their supersimplicity.

The following is similar to the result that the theory of fields with a derivation *and* an automorphism (of the field-structure only) has no model-companion [14]. The obstruction lies in positive characteristics p, where all derivatives of elements with pth roots must be 0.

Theorem 3. *The theory* ω-DF *has no model-companion.*

Proof. We use that an $\forall\exists$ theory T has a model-companion if and only if the class of its *existentially closed* models is elementary, and in this case the model-companion is the theory of this class [2]. (A model $\mathfrak{A}$ of T is an **existentially closed** model, provided that if $\mathfrak{B} \models T$ and $\mathfrak{A} \subseteq \mathfrak{B}$, then $\mathfrak{A} \preccurlyeq_1 \mathfrak{B}$; that is, all quantifier-free formulas over A that are soluble in $\mathfrak{B}$ are soluble in $\mathfrak{A}$.) For each n in ω, the theory ω-DF has an existentially closed model $\mathfrak{A}_n$, whose underlying field includes $\mathbb{F}_p(\alpha)$, where α is transcendental; and in this model,

$$\partial_k \alpha = \begin{cases} 1, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

Then α has no pth root in $\mathfrak{A}_n$. Therefore, in a non-principal ultraproduct of the $\mathfrak{A}_n$, α has no pth root, although $\partial_n \alpha = 0$ for all n in ω, so that α does have a pth root in some extension. Thus, the ultraproduct is not an existentially closed model of ω-DF. Therefore, the class of existentially closed models of ω-DF is not elementary. □

It follows then by Theorem 1 that m-DCF $\not\subseteq$ $(m+1)$-DCF for at least one m. In fact this is so for all m since

$$m\text{-DCF} \vdash p = 0 \to \forall x \left(\bigwedge_{i<m} \partial_i x = 0 \to \exists y\ y^p = x \right)$$

(where $p = 0$ means $\overbrace{1 + \cdots + 1}^{p} = 0$, which is true precisely in models having characteristic p), but $(m+1)$-DCF does not entail this sentence since

$$(m+1)\text{-DCF} \vdash \exists x \ \big(\bigwedge_{i<m} \partial_i x = 0 \wedge \partial_m x \neq 0 \big).$$

However, this observation by itself is not enough to establish the last theorem. For, by the results of [15], it is possible for each T_k to have a model-companion ${T_k}^*$

while $\bigcup_{k\in\omega} T_k$ has a model-companion that is not $\bigcup_{k\in\omega} T_k{}^*$. We may even require T_{k+1} to be a conservative extension of T_k.

Indeed, if $k > 0$, then in the notation of [15], VS_k is the theory of vector-spaces with their scalar-fields in the signature $\{+, -, \mathbf{0}, \circ, 0, 1, *, P^k\}$, where $\circ$ is multiplication of scalars, $*$ is the action of the scalar-field on the vector-space, and P^k is k-ary linear dependence. In particular, P^2 may written also as $\|$. Then VS_k has a model-companion $\mathrm{VS}_k{}^*$, which is the theory of k-dimensional vector-spaces over algebraically closed fields [15, Thm 2.3]. Let $\mathrm{VS}_\omega = \bigcup_{1\leqslant k<\omega} \mathrm{VS}_k$. (This was called VS_∞ in [15].) This theory has the model-companion $\mathrm{VS}_\omega{}^*$, which is the theory of infinite-dimensional vector-spaces over algebraically closed fields [15, Thm 2.4]. In particular $\mathrm{VS}_\omega{}^*$ is not the union of the $\mathrm{VS}_k{}^*$, because these are mutually inconsistent. We now turn this into a result about chains:

Theorem 4. *If $1 \leqslant n < \omega$, let T_n be the theory axiomatized by $\mathrm{VS}_1 \cup \cdots \cup \mathrm{VS}_n$. Then T_n has a model-companion $T_n{}^*$, which is axiomatized by $T_n \cup \mathrm{VS}_n{}^*$. Also T_{n+1} is a conservative extension of T_n. However, the model-companion $\mathrm{VS}_\omega{}^*$ of the union VS_ω of the chain $(T_n\colon 1 \leqslant n < \omega)$ is not the union of the $T_n{}^*$.*

Proof. Every vector-space can be considered as a model of every VS_k and hence of every T_k. In particular, T_{n+1} is a conservative extension of T_n. If the theories $T_n{}^*$ are as claimed, then they are mutually inconsistent, and so $\mathrm{VS}_\omega{}^*$ is not their union. It remains to show that there are theories $T_n{}^*$ as claimed. We already know this when $n = 1$. For the other cases, if $1 \leqslant k < n$, we define the relations P^k in models of VS_n of dimension at least n.

Let $\mathrm{VS}_n^{\mathrm{m}}$ the theory of such models: that is, $\mathrm{VS}_n^{\mathrm{m}}$ is axiomatized by VS_n and the requirement that the space have dimension at least n. The relation P^1 is defined in models of $\mathrm{VS}_n^{\mathrm{m}}$ (and indeed in models of VS_n) by the quantifier-free formula $\boldsymbol{x} = \mathbf{0}$. If $n > 2$, then there are existential formulas that, in each model of $\mathrm{VS}_n^{\mathrm{m}}$, define the relation $\|$ and its complement [15, §2, p. 431]. More generally, if $1 \leqslant k < n - 1$, then, using existential formulas, we can define P^{k+1} and its complement in models of $T_k \cup \mathrm{VS}_n^{\mathrm{m}}$ or just $\mathrm{VS}_k \cup \mathrm{VS}_n^{\mathrm{m}}$. Indeed, $\neg P^{k+1}\boldsymbol{x}_0 \cdots \boldsymbol{x}_k$ is equivalent to $\exists(\boldsymbol{x}_{k+1}, \dots, \boldsymbol{x}_{n-1})\ \neg P^n\boldsymbol{x}_0 \cdots \boldsymbol{x}_{n-1}$, and $P^{k+1}\boldsymbol{x}_0 \cdots \boldsymbol{x}_k$ is equivalent to

$$\exists(\boldsymbol{x}_{k+1}, \dots, \boldsymbol{x}_n)\ \Bigl(P^k\boldsymbol{x}_1 \cdots \boldsymbol{x}_k \vee \Bigl(\neg P^n\boldsymbol{x}_1 \cdots \boldsymbol{x}_n \wedge \bigwedge_{j=k+1}^{n} P^n\boldsymbol{x}_0 \cdots \boldsymbol{x}_{j-1}\boldsymbol{x}_{j+1} \cdots \boldsymbol{x}_n\Bigr)\Bigr).$$

For, in a space of dimension at least n, if $(\boldsymbol{a}_0, \dots, \boldsymbol{a}_k)$ is linearly dependent but $(\boldsymbol{a}_1, \dots, \boldsymbol{a}_k)$ is not, this means precisely that $(\boldsymbol{a}_1, \dots, \boldsymbol{a}_n)$ is independent for some $(\boldsymbol{a}_{k+1}, \dots, \boldsymbol{a}_n)$ but $\boldsymbol{a}_0$ is a *unique* linear combination of $(\boldsymbol{a}_1, \dots, \boldsymbol{a}_n)$ and in fact of $(\boldsymbol{a}_1, \dots, \boldsymbol{a}_{j-1}, \boldsymbol{a}_{j+1}, \dots \boldsymbol{a}_n)$ whenever $k + 1 \leqslant j \leqslant n$ and (therefore) of $(\boldsymbol{a}_1, \dots, \boldsymbol{a}_k)$.

By [15, Lem 1.1, 1.2], if $1 \leqslant k < n - 1$, we now have that reduction from models of $T_{k+1} \cup \mathrm{VS}_n^{\mathrm{m}}$ to models of $T_k \cup \mathrm{VS}_n^{\mathrm{m}}$ is an equivalence of the categories $\mathrm{Mod}^{\subseteq}(T_{k+1} \cup \mathrm{VS}_n^{\mathrm{m}})$ and $\mathrm{Mod}^{\subseteq}(T_k \cup \mathrm{VS}_n^{\mathrm{m}})$. Combining these results for all k, we have that reduction from models of $T_{n-1} \cup \mathrm{VS}_n^{\mathrm{m}}$ to models of $\mathrm{VS}_n^{\mathrm{m}}$ is an equivalence of the categories $\mathrm{Mod}^{\subseteq}(T_{n-1} \cup \mathrm{VS}_n^{\mathrm{m}})$ and $\mathrm{Mod}^{\subseteq}(\mathrm{VS}_n^{\mathrm{m}})$. Since $\mathrm{VS}_n \subseteq \mathrm{VS}_n^{\mathrm{m}}$ and every model of VS_n embeds in a model of $\mathrm{VS}_n^{\mathrm{m}}$, the two theories have the same model-companion, namely $\mathrm{VS}_n{}^*$. Similarly, T_n and $T_{n-1} \cup \mathrm{VS}_n^{\mathrm{m}}$ have the same model-companion, and by Theorem 2 this is axiomatized by $T_n \cup \mathrm{VS}_n{}^*$. □

A one-sorted version of the last theorem can be developed as follows. Let $\mathrm{VS}^{\mathrm{r}}_n$ comprise the sentences of $\mathrm{VS}^{\mathrm{m}}_n$ having one-sorted signature $\{\mathbf{0}, -, +, P^n\}$ of the sort of vectors alone. It is not obvious that all models of $\mathrm{VS}^{\mathrm{r}}_n$ can be furnished with scalar-fields to make them models of $\mathrm{VS}^{\mathrm{m}}_n$ again, but this will be the case. By [15, Thm 1.1], it is the case when $n = 2$: reduction of models of $\mathrm{VS}^{\mathrm{m}}_2$ to models of $\mathrm{VS}^{\mathrm{r}}_2$ is an equivalence of the categories $\mathrm{Mod}^{\subseteq}(\mathrm{VS}^{\mathrm{m}}_2)$ and $\mathrm{Mod}^{\subseteq}(\mathrm{VS}^{\mathrm{r}}_2)$. This reduction is therefore **conservative**, by the definition of [15, p. 426]. It is said further in [15, p. 431] that reduction from $\mathrm{VS}^{\mathrm{m}}_n$ to $\mathrm{VS}^{\mathrm{r}}_n$ is conservative when $n > 2$, but the details are not spelled out. However, the claim can be established as follows. Immediately, reduction from $\mathrm{VS}_2 \cup \mathrm{VS}^{\mathrm{m}}_n$ to $\mathrm{VS}^{\mathrm{r}}_2 \cup \mathrm{VS}^{\mathrm{r}}_n$ is conservative. In particular, models of the latter set of sentences really are vector-spaces without their scalar-fields. It is noted in effect in the proof of Theorem 4 that reduction from $\mathrm{VS}_2 \cup \mathrm{VS}^{\mathrm{m}}_n$ to $\mathrm{VS}^{\mathrm{m}}_n$ is conservative. Furthermore, in models of the latter theory, the defining of parallelism and its complement is done with existential formulas *in the signature of vectors alone.* Therefore, reduction from $\mathrm{VS}^{\mathrm{r}}_2 \cup \mathrm{VS}^{\mathrm{r}}_n$ to $\mathrm{VS}^{\mathrm{r}}_n$ is conservative. We now have the following commutative diagram of reduction-functors, three of them being conservative, that is, being equivalences of categories.

$$\begin{array}{ccc}
\mathrm{Mod}^{\subseteq}(\mathrm{VS}_2 \cup \mathrm{VS}^{\mathrm{m}}_n) & \longrightarrow & \mathrm{Mod}^{\subseteq}(\mathrm{VS}^{\mathrm{m}}_n) \\
\downarrow & & \downarrow \\
\mathrm{Mod}^{\subseteq}(\mathrm{VS}^{\mathrm{r}}_2 \cup \mathrm{VS}^{\mathrm{r}}_n) & \longrightarrow & \mathrm{Mod}^{\subseteq}(\mathrm{VS}^{\mathrm{r}}_n)
\end{array}$$

Therefore, the remaining reduction, from $\mathrm{VS}^{\mathrm{m}}_n$ to $\mathrm{VS}^{\mathrm{r}}_n$, must be conservative.

Now there is a version of Theorem 4 where T_n is axiomatized by $\mathrm{VS}^{\mathrm{r}}_2 \cup \dots \cup \mathrm{VS}^{\mathrm{r}}_n$. Indeed, by Theorem 2, T_n has a model-companion, which is the theory (in the same signature) of n-dimensional vector-spaces over algebraically closed fields; and the union of the T_n has a model-companion, which is the theory of infinite-dimensional vector-spaces over algebraically closed fields; but this theory is not the union of the model-companions of the T_n.

The implication A $\Rightarrow$ B in the following is used implicitly in [1, 1.12, p. 3013] to establish the result used above, that if (K, σ) is a model of ACFA, then so is (K, σ^m), assuming $m \geqslant 1$.

Theorem 5. *Assuming as usual $T_0 \subseteq T_1$, where each T_k has signature $\mathscr{S}_k$, we consider the following conditions.*

A *For every model $\mathfrak{A}$ of T_1 and model $\mathfrak{B}$ of T_0 such that*

$$\mathfrak{A} \restriction \mathscr{S}_0 \subseteq \mathfrak{B}, \tag{‡}$$

there is a model $\mathfrak{C}$ of T_1 such that

$$\mathfrak{A} \subseteq \mathfrak{C}, \qquad \mathfrak{B} \subseteq \mathfrak{C} \restriction \mathscr{S}_0. \tag{§}$$

B *The reduct to $\mathscr{S}_0$ of every existentially closed model of T_1 is an existentially closed model of T_0.*

C *T_0 has the* Amalgamation Property: *if one model embeds in two others, then those two in turn embed in a fourth model, compatibly with the original embeddings.*

D *T_1 is $\forall\exists$ (so that every model embeds in an existentially closed model).*

We have the two implications

$$\mathrm{A} \implies \mathrm{B}, \qquad\qquad \mathrm{B}\ \&\ \mathrm{C}\ \&\ \mathrm{D} \implies \mathrm{A},$$

but there is no implication among the four conditions that does not follow from these. This is true, even if T_1 is required to be a conservative extension of T_0.

Proof. Suppose A holds. Let $\mathfrak{A}$ be an existentially closed model of T_1, and let $\mathfrak{B}$ be an arbitrary model of T_0 such that (‡) holds. By hypothesis, there is a model $\mathfrak{C}$ of T_1 such that (§) holds. Then $\mathfrak{A} \preccurlyeq_1 \mathfrak{C}$, and therefore $\mathfrak{A} \restriction \mathscr{S}_0 \preccurlyeq_1 \mathfrak{C} \restriction \mathscr{S}_0$, and *a fortiori* $\mathfrak{A} \restriction \mathscr{S}_0 \preccurlyeq_1 \mathfrak{B}$. Therefore, $\mathfrak{A} \restriction \mathscr{S}_0$ must be an existentially closed model of T_0. Thus B holds.

Suppose conversely that B holds along with C and D. Let $\mathfrak{A} \models T_1$ and $\mathfrak{B} \models T_0$ such that (‡) holds. We establish the consistency of $T_1 \cup \operatorname{diag}(\mathfrak{A}) \cup \operatorname{diag}(\mathfrak{B})$. It is enough to show the consistency of

$$(\P) \qquad T_1 \cup \operatorname{diag}(\mathfrak{A}) \cup \{\exists \boldsymbol{x}\ \varphi(\boldsymbol{x})\},$$

where φ is an arbitrary quantifier-free formula of $\mathscr{S}_0(A)$ that is soluble in $\mathfrak{B}$. By D, there is an existentially closed model $\mathfrak{C}$ of T_1 that extends $\mathfrak{A}$. By B then, $\mathfrak{C} \restriction \mathscr{S}_0$ is an existentially closed model of T_0 that extends $\mathfrak{A} \restriction \mathscr{S}_0$. By C, both $\mathfrak{B}$ and $\mathfrak{C} \restriction \mathscr{S}_0$ embed over $\mathfrak{A} \restriction \mathscr{S}_0$ in a model of T_0. In particular, φ will be soluble in this model. Therefore, φ is already soluble in $\mathfrak{C} \restriction \mathscr{S}_0$ itself. Thus $\mathfrak{C}$ is a model of (¶). Therefore, A holds.

The foregoing arguments eliminate the five possibilities marked X on Table 1, where 0 means false, and 1, true. We give examples of each of the remaining cases, numbered according to Table 1. In each example, T_0 will be the reduct of T_1 to $\mathscr{S}_0$. We shall denote by $\mathscr{S}_{\mathrm{f}}$ the signature $\{+, \cdot, -, 0, 1\}$ of fields; and by $\mathscr{S}_{\mathrm{vs}}$, the signature $\{+, -, \mathbf{0}, \circ, 0, 1, *\}$ of vector-spaces as two-sorted structures.

Table 1

	1	X	2	3	4	X	5	6	7	X	8	9	10	X	X	11
A	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
B	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
C	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
D	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1

1. We first give an example in which none of the four lettered conditions hold. Let $\mathscr{S}_0 = \mathscr{S}_{\mathrm{f}} \cup \{a, b\}$ and $\mathscr{S}_1 = \mathscr{S}_0 \cup \{c\}$. Let T_1 be the theory of fields of characteristic p with distinguished elements a, b, and c such that $\{a, c\}$ or $\{b, c\}$ is p-independent, and if $\{b, c\}$ is p-independent, then so is $\{b, c, d\}$ for some d. Then T_0 is the theory of fields of characteristic p in which, for some c, $\{a, c\}$ or $\{b, c\}$ is p-independent, and if $\{b, c\}$ is p-independent, then so is $\{b, c, d\}$ for some d. The negations of the four lettered conditions are established as follows. Throughout, a, b, c, and d will be algebraically independent over $\mathbb{F}_p$.

¬A. We have

$$(\mathbb{F}_p(a, b^{1/p}, c), a, b, c) \models T_1, \qquad\qquad (\mathbb{F}_p(a, b^{1/p}, c^{1/p}), a, b) \models T_0,$$

but if $(\mathbb{F}_p(a, b^{1/p}, c), a, b, c)$ is a substructure of a model (K, a, b, c) of T_1, then K cannot contain $c^{1/p}$.

¬B. T_0 has no existentially closed models, since an element of a model that is p-independent from a or b will always have a pth root in some extension. Similarly, no model of T_1 in which $\{a, c\}$ is not p-independent is existentially closed. But T_1 does have existentially closed models, which are just the separably closed fields of characteristic p with p-basis $\{a, c\}$ and with an additional element b.

¬C. T_0 does not have the Amalgamation Property, since $(\mathbb{F}_p(a, b^{1/p}, c), a, b)$ and $(\mathbb{F}_p(a^{1/p}, b, c, d), a, b)$ are models that do not embed in the same model over the common substructure $(\mathbb{F}_p(a, b, c), a, b)$, which is a model of T_0.

¬D. T_1 is not $\forall\exists$, since, as we have already noted, models in which $\{a, c\}$ is not p-independent do not embed in existentially closed models.

2. For an example of the column headed by 2 in Table 1, we let $\mathscr{S}_0$ and $\mathscr{S}_1$ be as in column 1; but now T_1 is the theory of fields of characteristic p with distinguished elements a, b, and c such that $\{a, c, d\}$ or $\{b, c, d\}$ is p-independent for some d. This ensures that T_1 has no existentially closed models, so B holds vacuously; but the other three conditions still fail.

3. T_0 and T_1 are the same theory, so A and B hold trivially; and this theory is the theory of vector-spaces of dimension at least 2, in the signature $\mathscr{S}_{\mathrm{vs}}$, so the theory neither has the Amalgamation Property nor is $\forall\exists$.

4. T_1 is DF_p with the additional requirement that the field have p-dimension at least 2; and $\mathscr{S}_0 = \mathscr{S}_{\mathrm{f}}$, so T_0 is the theory of fields of characteristic p with p-dimension at least 2. The latter theory has the Amalgamation Property, but the other conditions fail. Indeed, let $(\mathbb{F}_p(a, b), D)$ be the model of T_1 in which $Da = 1$ and $Db = 0$. Then the field $\mathbb{F}_p(a, b)$ embeds in $\mathbb{F}_p(a^{1/p}, b)$, which is a model of T_0, but D does not extend to this field. Also, T_0 has no existentially closed models, but T_1 does, and indeed it has a model-companion, namely DCF_p. Also T_1 is not $\forall\exists$, since T_0 is not: there is a chain of models of the latter, whose union is not a model, and we can make the structures in the chain into models of T_1 by adding the zero derivation.

5. $\mathscr{S}_0 = \mathscr{S}_{\mathrm{f}}$, and $\mathscr{S}_1 = \mathscr{S}_0 \cup \{a\}$. T_1 is the theory of fields of characteristic p with distinguished element a, which is p-independent from another element; so T_0 is (as in 4) the theory of fields of characteristic p with p-dimension at least 2. Then we already have that C holds. But A fails: just let $\mathfrak{A}$ be $(\mathbb{F}_p(a, b), a)$, and let $\mathfrak{B}$ be $\mathbb{F}_p(a^{1/p}, b)$. Also T_1 has no existentially closed models, so B holds trivially, but T_1 is not $\forall\exists$.

6. T_0 and T_1 are the same, namely the theory of fields of characteristic p of positive p-dimension, in the signature of fields, so this theory has the Amalgamation Property but is not $\forall\exists$.

7. $\mathscr{S}_0 = \mathscr{S}_{\mathrm{vs}}$, $\mathscr{S}_1 = \mathscr{S}_0 \cup \{\parallel, \boldsymbol{a}, \boldsymbol{b}\}$, and T_1 is axiomatized by $\mathrm{VS}_2 \cup \{\boldsymbol{a} \nparallel \boldsymbol{b}\}$, so it is $\forall\exists$. Then T_0 is the theory of vector-spaces of dimension at least 2. As in Theorem 4 above, T_1 has a model-companion, namely the theory of vector-spaces over algebraically closed fields with basis $\{\boldsymbol{a}, \boldsymbol{b}\}$. But T_0 has no existentially closed models, since for all independent vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in some model, the equation

$$x * \boldsymbol{a} + y * \boldsymbol{b} = \boldsymbol{0} \qquad (\parallel)$$

is always soluble in some extension. Thus B fails. Then T_0 also does not have the Amalgamation Property, since the solutions of $(\parallel)$ may satisfy $2x^2 = y^2$ in one extension, but $3x^2 = y^2$ in another. Similarly, A fails, since the reduct to $\mathscr{S}_0$ of a model of T_1 may embed in a model of T_0 in which $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel.

8. $\mathscr{S}_0 = \mathscr{S}_{\mathrm{vs}} \cup \{\|\}$, $\mathscr{S}_1 = \mathscr{S}_0 \cup \{\boldsymbol{a}, \boldsymbol{b}\}$, and T_1 is axiomatized by VS_2 together with

$$\forall x\ \forall y\ (x * \boldsymbol{a} + y * \boldsymbol{b} = \mathbf{0} \to 2x^2 = y^2). \tag{$**$}$$

Then T_0 is the theory of vector-spaces such that either the dimension is at least 2, or the scalar field contains $\sqrt{2}$. As in 7, T_0 does not have the Amalgamation Property. The theory T_1 is $\forall\exists$. It also has the model $(\mathbb{Q} * \boldsymbol{a} \oplus \mathbb{Q} * \boldsymbol{b}, \boldsymbol{a}, \boldsymbol{b})$, and $\mathbb{Q} * \boldsymbol{a} \oplus \mathbb{Q} * \boldsymbol{b}$ embeds in the model $\mathbb{Q}(\sqrt{2}, \sqrt{3}) * \boldsymbol{a}$ of T_0 when we let $\boldsymbol{b} = \sqrt{3} * \boldsymbol{a}$, but then the latter space embeds in no space in which $\boldsymbol{a}$ and $\boldsymbol{b}$ are as required by $(**)$, so A fails. Finally, T_1 has a model-companion, axiomatized by $\mathrm{VS_2}^*$ together with

$$\exists x\ \exists y\ (x * \boldsymbol{a} + y * \boldsymbol{b} = \mathbf{0} \land 2x^2 = y^2 \land x \neq 0),$$

and T_0 has a model-companion, which is just $\mathrm{VS_2}^*$, so B holds.

9. T_0 and T_1 are both VS_1.

10. $T_1 = \mathrm{DF}_p$, and T_0 is the reduct to $\mathscr{S}_{\mathrm{f}}$, namely field-theory in characteristic p.

11. T_0 and T_1 are both field-theory. □

Now let $\omega\text{-DCF}_0 = \bigcup_{m\in\omega} m\text{-DCF}_0$. As noted above, by Theorem 1, we shall have that $\omega\text{-DCF}_0$ is the model-companion of $\omega\text{-DF}_0$, once we know

$$m\text{-DCF}_0 \subseteq (m+1)\text{-DCF}_0. \tag{$\dagger\dagger$}$$

In fact this inclusion is Tracey McGrail's [10, Proposition 5.1.5], which is based ultimately (through her Lemma 2.2.17 and Theorem 2.2.18) on E. R. Kolchin's [5, Lemma III.5, p. 137, and Lemma IV.2, p. 167]. The same inclusion $(\dagger\dagger)$ is used in Omar León Sánchez's geometric characterization [7, Theorem 3.4; 8] of the models of $(m+1)\text{-DCF}_0$. In fact it is León Sánchez who has referred us to McGrail's result, but in his paper, the result is implicitly derived from [7, Corollary 2.5], which is Kolchin's [6, Ch. 0, §4, Corollary 1, p. 11]. In our terms, Kolchin's result is that, if $(L, \partial_0, \dots, \partial_{m-1}) \models m\text{-DCF}_0$, and L has a subring R that is closed under the ∂_i (where $i < m$), and ∂_m is a derivation from R to L that commutes with the other ∂_i, then $\partial_m = \tilde{\partial}_m \restriction R$ for some $\tilde{\partial}_m$ such that $(L, \partial_0, \dots, \partial_{m-1}, \tilde{\partial}_m) \models (m+1)\text{-DCF}_0$. In particular, Condition A of Theorem 5 holds when T_0 is $m\text{-DF}_0$ and T_1 is $(m+1)\text{-DF}_0$. Then Condition B holds, and therefore $(\dagger\dagger)$. In short, [7, Theorem 3.4] is an implicit application of Theorem 5. We can establish Kolchin's result "geometrically", using points rather than differential polynomials, as follows.

Lemma (Kolchin)**.** *Condition* A *of Theorem* 5 *holds when* T_0 *is* $m\text{-DF}_0$ *and* T_1 *is* $(m+1)\text{-DF}_0$.

Proof. Suppose $(L, \partial_0, \dots, \partial_{m-1})$ is a model of $m\text{-DF}_0$, and L has a subfield K that is closed under the ∂_i (where $i < m$), and there is also a derivation ∂_m on K such that $(K, \partial_0 \restriction K, \dots, \partial_{m-1} \restriction K, \partial_m)$ is a model of $(m+1)\text{-DF}_0$. We shall include $(L, \partial_0, \dots, \partial_{m-1})$ in another model of $m\text{-DF}_0$, namely a model that expands to a model of $(m+1)\text{-DF}_0$ that includes $(K, \partial_0, \dots, \partial_m)$.

If $K = L$, we are done. Suppose $a \in L \smallsetminus K$. We shall define a differential field $(K\langle a\rangle, \tilde{\partial}_0, \dots, \tilde{\partial}_m)$, where $a \in K\langle a\rangle$, and for each i in m,

$$\tilde{\partial}_i \restriction K\langle a\rangle \cap L = \partial_i \restriction K\langle a\rangle \cap L, \tag{$\ddagger\ddagger$}$$

and $\tilde{\partial}_m \restriction K = \partial_m$. Then we shall be able to repeat the process, in case $L \nsubseteq K\langle a\rangle$: We can work with an element of $L \smallsetminus K\langle a\rangle$ as we did with a. Ultimately we shall obtain the desired model of $(m+1)$-DF_0 with reduct that includes $(L, \partial_0, \dots, \partial_{m-1})$.

Considering ω^{m+1} as the set of $(m+1)$-tuples of natural numbers, we shall have

$$K\langle a\rangle = K(a^\sigma : \sigma \in \omega^{m+1}),$$

where

$$a^\sigma = \tilde{\partial}_0{}^{\sigma(0)} \dots \tilde{\partial}_m{}^{\sigma(m)} a. \tag{§§}$$

In particular then, by (‡‡), we must have

$$\sigma(m) = 0 \implies a^\sigma = \partial_0{}^{\sigma(0)} \dots \partial_{m-1}{}^{\sigma(m-1)} a.$$

Using this rule, we make the definition

$$K_1 = K(a^\sigma : \sigma(m) = 0).$$

We may assume that the derivations $\tilde{\partial}_i$ have been defined so far that

$$i < m \implies \tilde{\partial}_i \restriction K_1 = \partial_i \restriction K_1, \qquad \tilde{\partial}_m \restriction K = \partial_m \restriction K. \tag{¶¶}$$

Then (§§) holds when $\sigma(m) < 1$.

Now suppose that, for some positive j in ω, we have been able to define the field $K(a^\sigma : \sigma(m) < j)$, and for each i in m, we have been able to define $\tilde{\partial}_i$ as a derivation on this field, and we have been able to define $\tilde{\partial}_m$ as a derivation from $K(a^\sigma : \sigma(m) < j-1)$ to $K(a^\sigma : \sigma(m) < j)$ so that (¶¶) holds, and (§§) holds when $\sigma(m) < j$. We want to define the a^σ such that $\sigma(m) = j$, and we want to be able to extend the derivations $\tilde{\partial}_i$ appropriately.

If $i < m+1$, then, as in [16, §4.1], we let $\boldsymbol{i}$ denote the characteristic function of $\{i\}$ on $m+1$; that is, $\boldsymbol{i}$ will be the element of ω^{m+1} that takes the value 1 at i and 0 elsewhere. Considered as a product structure, ω^{m+1} inherits from ω the binary operations $-$ and $+$. For each i in $m+1$, we have a derivation $\tilde{\partial}_i$ from $K(a^\sigma : (\sigma + \boldsymbol{i})(m) < j)$ to $K(a^\sigma : \sigma(m) < j)$ such that (¶¶) holds, and also, if $\sigma(m) < j$, then

$$\sigma(i) > 0 \implies \tilde{\partial}_i a^{\sigma - \boldsymbol{i}} = a^\sigma. \tag{***}$$

We now define the a^σ, where $\sigma(m) = j$, so that, first of all, we can extend $\tilde{\partial}_m$ so that (***) holds when $\sigma(m) = j$ and $i = m$; but we must also ensure that (***) can hold also when $\sigma(m) = j$ and $i < m$. To do this, we shall have to make an inductive hypothesis, which is vacuously satisfied when $j = 1$. We shall also proceed recursively again. More precisely, we shall refine the recursion that we are already engaged in. We well-order the elements σ of ω^{m+1} by the linear ordering $\lhd$ determined by the left lexicographic ordering of the $(m+1)$-tuples

$$\big(\sigma(m), \sigma(0) + \dots + \sigma(m-1), \sigma(0), \sigma(1), \dots, \sigma(m-2)\big).$$

Then $(\omega^{m+1}, \lhd)$ has the order-type of the ordinal ω^2. This is a difference from the linear ordering defined in [16, §4.1] and elsewhere. However, for all σ and τ in ω^{m+1}, and all i in $m+1$, we still have

$$\sigma \lhd \tau \implies \sigma + \boldsymbol{i} \lhd \tau + \boldsymbol{i}.$$

We have assumed that, when $\tau = (0, \dots, 0, j)$, we have the field $K(a^\sigma : \sigma \lhd \tau)$, together with, for each i in $m+1$, a derivation $\tilde{\partial}_i$ from $K(a^\xi : \xi + \boldsymbol{i} \lhd \tau)$ to $K(a^\xi : \xi \lhd \tau)$ such that (¶¶) holds, and also, if $\sigma \lhd \tau$, then (***) holds. We have noted that we *can* have all of this when $\tau = (0, \dots, 0, 1)$. Suppose we have all of this for *some* τ in ω^{m+1} such that $(0, \dots, 0, 1) \trianglelefteq \tau$, that is, $\tau(m) > 0$. We want to define the extension $K(a^\sigma : \sigma \trianglelefteq \tau)$ of $K(a^\sigma : \sigma \triangleleft \tau)$ so that we can extend the $\tilde{\partial}_i$ appropriately. For defining a^τ, there are two cases to consider. We use the rules for derivations gathered, for example, in [14, Fact 1.1].

1. If $a^{\tau-\boldsymbol{m}}$ is algebraic over $K(a^\xi : \xi \triangleleft \tau - \boldsymbol{m})$, then the derivative $\tilde{\partial}_m a^{\tau-\boldsymbol{m}}$ is determined as an element of $K(a^\xi : \xi \triangleleft \tau)$. We let a^τ be this element.
2. If $a^{\tau-\boldsymbol{m}}$ is not algebraic over $K(a^\xi : \xi \triangleleft \tau - \boldsymbol{m})$, then we let a^τ be transcendental over $L(a^\xi : \xi \triangleleft \tau)$. We are then free to define $\tilde{\partial}_m a^{\tau-\boldsymbol{m}}$ as a^τ. (We require a^τ to be transcendental over $L(a^\xi : \xi \triangleleft \tau)$, and not just over $K(a^\xi : \xi \triangleleft \tau)$, so that we can establish (‡‡) later.)

We now check that, when $i < m$ and $\tau(i) > 0$, we can define $\tilde{\partial}_i a^{\tau-\boldsymbol{i}}$ as a^τ. Here we make the inductive hypothesis mentioned above, namely that the foregoing two-part definition of a^τ was already used to define $a^{\tau-\boldsymbol{i}}$. Again we consider two cases.

1. Suppose $a^{\tau-\boldsymbol{i}}$ is algebraic over $K(a^\xi : \xi \triangleleft \tau - \boldsymbol{i})$. Then $\tilde{\partial}_i a^{\tau-\boldsymbol{i}}$ is determined as an element of $K(a^\xi : \xi \triangleleft \tau)$. Thus the value of the bracket $[\tilde{\partial}_i, \tilde{\partial}_m]$ at $a^{\tau-\boldsymbol{i}-\boldsymbol{m}}$ is determined. Indeed, we have
$$[\tilde{\partial}_i, \tilde{\partial}_m] a^{\tau-\boldsymbol{i}-\boldsymbol{m}} = \tilde{\partial}_i \tilde{\partial}_m a^{\tau-\boldsymbol{i}-\boldsymbol{m}} - \tilde{\partial}_m \tilde{\partial}_i a^{\tau-\boldsymbol{i}-\boldsymbol{m}} = \tilde{\partial}_i a^{\tau-\boldsymbol{i}} - a^\tau.$$
By inductive hypothesis, since $a^{\tau-\boldsymbol{i}}$ is algebraic over $K(a^\xi : \xi \triangleleft \tau - \boldsymbol{i})$, also $a^{\tau-\boldsymbol{i}-\boldsymbol{m}}$ must be algebraic over $K(a^\xi : \xi \triangleleft \tau - \boldsymbol{i} - \boldsymbol{m})$. Since the bracket is 0 on this field, it must be 0 at $a^{\tau-\boldsymbol{i}-\boldsymbol{m}}$ as well (see [16, Lem. 4.2]).
2. If $a^{\tau-\boldsymbol{i}}$ is transcendental over $K(a^\xi : \xi \triangleleft \tau - \boldsymbol{i})$, then, since we are given $\tilde{\partial}_i$ as a derivation whose domain is this field, we are free to define $\tilde{\partial}_i a^{\tau-\boldsymbol{i}}$ as a^τ.

Thus we have obtained $K(a^\xi : \xi \trianglelefteq \tau)$ as desired. By induction, we obtain the differential field $(K(a^\sigma : \sigma \in \omega^{m+1}), \tilde{\partial}_0, \dots, \tilde{\partial}_m)$ such that (§§) and (¶¶) hold.

It remains to check that (‡‡) holds. It is enough to show

$$K\langle a \rangle \cap L \subseteq K_1. \tag{†††}$$

(We have the reverse inclusion.) Suppose $\tau \in \omega^{m+1}$ and $\tau(m) > 0$. By the definition of a^τ,

$$a^\tau \in K(a^\sigma : \sigma \triangleleft \tau)^{\text{alg}} \implies a^\tau \in K(a^\sigma : \sigma \triangleleft \tau), \tag{‡‡‡}$$

$$a^\tau \notin K(a^\sigma : \sigma \triangleleft \tau)^{\text{alg}} \implies a^\tau \notin L(a^\sigma : \sigma \triangleleft \tau)^{\text{alg}}. \tag{§§§}$$

Suppose $b \in K\langle a \rangle \cap L$. Since $b \in K\langle a \rangle$, we have, for some τ in ω^{m+1}, that b is a rational function over K_1 of those a^σ such that $\boldsymbol{m} \trianglelefteq \sigma \trianglelefteq \tau$. But then, by (‡‡‡), we do not need any a^σ that is algebraic over $K(a^\xi : \xi \triangleleft \sigma)$, since it actually belongs to this field. When we throw out all such a^σ, then, by (§§§), those that remain are algebraically independent over L. Thus we have

$$b \in K_1(a^{\sigma_0}, \dots, a^{\sigma_{n-1}}) \cap L$$

for some σ_j in ω^{m+1} such that $(a^{\sigma_0}, \dots, a^{\sigma_{n-1}})$ is algebraically independent over L. Therefore we may assume $n = 0$ and $b \in K_1$. Thus (†††) holds, and we have the differential field $(K\langle a \rangle, \tilde{\partial}_0, \dots, \tilde{\partial}_m)$ fully as desired.

We have to be able to repeat this contruction, in case $L \not\subseteq K\langle a \rangle$. If $b \in L \smallsetminus K\langle a \rangle$, we have to be able to construct $K\langle a, b \rangle$, and so on. Let $L\langle a \rangle$ be the compositum of $K\langle a \rangle$ and L. Since m-DF_0 has the Amalgamation Property, we can extend the $\tilde{\partial}_i$, where $i < m$, to commutating derivations on the field $L\langle a\rangle$ that extend the original ∂_i on L. Thus we have a model $(L\langle a\rangle, \tilde{\partial}_0, \dots, \tilde{\partial}_{m-1})$ of m-DF$_0$ and a model $(K\langle a\rangle, \tilde{\partial}_0 \restriction K\langle a\rangle, \dots, \tilde{\partial}_{m-1} \restriction K\langle a\rangle, \tilde{\partial}_m)$ of $(m+1)$-DF$_0$ that include, respectively, the models that we started with. Now we can continue as before, ultimately extending the domain of $\tilde{\partial}_m$ to include all of L. At limit stages of this process, we take unions, which is no problem, since m-DF$_0$ and $(m+1)$-DF$_0$ are $\forall\exists$. □

Theorem 6. *The theory* ω-DF$_0$ *has a model-companion, which is* ω-DCF$_0$*. This theory admits full elimination of quantifiers, is complete, and is properly stable.*

Proof. By Kolchin's Lemma and Theorem 5, condition B of Theorem 5 holds when T_0 is m-DF$_0$ and T_1 is $(m+1)$-DF$_0$: this means (††). Since m is arbitrary, it follows by Theorem 1 that ω-DCF$_0$ is the model-companion of ω-DF$_0$. Since the m-DCF$_0$ have the properties of quantifier-elimination, completeness, and stability [10], the observations of Medvedev noted earlier allow us to conclude that ω-DCF$_0$ also has these properties. Although each m-DCF$_0$ is actually ω-stable, ω-DCF$_0$ is not even superstable, since if A is a set of constants (in the sense that all of their derivatives are 0), then as σ ranges over A^ω, the sets $\{\partial_m x = \sigma(m) \colon m \in \omega\}$ belong to distinct complete types. □

Let us note by the way that, in Kolchin's Lemma, we cannot establish condition A of Theorem 5 in the stronger form in which the structure $\mathfrak{C}$ is required to be a mere *expansion* to $\mathscr{S}_1$ of $\mathfrak{B}$:

Proposition. *If* $m > 0$*, there is a model* $\mathfrak{K}$ *of* $(m+1)$-DF$_0$ *with a reduct that is included in a model* $\mathfrak{L}$ *of* m-DF$_0$*, while* $\mathfrak{L}$ *does not expand to a model of* $(m+1)$-DF$_0$ *that includes* $\mathfrak{K}$*.*

Proof. We generalize the example of [4] repeated in [13, Ex. 1.2, p. 927]. Suppose K is a pure transcendental extension $\mathbb{Q}(a^\sigma \colon \sigma \in \omega^{m+1})$ of $\mathbb{Q}$. We make this into a model of $(m+1)$-DF$_0$ by requiring $\partial_i a^\sigma = a^{\sigma+\boldsymbol{i}}$ in each case. Let L be the pure transcendental extension $K(b^\tau \colon \tau \in \omega^{m-1})$ of K. We make this into a model of m-DF$_0$ by extending the ∂_i so that, if $i < m-1$, we have $\partial_i b^\tau = b^{\tau+\boldsymbol{i}}$, while $\partial_{m-1} b^\tau$ is the element $a^{(\tau,0,0)}$ of K. Note that indeed if $i < m-1$, then

$$[\partial_i, \partial_{m-1}] b^\tau = \partial_i a^{(\tau,0,0)} - \partial_{m-1} b^{\tau+\boldsymbol{i}} = 0.$$

Suppose, if possible, ∂_m extends to L as well so as to commute with the other ∂_i. Then for any τ in ω^{m-1} we have $\partial_m b^\tau = f(b^\xi \colon \xi \in \omega^{m-1})$ for some polynomial f over K. But then, writing $\partial_\eta f$ for the derivative of f with respect to the variable indexed by η, we have, as by [14, Fact 1.1(0)],

$$\begin{aligned} a^{(\tau,0,1)} &= \partial_m \partial_{m-1} b^\tau \\ &= \partial_{m-1} \partial_m b^\tau \\ &= \partial_{m-1}(f(b^\xi \colon \xi \in \omega^{m-1})) \\ &= \sum_{\eta \in \omega^{m-1}} \partial_\eta f(b^\xi \colon \xi \in \omega^{m-1}) \cdot a^{(\eta,0,0)} + f^{\partial_{m-1}}(b^\xi \colon \xi \in \omega^{m-1}), \end{aligned}$$

where the sum has only finitely many non-zero terms. The polynomial expression $f^{\partial_{m-1}}(b^\xi \colon \xi \in \omega^{m-1})$ cannot have $a^{(\tau,0,1)}$ as a constant term, since this is not $\partial_{m-1} x$ for any x in K. Thus we have obtained an algebraic relation among the b^σ and a^τ; but there can be no such relation. □

Finally, the union of a chain of non-companionable theories may be companionable:

Theorem 7. *In the signature* $\{f\} \cup \{c_k : k \in \omega\}$, *where* f *is a singulary operation-symbol and the* c_k *are constant-symbols, let* T_0 *be axiomatized by the sentences*

$$\forall x\ \forall y\ (fx = fy \to x = y),$$

and, for each k *in* ω,

$$\forall x\ (f^{k+1}x \neq x), \quad \forall x\ (fx = c_k \to x = c_{k+1}), \quad fc_{k+2} = c_{k+1} \to fc_{k+1} = c_k.$$

For each n *in* ω, *let* T_{n+1} *be axiomatized by*

$$T_n \cup \{fc_{n+1} = c_n\}.$$

Then

(1) *each* T_n *is universally axiomatized, and* a fortiori $\forall\exists$, *so it does have existentially closed models;*
(2) *each* T_n *has the Amalgamation Property;*
(3) *every existentially closed model of* T_{n+1} *is an existentially closed model of* T_n;
(4) *no* T_n *is companionable;*
(5) $\bigcup_{n\in\omega} T_n$ *is companionable.*

Proof. Let $\mathfrak{A}_m$ be the model of T_0 with universe $\omega \times \omega$ such that

$$f^{\mathfrak{A}_m}(k, \ell) = (k, \ell + 1), \qquad {c_k}^{\mathfrak{A}_m} = \begin{cases} (k - m, 0), & \text{if } k > m, \\ (0, m - k), & \text{if } k \leqslant m. \end{cases}$$

Let $\mathfrak{A}_\omega$ be the model of T_0 with universe $\mathbb{Z}$ such that

$$f^{\mathfrak{A}_\omega} k = k + 1, \qquad {c_k}^{\mathfrak{A}_\omega} = -k.$$

Then $\mathfrak{A}_m$ is a model of each T_k such that $k \leqslant m$, and $\mathfrak{A}_\omega$ is a model of each T_k. Moreover, each model of T_k consists of a copy of some $\mathfrak{A}_\beta$ such that $k \leqslant \beta \leqslant \omega$, along with some (or no) disjoint copies of ω and $\mathbb{Z}$ in which f is interpreted as $x \mapsto x + 1$. Conversely, every structure of this form is a model of T_k. The β such that $\mathfrak{A}_\beta$ embeds in a given model of T_k is uniquely determined by that model. Consequently, T_k has the Amalgamation Property. Also, a model of T_k is an existentially closed model if and only if includes no copies of ω (outside the embedded $\mathfrak{A}_\beta$): this establishes that every existentially closed model of T_{k+1} is an existentially closed model of T_k.

The existentially closed models of T_k are those models that omit the type $\{\forall y\ fy \neq x\} \cup \{x \neq c_j : j \in \omega\}$. In particular, $\mathfrak{A}_m$ is an existentially closed model of T_k if $k \leqslant m$, but $\mathfrak{A}_m$ is elementarily equivalent to a structure that realizes the given type. Thus T_k is not companionable.

Finally, the model-companion of $\bigcup_{k\in\omega} T_k$ is axiomatized by this theory, together with $\forall x\ \exists y\ fy = x$. □

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