

## ON BIHARMONIC HYPERSURFACES WITH CONSTANT SCALAR CURVATURES IN $\mathbb{S}^5$

YU FU

(Communicated by Michael Wolf)

ABSTRACT. We prove that proper biharmonic hypersurfaces with constant scalar curvature in Euclidean sphere  $\mathbb{S}^5$  must have constant mean curvature. Moreover, we also show that there exist no proper biharmonic hypersurfaces with constant scalar curvature in Euclidean space  $\mathbb{E}^5$  or hyperbolic space  $\mathbb{H}^5$ , which give affirmative partial answers to Chen’s conjecture and the Generalized Chen’s conjecture.

### 1. INTRODUCTION

Biharmonic maps  $\phi : (M^n, g) \longrightarrow (\bar{M}^m, \langle \cdot, \cdot \rangle)$  between Riemannian manifolds are critical points of the bienergy functional

$$E_2(\phi) = \frac{1}{2} \int_M |\tau(\phi)|^2 v_g,$$

where  $\tau(\phi) = \text{trace} \nabla d\phi$  is the tension field of  $\phi$  that vanishes for harmonic maps. The Euler-Lagrange equation associated to the bienergy, which characterizes biharmonic maps, is given by the vanishing of the bitension field

$$\tau_2(\phi) = -\Delta \tau(\phi) - \text{trace } R^{\bar{M}}(d\phi, \tau(\phi))d\phi = 0,$$

where  $R^{\bar{M}}$  is the curvature tensor of  $\bar{M}^m$ . The above equation shows that  $\phi$  is a biharmonic map if and only if its bitension field  $\tau_2(\phi)$  vanishes. Equivalently, for an immersion  $\phi : (M^n, g) \longrightarrow (\bar{M}^m, \langle \cdot, \cdot \rangle)$  between Riemannian manifolds, the mean curvature vector field  $\vec{H}$  satisfies the following fourth order elliptic semi-linear PDE:

$$(1.1) \quad \Delta \vec{H} + \text{trace } R^{\bar{M}}(d\phi, \vec{H})d\phi = 0.$$

In view of (1.1), any minimal immersion, i.e. immersion satisfying  $\vec{H} = 0$ , is biharmonic. The non-harmonic biharmonic immersions are called proper biharmonic.

In a different setting, B. Y. Chen in the middle of the 1980s initiated the study of biharmonic submanifolds in a Euclidean space by the condition  $\Delta \vec{H} = 0$ , where  $\Delta$  is the rough Laplacian operator of submanifolds with respect to the induced metric. Both notions of biharmonic submanifolds in Euclidean spaces coincide with each other.

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Received by the editors February 21, 2014 and, in revised form, September 28, 2014.

2010 *Mathematics Subject Classification*. Primary 53D12, 53C40; Secondary 53C42.

*Key words and phrases*. Biharmonic maps, biharmonic submanifolds, Chen’s conjecture, Generalized Chen’s conjecture.

The author was supported by the NSFC (No.11326068, 71271045,11301059), the project funded by China Postdoctoral Science Foundation (No.2014M560216), and the Excellent Innovation Talents Project of DUFE (No. DUFE2014R26).

Nowadays, the study of biharmonic submanifolds is becoming a very active subject. There is a challenging biharmonic conjecture of B. Y. Chen made in 1991 [8]:

**Chen's conjecture:** *The only biharmonic submanifolds of Euclidean spaces are the minimal ones.*

Due to some non-existence results, Caddeo, Montaldo and Oniciuc [6] made the following Generalized Chen's conjecture in 2001:

**Generalized Chen's conjecture:** *Every biharmonic submanifold of a Riemannian manifold with non-positive sectional curvature is minimal.*

Up to now, Chen's conjecture is still open. Recently, the Generalized Chen's conjecture was proved to be wrong by Y. L. Ou and L. Tang in [19], who constructed examples of proper-biharmonic hypersurfaces in a 5-dimensional space of non-constant negative sectional curvature. However, the Generalized Chen's conjecture is still open in its full generality for ambient spaces with constant sectional curvature. For more recent developments of Chen's conjecture and the Generalized Chen's conjecture, see, for instance [1–3, 10–15, 17, 18].

In contrast, the class of proper biharmonic submanifolds in Euclidean spheres is rather rich and quite interesting. The complete classifications of biharmonic hypersurfaces in  $\mathbb{S}^3$  and  $\mathbb{S}^4$  were obtained in [5, 6]. Moreover, the authors in [4] classified biharmonic hypersurfaces with at most two distinct principal curvatures in  $\mathbb{S}^n$  with arbitrary dimension. Very recently, biharmonic hypersurfaces with three distinct principal curvatures in  $\mathbb{S}^n$  were classified by the author in [16].

In the present paper, we prove that a biharmonic hypersurface with constant scalar curvature in the space forms  $\mathbb{E}^5(c)$  necessarily has constant mean curvature. As an application of this result, we show that biharmonic hypersurfaces with constant scalar curvature in Euclidean space  $\mathbb{E}^5$  and hyperbolic space  $\mathbb{H}^5$  have to be minimal. Hence, these results give affirmative partial answers to Chen's conjecture and the Generalized Chen's conjecture.

## 2. PRELIMINARIES

Let  $x : M^n \rightarrow \mathbb{E}^{n+1}(c)$  be an isometric immersion of a hypersurface  $M^n$  into a space form  $\mathbb{E}^{n+1}(c)$  with constant sectional curvature  $c$ . Denote the Levi-Civita connections of  $M^n$  and  $\mathbb{E}^{n+1}(c)$  by  $\nabla$  and  $\tilde{\nabla}$ , respectively. Let  $X$  and  $Y$  denote vector fields tangent to  $M^n$  and let  $\xi$  be a unit normal vector field. Then the Gauss and Weingarten formulas (cf. [9, 10]) are given, respectively, by

$$(2.1) \quad \tilde{\nabla}_X Y = \nabla_X Y + h(X, Y),$$

$$(2.2) \quad \tilde{\nabla}_X \xi = -AX,$$

where  $h$  is the second fundamental form, and  $A$  is the Weingarten operator. It is well known that the second fundamental form  $h$  and the Weingarten operator  $A$  are related by

$$(2.3) \quad \langle h(X, Y), \xi \rangle = \langle AX, Y \rangle.$$

The mean curvature vector field  $\vec{H}$  is given by

$$(2.4) \quad \vec{H} = \frac{1}{n} \text{trace } h.$$

Moreover, the Gauss and Codazzi equations are given, respectively, by

$$R(X, Y)Z = c(\langle Y, Z \rangle X - \langle X, Z \rangle Y) + \langle AY, Z \rangle AX - \langle AX, Z \rangle AY,$$

$$(\nabla_X A)Y = (\nabla_Y A)X,$$

where  $R$  is the curvature tensor of the hypersurface  $M^n$  and  $(\nabla_X A)Y$  is defined by

$$(2.5) \quad (\nabla_X A)Y = \nabla_X (AY) - A(\nabla_X Y)$$

for all  $X, Y, Z$  tangent to  $M^n$ .

Assume that  $\overline{H} = H\xi$  and  $H$  denotes the mean curvature.

By identifying the tangent and the normal parts of the biharmonic condition (1.1) for hypersurfaces in a space form  $\mathbb{E}^{n+1}(c)$ , we obtain the following characterization result for  $M^n$  to be biharmonic (see also [5, 7]).

**Theorem 2.1.** *The immersion  $x : M^n \rightarrow \mathbb{E}^{n+1}(c)$  of a hypersurface  $M^n$  in an  $n + 1$ -dimensional space form  $\mathbb{E}^{n+1}(c)$  is biharmonic if and only if*

$$(2.6) \quad \begin{cases} \Delta H + H \text{trace } A^2 = ncH, \\ 2A \text{grad} H + n H \text{grad} H = 0. \end{cases}$$

Recall a result on biharmonic hypersurfaces with at most three distinct principal curvatures in  $\mathbb{E}^{n+1}(c)$  in [16] for later use.

**Theorem 2.2.** *Let  $M^n$  be a proper biharmonic hypersurface with at most three distinct principal curvatures in  $\mathbb{E}^{n+1}(c)$ . Then  $M^n$  has constant mean curvature.*

### 3. BIHARMONIC HYPERSURFACES WITH CONSTANT GAUSS SCALAR CURVATURE IN $\mathbb{E}^5(c)$

We restrict ourselves to biharmonic hypersurfaces  $M$  in the 5-dimensional space form  $\mathbb{E}^5(c)$ .

Assume that the mean curvature  $H$  is not constant.

It follows from the second equation of (2.6) that  $\text{grad } H$  is an eigenvector of the shape operator  $A$  with the corresponding principal curvature  $-2H$ . Therefore, without loss of generality, we choose  $e_1$  such that  $e_1$  is parallel to  $\text{grad } H$ , and hence the shape operator  $A$  of  $M$  takes the following form with respect to some suitable orthonormal frame  $\{e_1, e_2, e_3, e_4\}$ :

$$(3.1) \quad Ae_i = \lambda_i e_i,$$

where  $\lambda_1 = -2H$ .

Denote by  $R$  the scalar curvature and by  $B$  the squared length of the second fundamental form  $h$  of  $M$ . It follows from (3.1) that  $B$  is given by

$$(3.2) \quad B = \sum_{i=1}^4 \lambda_i^2 = 4H^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2.$$

From the Gauss equation, the scalar curvature  $R$  is given by

$$(3.3) \quad R = 12c + 16H^2 - B = 12c + 12H^2 - \lambda_2^2 - \lambda_3^2 - \lambda_4^2.$$

We compute  $\text{grad } H$  as

$$\text{grad } H = \sum_{i=1}^4 e_i(H)e_i.$$

Since  $e_1$  is parallel to  $\text{grad } H$ , it follows that

$$(3.4) \quad e_1(H) \neq 0, \quad e_2(H) = e_3(H) = e_4(H) = 0.$$

We set

$$(3.5) \quad \nabla_{e_i} e_j = \sum_{k=1}^4 \omega_{ij}^k e_k, \quad i, j = 1, 2, 3, 4.$$

The compatibility conditions  $\nabla_{e_k} \langle e_i, e_i \rangle = 0$  and  $\nabla_{e_k} \langle e_i, e_j \rangle = 0$  ( $i \neq j$ ) give, respectively, that

$$(3.6) \quad \omega_{ki}^i = 0, \quad \omega_{ki}^j + \omega_{kj}^i = 0,$$

for  $i \neq j$  and  $i, j, k = 1, 2, 3, 4$ . From (3.1) and (3.4), the Codazzi equation leads to

$$(3.7) \quad e_i(\lambda_j) = (\lambda_i - \lambda_j)\omega_{ji}^j,$$

$$(3.8) \quad (\lambda_i - \lambda_j)\omega_{ki}^j = (\lambda_k - \lambda_j)\omega_{ik}^j$$

for distinct  $i, j, k = 1, 2, 3, 4$ .

Since  $\lambda_1 = -2H$ , from (3.4) we compute that

$$[e_2, e_3](\lambda_1) = [e_3, e_4](\lambda_1) = [e_2, e_4](\lambda_1) = 0,$$

which yields directly

$$(3.9) \quad \omega_{ij}^1 = \omega_{ji}^1, \quad i, j = 2, 3, 4 \text{ and } i \neq j.$$

Now we claim that  $\lambda_j \neq \lambda_1$  for  $j = 2, 3, 4$ . In fact, if  $\lambda_j = \lambda_1$  for  $j \neq 1$ , by putting  $i = 1$  in (3.7) we have that

$$(3.10) \quad 0 = (\lambda_1 - \lambda_j)\omega_{j1}^j = e_1(\lambda_j) = e_1(\lambda_1).$$

However, (3.10) contradicts the first expression of (3.4).

According to Theorem 2.1, we only need to deal with the case when  $M$  has four distinct principal curvatures. Hence, we assume that  $M$  has our distinct principal curvatures in the following.

By the definition (2.4) of the mean curvature vector field  $\vec{H}$  and  $\lambda_1 = -2H$ , we have

$$(3.11) \quad \lambda_2 + \lambda_3 + \lambda_4 = 6H$$

for distinct  $\lambda_2, \lambda_3, \lambda_4$  and  $\lambda_i \neq -2H$ .

We now state a lemma to express the connection coefficients of  $M$ .

**Lemma 3.1.** *Let  $M$  be a biharmonic hypersurface with four distinct principal curvatures in space forms  $\mathbb{E}^5(c)$ , whose shape operator is given by (3.1) with respect to an orthonormal frame  $\{e_1, e_2, e_3, e_4\}$ . Then we have*

$$\nabla_{e_1} e_i = 0, \quad i = 1, 2, 3, 4,$$

$$\nabla_{e_i} e_1 = -\omega_{ii}^1 e_i, \quad i = 2, 3, 4,$$

$$\nabla_{e_i} e_i = \sum_{k=1, k \neq i}^4 \omega_{ii}^k e_k, \quad i = 2, 3, 4,$$

$$\nabla_{e_i} e_j = -\omega_{ii}^j e_i + \omega_{ij}^k e_k \text{ for distinct } i, j, k = 2, 3, 4,$$

where

$$\omega_{ii}^j = -\frac{e_j(\lambda_i)}{\lambda_j - \lambda_i}.$$

*Proof.* Consider the equations (3.7) and (3.8).

By putting  $j = 1$  and  $i = 2, 3, 4$  in (3.7), from (3.4) we have  $\omega_{1i}^1 = 0$ , which together with the first expression of (3.6) gives

$$(3.12) \quad \omega_{1i}^1 = 0, \quad i = 1, 2, 3, 4.$$

Combining (3.12) with the second expression of (3.6) gives

$$(3.13) \quad \omega_{11}^i = 0, \quad i = 1, 2, 3, 4.$$

By putting  $j = 1, i, k = 2, 3, 4$  in (3.8), and applying (3.9) we have

$$(3.14) \quad \omega_{ij}^1 = \omega_{ji}^1 = 0,$$

which together with the second expression of (3.6) yields

$$(3.15) \quad \omega_{i1}^j = 0, \quad i, j = 2, 3, 4, \text{ and } i \neq j.$$

By applying (3.8) again, from (3.15) it follows that

$$(3.16) \quad \omega_{1i}^j = 0, \quad i, j = 2, 3, 4, \text{ and } i \neq j.$$

Combining (3.12)–(3.16) with (3.6) and (3.7), we complete the proof of Lemma 3.1. □

Since the Gauss curvature tensor  $R(X, Y)Z$  is defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z,$$

we could compute the curvature tensor  $R$  by Lemma 3.1. On the other hand, by applying the Gauss equation for different values of  $X, Y$  and  $Z$ , and by comparing the coefficients with respect to the orthonormal basis  $\{e_1, e_2, e_3, e_4\}$ , we get the following:

•  $X = e_1, Y = e_2, Z = e_1,$   
 (3.17) 
$$e_1(\omega_{22}^1) - (\omega_{22}^1)^2 = \lambda_1 \lambda_2 + c;$$

•  $X = e_1, Y = e_3, Z = e_1,$   
 (3.18) 
$$e_1(\omega_{33}^1) - (\omega_{33}^1)^2 = \lambda_1 \lambda_3 + c;$$

•  $X = e_1, Y = e_4, Z = e_1,$   
 (3.19) 
$$e_1(\omega_{44}^1) - (\omega_{44}^1)^2 = \lambda_1 \lambda_4 + c;$$

•  $X = e_1, Y = e_3, Z = e_3,$   
 (3.20) 
$$e_1(\omega_{33}^2) = \omega_{33}^1 \omega_{33}^2;$$

•  $X = e_1, Y = e_4, Z = e_4,$   
 (3.21) 
$$e_1(\omega_{44}^2) = \omega_{44}^1 \omega_{44}^2;$$

•  $X = e_2, Y = e_3, Z = e_3,$   
 (3.22) 
$$e_2(\omega_{33}^1) = -\omega_{22}^1 \omega_{33}^2 + \omega_{33}^1 \omega_{33}^2;$$

•  $X = e_2, Y = e_4, Z = e_4,$   
 (3.23) 
$$e_2(\omega_{44}^1) = -\omega_{22}^1 \omega_{44}^2 + \omega_{44}^1 \omega_{44}^2;$$

•  $X = e_2, Y = e_3, Z = e_2,$   
 (3.24) 
$$\begin{aligned} & -e_2(\omega_{33}^2) - e_3(\omega_{22}^3) + \omega_{22}^4 \omega_{33}^4 + (\omega_{22}^3)^2 + (\omega_{33}^2)^2 \\ & + \omega_{22}^1 \omega_{33}^1 - \omega_{24}^3 \omega_{34}^2 - \omega_{24}^3 \omega_{43}^2 + \omega_{34}^2 \omega_{43}^2 = -(c + \lambda_2 \lambda_3); \end{aligned}$$

- $X = e_2, Y = e_4, Z = e_2,$

$$(3.25) \quad \begin{aligned} & -e_4(\omega_{22}^4) - e_2(\omega_{44}^2) + \omega_{22}^3\omega_{44}^3 + (\omega_{22}^4)^2 + (\omega_{44}^2)^2 \\ & + \omega_{22}^1\omega_{44}^1 + \omega_{24}^3\omega_{34}^2 + \omega_{24}^3\omega_{43}^2 + \omega_{34}^2\omega_{43}^2 = -(c + \lambda_2\lambda_4); \end{aligned}$$

- $X = e_3, Y = e_4, Z = e_3,$

$$(3.26) \quad \begin{aligned} & -e_3(\omega_{44}^3) - e_4(\omega_{33}^4) + \omega_{33}^2\omega_{44}^2 + (\omega_{33}^4)^2 + (\omega_{44}^3)^2 \\ & + \omega_{33}^1\omega_{44}^1 + \omega_{24}^3\omega_{34}^2 - \omega_{24}^3\omega_{43}^2 - \omega_{34}^2\omega_{43}^2 = -(c + \lambda_3\lambda_4). \end{aligned}$$

Note that in the above we only state the equations useful for later use.

Consider the first equation of (2.6). It follows from (3.1), (3.3), and Lemma 3.1 that

$$(3.27) \quad -e_1e_1(H) + (\omega_{22}^1 + \omega_{33}^1 + \omega_{44}^1)e_1(H) + H(8c + 16H^2 - R) = 0.$$

Let us compute  $[e_1, e_i](H) = (\nabla_{e_1}e_i - \nabla_{e_i}e_1)(H)$  for  $i = 2, 3, 4$ . From (3.4) and Lemma 3.1, it follows that

$$(3.28) \quad e_i e_1(H) = 0, \quad i = 2, 3, 4.$$

**Lemma 3.2.** *Let  $M$  be a biharmonic hypersurface with four distinct principal curvatures in space forms  $\mathbb{E}^5(c)$ , and whose shape operator is given by (3.1) with respect to an orthonormal frame  $\{e_1, e_2, e_3, e_4\}$ . If the scalar curvature  $R$  is constant, then  $e_i(\lambda_j) = 0$  for  $i = 2, 3, 4$  and  $j = 1, 2, 3, 4$ .*

*Proof.* By the hypothesis, the scalar curvature  $R$  is constant. Differentiating (3.27) along  $e_2$ , from (3.28) we have

$$(3.29) \quad e_2(\omega_{22}^1 + \omega_{33}^1 + \omega_{44}^1) = 0.$$

On the other hand, differentiating (3.11) along  $e_1$ , by (3.7) and the second equation of (3.6) we obtain

$$(3.30) \quad (\lambda_1 - \lambda_2)\omega_{22}^1 + (\lambda_1 - \lambda_3)\omega_{33}^1 + (\lambda_1 - \lambda_4)\omega_{44}^1 = -6e_1(H),$$

which reduces to

$$\omega_{22}^1 + \omega_{33}^1 + \omega_{44}^1 = -\frac{\lambda_2 - \lambda_3}{\lambda_1 - \lambda_2}\omega_{33}^1 - \frac{\lambda_2 - \lambda_4}{\lambda_1 - \lambda_2}\omega_{44}^1 - \frac{6e_1(H)}{\lambda_1 - \lambda_2}.$$

Now letting  $e_2$  act on both sides of the above equation, by (3.11), (3.4), (3.22) and (3.23) we obtain

$$\begin{aligned}
 (3.31) \quad e_2(\omega_{22}^1 + \omega_{33}^1 + \omega_{44}^1) &= \left( \frac{2e_2(\lambda_3) + e_2(\lambda_4)}{\lambda_1 - \lambda_2} + \frac{(\lambda_2 - \lambda_3)(e_2(\lambda_3) + e_2(\lambda_4))}{(\lambda_1 - \lambda_2)^2} \right) \omega_{33}^1 \\
 &+ \left( \frac{e_2(\lambda_3) + 2e_2(\lambda_4)}{\lambda_1 - \lambda_2} + \frac{(\lambda_2 - \lambda_4)(e_2(\lambda_3) + e_2(\lambda_4))}{(\lambda_1 - \lambda_2)^2} \right) \omega_{44}^1 \\
 &+ \frac{6e_1(H)(e_2(\lambda_3) + e_2(\lambda_4))}{(\lambda_1 - \lambda_2)^2} + \frac{\lambda_2 - \lambda_3}{\lambda_1 - \lambda_2} \omega_{33}^2 (\omega_{22}^1 - \omega_{33}^1) \\
 &+ \frac{\lambda_2 - \lambda_4}{\lambda_1 - \lambda_2} \omega_{44}^2 (\omega_{22}^1 - \omega_{44}^1) \\
 &= \frac{(2\lambda_1 - \lambda_2 - \lambda_3)e_2(\lambda_3) + (\lambda_1 - \lambda_3)e_2(\lambda_4)}{(\lambda_1 - \lambda_2)^2} \omega_{33}^1 \\
 &+ \frac{(\lambda_1 - \lambda_4)e_2(\lambda_3) + (2\lambda_1 - \lambda_2 - \lambda_4)e_2(\lambda_4)}{(\lambda_1 - \lambda_2)^2} \omega_{44}^1 \\
 &- \frac{(\lambda_1 - \lambda_2)\omega_{22}^1 + (\lambda_1 - \lambda_3)\omega_{33}^1 + (\lambda_1 - \lambda_4)\omega_{44}^1}{(\lambda_1 - \lambda_2)^2} (e_2(\lambda_3) + e_2(\lambda_4)) \\
 &+ \frac{\lambda_2 - \lambda_3}{\lambda_1 - \lambda_2} \omega_{33}^2 (\omega_{22}^1 - \omega_{33}^1) + \frac{\lambda_2 - \lambda_4}{\lambda_1 - \lambda_2} \omega_{44}^2 (\omega_{22}^1 - \omega_{44}^1).
 \end{aligned}$$

From (3.7) and the second expression of (3.6), we have  $e_2(\lambda_3) = -(\lambda_2 - \lambda_3)\omega_{33}^2$  and  $e_2(\lambda_4) = -(\lambda_2 - \lambda_4)\omega_{44}^2$ . Substituting these into (3.31) gives

$$(3.32) \quad e_2(\omega_{22}^1 + \omega_{33}^1 + \omega_{44}^1) = \frac{2(\lambda_2 - \lambda_3)}{\lambda_1 - \lambda_2} (\omega_{22}^1 - \omega_{33}^1) \omega_{33}^2 + \frac{2(\lambda_2 - \lambda_4)}{\lambda_1 - \lambda_2} (\omega_{22}^1 - \omega_{44}^1) \omega_{44}^2.$$

Combining (3.32) with (3.29) gives

$$(3.33) \quad (\lambda_2 - \lambda_3)(\omega_{22}^1 - \omega_{33}^1) \omega_{33}^2 + (\lambda_2 - \lambda_4)(\omega_{22}^1 - \omega_{44}^1) \omega_{44}^2 = 0.$$

Moreover, differentiating (3.3) along  $e_2$ , by (3.11) and (3.7) we have

$$(3.34) \quad (\lambda_2 - \lambda_3)^2 \omega_{33}^2 + (\lambda_2 - \lambda_4)^2 \omega_{44}^2 = 0.$$

Differentiating (3.34) along  $e_1$ , by applying (3.7), the second expression of (3.6), (3.20) and (3.21) we obtain

$$\begin{aligned}
 (3.35) \quad (\lambda_2 - \lambda_3) &\left[ 2(\lambda_1 - \lambda_2)\omega_{22}^1 - (2\lambda_1 + \lambda_2 - 3\lambda_3)\omega_{33}^1 \right] \omega_{33}^2 \\
 &+ (\lambda_2 - \lambda_4) \left[ 2(\lambda_1 - \lambda_2)\omega_{22}^1 - (2\lambda_1 + \lambda_2 - 3\lambda_4)\omega_{44}^1 \right] \omega_{44}^2 = 0.
 \end{aligned}$$

We claim that  $\omega_{33}^2 = \omega_{44}^2 = 0$ .

In fact, if one of  $\omega_{33}^2$  and  $\omega_{44}^2$  is not vanishing, (3.33) and (3.34) imply that

$$(3.36) \quad (\lambda_3 - \lambda_4)\omega_{22}^1 - (\lambda_2 - \lambda_4)\omega_{33}^1 + (\lambda_2 - \lambda_3)\omega_{44}^1 = 0.$$

Also, (3.34) and (3.35) reduce to

$$\begin{aligned}
 (3.37) \quad 2(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)\omega_{22}^1 &- (\lambda_2 - \lambda_4)(2\lambda_1 + \lambda_2 - 3\lambda_3)\omega_{33}^1 \\
 &+ (\lambda_2 - \lambda_3)(2\lambda_1 + \lambda_2 - 3\lambda_4)\omega_{44}^1 = 0.
 \end{aligned}$$

Eliminating  $\omega_{22}^1$  between (3.36) and (3.37) gives

$$3(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4)(\omega_{33}^1 - \omega_{44}^1) = 0,$$

which yields

$$(3.38) \quad \omega_{33}^1 = \omega_{44}^1.$$

Substituting (3.38) into (3.36), we obtain

$$(3.39) \quad \omega_{22}^1 = \omega_{33}^1.$$

Letting  $e_1$  act on both sides of (3.3) and (3.11), by using (3.7) and the second expression of (3.6) we obtain a relation

$$(3.40) \quad (\lambda_1 - \lambda_2)(2\lambda_2 - \lambda_3 - \lambda_4)\omega_{22}^1 - (\lambda_1 - \lambda_3)(\lambda_2 - 2\lambda_3 + \lambda_4)\omega_{33}^1 - (\lambda_1 - \lambda_4)(\lambda_2 + \lambda_3 - 2\lambda_4)\omega_{44}^1 = 0,$$

which together with (3.38) and (3.39) yields

$$(3.41) \quad [(\lambda_2 - \lambda_3)^2 + (\lambda_2 - \lambda_4)^2 + (\lambda_3 - \lambda_4)^2]\omega_{22}^1 = 0.$$

Since the principal curvatures  $\lambda_i$  ( $i = 1, 2, 3, 4$ ) are mutually different, it follows from (3.41), (3.38) and (3.39) that

$$(3.42) \quad \omega_{22}^1 = \omega_{33}^1 = \omega_{44}^1 = 0.$$

Combining (3.30) with (3.42) gives  $e_1(H) = 0$ , which contradicts the first expression of (3.4).

Therefore, we conclude  $\omega_{33}^2 = \omega_{44}^2 = 0$ . By (3.7), (3.4) and (3.11), we obtain  $e_2(\lambda_i) = 0$  for  $i = 1, 2, 3, 4$ .

With some similar discussions, we could show that  $e_3(\lambda_i) = e_4(\lambda_i) = 0$  for  $i = 1, 2, 3, 4$  as well. This completes the proof of Lemma 3.2. □

We are ready to state the main theorem.

**Theorem 3.3.** *Let  $M$  be a proper biharmonic hypersurface with constant scalar curvature in  $\mathbb{E}^5(c)$ . Then  $M$  has constant mean curvature.*

*Proof.* By Lemma 3.2, equations (3.24)–(3.26), respectively, reduce to

$$(3.43) \quad \omega_{22}^1\omega_{33}^1 - \omega_{24}^3\omega_{34}^2 - \omega_{24}^3\omega_{43}^2 + \omega_{34}^2\omega_{43}^2 = -(c + \lambda_2\lambda_3),$$

$$(3.44) \quad \omega_{22}^1\omega_{44}^1 + \omega_{24}^3\omega_{34}^2 + \omega_{24}^3\omega_{43}^2 + \omega_{34}^2\omega_{43}^2 = -(c + \lambda_2\lambda_4),$$

$$(3.45) \quad \omega_{33}^1\omega_{44}^1 + \omega_{24}^3\omega_{34}^2 - \omega_{24}^3\omega_{43}^2 - \omega_{34}^2\omega_{43}^2 = -(c + \lambda_3\lambda_4).$$

Moreover, it follows from (3.8) and the second expression of (3.6) that

$$(3.46) \quad \omega_{24}^3\omega_{34}^2 = \omega_{24}^3\omega_{43}^2 - \omega_{34}^2\omega_{43}^2,$$

$$(3.47) \quad (\lambda_3 - \lambda_4)\omega_{24}^3 = (\lambda_2 - \lambda_4)\omega_{34}^2.$$

Eliminating  $\omega_{24}^3$ ,  $\omega_{34}^2$  and  $\omega_{43}^2$  from (3.43)–(3.45) by using (3.46), (3.47), (3.11) and (3.3), we obtain

$$(3.48) \quad \omega_{22}^1\omega_{33}^1 + \omega_{22}^1\omega_{44}^1 + \omega_{33}^1\omega_{44}^1 = -12H^2 + 3c - \frac{1}{2}R,$$

$$(3.49) \quad \lambda_3\omega_{22}^1\omega_{44}^1 + \lambda_2\omega_{33}^1\omega_{44}^1 + \lambda_4\omega_{22}^1\omega_{33}^1 = -6cH - 3\lambda_2\lambda_3\lambda_4.$$

By (3.7) and the second expression of (3.6), we rewrite (3.17)–(3.19), respectively, as follows:

$$(3.50) \quad e_1e_1(\lambda_2) + \omega_{22}^1e_1(\lambda_1) + 2(\lambda_1 - \lambda_2)(\omega_{22}^1)^2 + (\lambda_1 - \lambda_2)(\lambda_1\lambda_2 + c) = 0,$$

$$(3.51) \quad e_1e_1(\lambda_3) + \omega_{33}^1e_1(\lambda_1) + 2(\lambda_1 - \lambda_3)(\omega_{33}^1)^2 + (\lambda_1 - \lambda_3)(\lambda_1\lambda_3 + c) = 0,$$

$$(3.52) \quad e_1e_1(\lambda_4) + \omega_{44}^1e_1(\lambda_1) + 2(\lambda_1 - \lambda_4)(\omega_{44}^1)^2 + (\lambda_1 - \lambda_4)(\lambda_1\lambda_4 + c) = 0.$$

Since  $\lambda_1 = -2H$ , eliminating  $e_1e_1(H)$  from (3.27) and (3.50)–(3.52), by (3.30), (3.11) and (3.3) we have

$$(3.53) \quad 4(\omega_{22}^1 + \omega_{33}^1 + \omega_{44}^1)e_1(H) + 48H^3 - 66cH + 9RH - 3\lambda_2\lambda_3\lambda_4 = 0.$$

It follows from (3.53) that (3.27) reduces to

$$(3.54) \quad 4e_1e_1(H) - 16H^3 - 98cH + 13RH - 3\lambda_2\lambda_3\lambda_4 = 0.$$

Now, by the fact  $\lambda_1 = -2H$  and (3.11), (3.40) becomes

$$(3.55) \quad (\lambda_2^2 - 4H^2)\omega_{22}^1 + (\lambda_3^2 - 4H^2)\omega_{33}^1 + (\lambda_4^2 - 4H^2)\omega_{44}^1 = 0.$$

From (3.3) and (3.11), we have that

$$(3.56) \quad \lambda_3\lambda_4 = \frac{1}{2}R - 6c + 12H^2 - 6H\lambda_2 + \lambda_2^2,$$

$$(3.57) \quad \lambda_2\lambda_4 = \frac{1}{2}R - 6c + 12H^2 - 6H\lambda_3 + \lambda_3^2,$$

$$(3.58) \quad \lambda_3\lambda_4 = \frac{1}{2}R - 6c + 12H^2 - 6H\lambda_4 + \lambda_4^2.$$

Hence, from (3.55)–(3.58), (3.7), (3.30) we get

$$(3.59) \quad \begin{aligned} e_1(\lambda_2\lambda_3\lambda_4) &= -(\lambda_1 - \lambda_2)\lambda_3\lambda_4\omega_{22}^1 - (\lambda_1 - \lambda_3)\lambda_2\lambda_4\omega_{33}^1 - (\lambda_1 - \lambda_4)\lambda_2\lambda_3\omega_{44}^1 \\ &= (56H^3 + RH - 12cH + \lambda_2\lambda_3\lambda_4)(\omega_{22}^1 + \omega_{33}^1 + \omega_{44}^1) - 72H^2e_1(H). \end{aligned}$$

Differentiating (3.53) along  $e_1$ , by using (3.17)–(3.19), (3.54), (3.53) and (3.59) we obtain that

$$(3.60) \quad \begin{aligned} (200H^3 + 25RH - 200cH - 3\lambda_2\lambda_3\lambda_4)(\omega_{22}^1 + \omega_{33}^1 + \omega_{44}^1) \\ = (160H^2 + 13R - 78c)e_1(H). \end{aligned}$$

Combining (3.60) with (3.53) gives

$$(3.61) \quad \begin{aligned} 4(e_1(H))^2(160H^2 + 13R - 78c) &= -(48H^3 - 66cH + 9RH \\ &\quad - 3\lambda_2\lambda_3\lambda_4)(200H^3 + 25RH - 200cH - 3\lambda_2\lambda_3\lambda_4). \end{aligned}$$

Now differentiating (3.61) along  $e_1$ , using (3.54), (3.59), (3.60), (3.61) we have an algebraic equation concerning  $H$  and  $\lambda_2\lambda_3\lambda_4$  with constant coefficients

$$(3.62) \quad \begin{aligned} &2040217600H^{10} + (659304960R - 4882549760c)H^8 \\ &+ (3730891264c^2 - 1021023488cR + 69428224R^2)H^6 \\ &+ (-987669696c^3 - 55470688cR^2 + 407658368c^2R + 2493816R^3)H^4 \\ &+ (115086816c^4 - 55092024c^3R + 9593272c^2R^2 - 716326cR^3 + 19162R^4)H^2 \\ &- 74403840H^7K + (105242112c - 15432192R)H^5K \\ &+ (-927984R^2 + 12200976cR - 38310432c^2)H^3K \\ &+ (11289096c^3 - 4544436c^2R + 602004cR^2 - 26364R^3)HK \\ &+ 403200H^4K^2 + (133488c + 16632R)H^2K^2 + 8640HK^3 \\ &+ (186732c^2 - 54990Rc + 3978R^2)K^2 = 0, \end{aligned}$$

where  $K = \lambda_2\lambda_3\lambda_4$ .

If  $K$  is a constant, then (3.62) reduces to an algebraic equation of  $H$  with constant coefficients. Thus, the real function  $H$  satisfies a polynomial equation  $q(H) = 0$  with constant coefficients, therefore it must be a constant. We get a contradiction.

Assume that  $K$  is not constant. Consider an integral curve of  $e_1$  passing through  $p = \gamma(t_0)$  as  $\gamma(t), t \in I$ . According to Lemma 3.2, we can assume  $t = t(K)$  and  $H = H(K)$  in some neighborhood of  $K_0 = K(t_0)$ .

Note that

$$(3.63) \quad \frac{dH}{dK} = \frac{dH}{dt} \frac{dt}{dK} = \frac{e_1(H)}{e_1(K)}.$$

In fact, equations (3.59) and (3.60) could yield

$$(3.64) \quad \frac{e_1(K)}{e_1(H)} = \frac{(56H^3 + RH - 12cH + K)(160H^2 + 13R - 78c)}{(200H^3 + 25RH - 200cH - 3K)} - 72H^2.$$

Differentiating (3.62) with respect to  $K$  and substituting  $\frac{dH}{dK}$  from (3.63) and (3.64), we get another independent algebraic equation of  $H$  and  $K$

$$(3.65) \quad \sum_{i=0}^4 q_i(H)K^i = 0,$$

where  $q_i(H)$  is a polynomial concerning function  $H$ .

We may eliminate  $K^4, K^3, K^2$  and  $K$  from equations (3.62) and (3.65) gradually. At last, we obtain a non-trivial algebraic polynomial equation of  $H$  with constant coefficients. Therefore, we conclude that the real function  $H$  must be a constant, which contradicts our original assumption. This completes the proof of Theorem 3.3.  $\square$

As a corollary, we immediately get the following characterization theorem.

**Theorem 3.4.** *Every biharmonic hypersurface with constant scalar curvature in the 5-dimensional sphere  $S^5$  has constant mean curvature.*

*Remark 3.5.* Since all the known examples of proper biharmonic submanifolds in  $S^n$  have constant mean curvature, Balmuş-Montaldo-Oniciuc in [4] conjectured that *the proper biharmonic hypersurfaces in  $S^{n+1}$  must have constant mean curvature.* Hence, Theorem 3.4 gives an affirmative partial answer to this conjecture.

Consider the cases  $c = 0, -1$  in the calculation above. From the first equation of (2.6), one can easily obtain the following:

**Theorem 3.6.** *There exist no proper biharmonic hypersurfaces with constant scalar curvature in the 5-dimensional Euclidean space  $E^5$  or hyperbolic space  $H^5$ .*

*Remark 3.7.* Theorem 3.6 gives affirmative partial answers to Chen's conjecture and the Generalized Chen's conjecture.

We end this paper with a further remark.

*Remark 3.8.* Replace the condition *constant scalar curvature* by *constant length of the second fundamental form* in Theorem 3.3. In view of expressions (3.2) and (3.3), with quite similar argument as above we could obtain similar conclusions.

#### ACKNOWLEDGMENTS

The author would like to thank the referee for the valuable comments and suggestions to improve this article.

## REFERENCES

- [1] Luis J. Alías, S. Carolina García-Martínez, and Marco Rigoli, *Biharmonic hypersurfaces in complete Riemannian manifolds*, Pacific J. Math. **263** (2013), no. 1, 1–12, DOI 10.2140/pjm.2013.263.1. MR3069073
- [2] A. Balmuş, *Biharmonic maps and submanifolds*, PhD thesis, Università degli Studi di Cagliari, Italy, 2007.
- [3] Adina Balmuş, Stefano Montaldo, and Cezar Oniciuc, *Biharmonic PNMC submanifolds in spheres*, Ark. Mat. **51** (2013), no. 2, 197–221, DOI 10.1007/s11512-012-0169-5. MR3090194
- [4] A. Balmuş, S. Montaldo, and C. Oniciuc, *Classification results for biharmonic submanifolds in spheres*, Israel J. Math. **168** (2008), 201–220, DOI 10.1007/s11856-008-1064-4. MR2448058 (2009j:53079)
- [5] Adina Balmuş, Stefano Montaldo, and Cezar Oniciuc, *Biharmonic hypersurfaces in 4-dimensional space forms*, Math. Nachr. **283** (2010), no. 12, 1696–1705, DOI 10.1002/mana.200710176. MR2560665 (2012b:53131)
- [6] R. Caddeo, S. Montaldo, and C. Oniciuc, *Biharmonic submanifolds of  $S^3$* , Internat. J. Math. **12** (2001), no. 8, 867–876, DOI 10.1142/S0129167X01001027. MR1863283 (2002k:53123)
- [7] R. Caddeo, S. Montaldo, and C. Oniciuc, *Biharmonic submanifolds in spheres*, Israel J. Math. **130** (2002), 109–123, DOI 10.1007/BF02764073. MR1919374 (2003c:53090)
- [8] Bang-Yen Chen, *Some open problems and conjectures on submanifolds of finite type*, Soochow J. Math. **17** (1991), no. 2, 169–188. MR1143504 (92m:53091)
- [9] Bang-Yen Chen, *Pseudo-Riemannian geometry,  $\delta$ -invariants and applications*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2011. With a foreword by Leopold Verstraelen. MR2799371 (2012d:53223)
- [10] Bang-Yen Chen, *Total mean curvature and submanifolds of finite type*, Series in Pure Mathematics, vol. 1, World Scientific Publishing Co., Singapore, 1984. MR749575 (86b:53053)
- [11] Bang-Yen Chen, *Some open problems and conjectures on submanifolds of finite type: recent development*, Tamkang J. Math. **45** (2014), no. 1, 87–108, DOI 10.5556/j.tkmj.45.2014.1564. MR3188077
- [12] Filip Defever, *Hypersurfaces of  $E^4$  with harmonic mean curvature vector*, Math. Nachr. **196** (1998), 61–69, DOI 10.1002/mana.19981960104. MR1657990 (99j:53004)
- [13] Ivko Dimitrić, *Submanifolds of  $E^m$  with harmonic mean curvature vector*, Bull. Inst. Math. Acad. Sinica **20** (1992), no. 1, 53–65. MR1166218 (93g:53087)
- [14] Th. Hasanis and Th. Vlachos, *Hypersurfaces in  $E^4$  with harmonic mean curvature vector field*, Math. Nachr. **172** (1995), 145–169, DOI 10.1002/mana.19951720112. MR1330627 (96c:53085)
- [15] Y. Fu, *Biharmonic hypersurfaces with three distinct principal curvatures in Euclidean space*, accepted in Tohoku Math J. (2014).
- [16] Y. Fu, *Biharmonic hypersurfaces with three distinct principal curvatures in spheres*, Math. Nachr. **288** (2015), No.7, 763–774, DOI 10.1002/mana.201400101
- [17] Guo Ying Jiang, *2-harmonic maps and their first and second variational formulas* (Chinese), Chinese Ann. Math. Ser. A **7** (1986), no. 4, 389–402. An English summary appears in Chinese Ann. Math. Ser. B **7** (1986), no. 4, 523. MR886529 (88i:58039)
- [18] Ye-Lin Ou, *Biharmonic hypersurfaces in Riemannian manifolds*, Pacific J. Math. **248** (2010), no. 1, 217–232, DOI 10.2140/pjm.2010.248.217. MR2734173 (2011i:53097)
- [19] Ye-Lin Ou and Liang Tang, *On the generalized Chen's conjecture on biharmonic submanifolds*, Michigan Math. J. **61** (2012), no. 3, 531–542, DOI 10.1307/mmj/1347040257. MR2975260

SCHOOL OF MATHEMATICS AND QUANTITATIVE ECONOMICS, DONGBEI UNIVERSITY OF FINANCE AND ECONOMICS, DALIAN 116025, PEOPLE'S REPUBLIC OF CHINA

*E-mail address:* yufudufe@gmail.com; yufu@dufe.edu.cn