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THE IMAGES OF MULTILINEAR POLYNOMIALS EVALUATED ON 3×3 MATRICES

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Abstract. Let p be a multilinear polynomial in several noncommuting variables, with coefficients in an algebraically closed field K of arbitrary characteristic. In this paper we classify the possible images of p evaluated on 3×3 matrices. The image is one of the following:

- $\{0\}$,
- the set of scalar matrices,
- a (Zariski-)dense subset of $\mathrm{sl}_3(K)$, the matrices of trace 0,
- a dense subset of $M_3(K)$,
- the set of 3-scalar matrices (i.e., matrices having eigenvalues $(\beta, \beta\varepsilon, \beta\varepsilon^2)$ where ε is a cube root of 1), or
- the set of scalars plus 3-scalar matrices.

1. Introduction

This paper is the continuation of [KBMR12], in which we considered the question, reputedly raised by Kaplansky, of the possible image set $\operatorname{Im} p$ of a polynomial p on matrices.

Conjecture 1. *If p is a multilinear polynomial evaluated on the matrix ring $M_n(K)$, then $\operatorname{Im} p$ is either $\{0\}$, K (viewed as K the set of scalar matrices), $\mathrm{sl}_n(K)$, or $M_n(K)$.*

Here $\mathrm{sl}_n(K)$ is the set of matrices of trace zero.

This subject was investigated by many authors (see [AM57], [BK09], [Chu90], [Kul00], [Ku2], [LZ09]). For review and basic terminology we refer to our previous paper [KBMR12]. (Connections between images of polynomials on algebras and word equations are discussed in [KBKP13]; also see [Lar04], [LS09], [Sha09].)

Recall that a polynomial p (written as a sum of monomials) is called *semi-homogeneous of weighted degree* d with (integer) *weights* $(w_1, \dots, w_m)$ if for each monomial h of p, taking $d_{j,h}$ to be the degree of x_j in h, we have

$$d_{1,h}w_1 + \cdots + d_{n,h}w_n = d.$$

A semi-homogeneous polynomial with weights $(1, 1, \dots, 1)$ is called *homogeneous* of degree d.

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In [KBMR12] we settled Conjecture 1 for $n = 2$ and classified the possible images for semi-homogeneous polynomials:

Theorem 1. *Let $p(x_1, \dots, x_m)$ be a semi-homogeneous polynomial evaluated on the algebra $M_2(K)$ of 2×2 matrices over a quadratically closed field. Then $\operatorname{Im} p$ is either $\{0\}$, K, $\mathrm{sl}_2(K)$, the set of all nonnilpotent matrices in $\mathrm{sl}_2(K)$, or a dense subset of $M_2(K)$ (with respect to Zariski topology).*

A homogeneous polynomial p is called *multilinear* if $d_{j,h} = 1$ for each $1 \le j \le n$ and each monomial h of p (and thus $d = n$).

Examples were given in [KBMR12] of homogeneous (but not multilinear) polynomials whose images do not belong to the classification of Theorem 1.

Our research in this paper continues for the 3×3 case, yielding the following:

Theorem 2. *If p is a multilinear polynomial evaluated on 3×3 matrices, then $\operatorname{Im} p$ is one of the following:*

- $\{0\}$,
- *the set of scalar matrices,*
- $\mathrm{sl}_3(K)$ *(perhaps lacking the diagonalizable matrices of discriminant 0; cf. Remark 7),*
- *a dense subset of $M_3(K)$,*
- *the set of 3-scalar matrices, or*
- *the set of scalars plus 3-scalar matrices.*

2. Images of polynomials

For any polynomial $p \in K\langle x_1, \dots, x_m\rangle$, the *image* of p (in R) is defined as

$$\operatorname{Im} p = \{r \in R : \text{ there exist } a_1, \dots, a_m \in R \text{ such that } p(a_1, \dots, a_m) = r\}.$$

Remark 1. $\operatorname{Im} p$ is invariant under conjugation, since

$$ap(x_1, \dots, x_m)a^{-1} = p(ax_1a^{-1}, ax_2a^{-1}, \dots, ax_ma^{-1}) \in \operatorname{Im} p,$$

for any nonsingular $a \in M_n(K)$.

We recall the following lemmas (for arbitrary n) proved in [KBMR12]:

Lemma 1 ([KBMR12, Lemma 4])**.** *If a_i are matrix units, then $p(a_1, \dots, a_m)$ is either 0, $c \cdot e_{ij}$ for some $i \ne j$, or a diagonal matrix.*

Lemma 2 ([KBMR12, Lemma 5])**.** *The linear span of $\operatorname{Im} p$ is either $\{0\}$, K, sl_n, or $M_n(K)$. If $\operatorname{Im} p$ is not $\{0\}$ or the set of scalar matrices, then for any $i \ne j$ the matrix unit e_{ij} belongs to $\operatorname{Im} p$.*

Another major tool is Amitsur's Theorem [Row80, Theorem 3.2.6, p. 176], that the algebra of generic $n \times n$ matrices (generated by matrices $Y_k = (\xi_{i,j}^{(k)})$ whose entries $\{\xi_{i,j}^{(k)}, 1 \le i, j \le n\}$ are commuting indeterminates) is a noncommutative domain UD whose ring of fractions with respect to the center is a division algebra which we denote as $\widetilde{\mathrm{UD}}$ of dimension n^2 over its center $F_1 := \mathrm{Cent}(\widetilde{\mathrm{UD}})$.

Lemma 3. *Let K be an algebraically closed field of characteristic 0, let I be an ideal of $K[X_1, \dots, X_n]$, and let $V(I) = \{x \in K^n : f(x) = 0 \ \forall f \in I\}$. Let*

$\pi\colon K^n \to K^{n-1}$ be the projection onto the first $n-1$ coordinates. Let I' denote the ideal $I \cap K[X_1, \dots, X_{n-1}]$ of $K[X_1, \dots, X_{n-1}]$. Then:

(1) $\pi(V(I))$ is a Zariski dense subset of $V(I')$;

(2) if there exists a Zariski dense subset W of $V(I')$ such that the preimage $\pi^{-1}(p) \cap V(I)$ of each point $p \in W$ consists of one point, then there exists a rational K-valued function ϕ on $V(I')$ such that all points of a Zariski-dense subset of $V(I)$ have the form $(p, \phi(p))$ where $p \in V(I')$.

If $\mathrm{Char}\,(K) = k > 0$*, then there exists a nonnegative integer ℓ such that $(p, a) \in V(I)$ satisfies $\phi(p) = a^{k^\ell}$ on a Zariski-dense subset of $V(I)$.*

Proof. (1) is by [CLO07, Chapter 3, §2, Theorem 3] and the subsequent remarks.

To prove (2), note that by (1) π induces a field homomorphism (hence an embedding) $K(V(I')) \to K(V(I))$ between the fields of rational functions on the respective varieties. It is enough to show that this is an isomorphism. Indeed, $K(V(I))$ is generated by $K(V(I'))$ and X_n. Moreover, X_n is algebraic over $K(V(I'))$. Let h be the minimal polynomial of X_n over $K(V(I'))$, of degree d. The derivative h' has degree $d-1$, and the discriminant $\mathrm{Discr}(h)$ is, up to a scalar, the resultant of h and h'; it is nonzero since h is irreducible implies that h and h' are relatively prime. Now let U be the open subset of $V(I')$ in which $\mathrm{Discr}(h) \neq 0$ and the coefficients of h are defined. Then each point of U has precisely d distinct π-preimages in $V(I)$. It follows that $d = 1$, as required.

If $\mathrm{Char}\,(K) = k > 0$, we take ℓ such that $h(x) = h_1(x^{k^\ell})$ but h_1' is not identically zero. ☐

Remark 2. Assume $\mathrm{Char}\,(K) = 0$. For a commuting indeterminate t, suppose $f(x_1, \dots, x_m; t)$ is a polynomial taking values under matrix substitutions for the x_i and scalars for t. If there exists a unique t_0 such that $f(x_1, \dots, x_m; t_0) = 0$, then t_0 is a rational function with respect to the entries of x_i. If this t_0 is fixed under simultaneous conjugation of the matrices $x_1, \dots, x_m$, then t_0 is in the center of Amitsur's division algebra $\widetilde{\mathrm{UD}}$, implying $f \in \widetilde{\mathrm{UD}}$. If $\mathrm{Char}\,(K) > 0$, then $t_0^{q^\ell}$ is a rational function for some $\ell \in \mathbb{N}_0$, in the notation of Lemma 3.

Remark 3. In Remark 2 we could take a system of polynomial equations and polynomial inequalities. If t_0 is unique, then it is a rational function (or $t_0^{k^\ell}$ if $\mathrm{Char}\,(K) = k$).

In fact, we need a slight modification of Amitsur's Theorem, which is well known. Viewing

$$\widetilde{\mathrm{UD}} \subseteq M_n\left(F(\xi_{i,j}^{(k)}) : 1 \le i, j \le n,\ k \ge 1\right)$$

we can define the reduced characteristic coefficients of elements of $\widetilde{\mathrm{UD}}$, which by [Row08, Remark 24.67] lie in F_1.

Lemma 4. *Assume* $\mathrm{Char}\,(K) = 0$*. If an element a of $\widetilde{\mathrm{UD}}$ has a unique eigenvalue α (i.e., of multiplicity n), then a is scalar. If* $\mathrm{Char}\,(K) = k \neq 0$*, then a is k^ℓ-scalar for some l.*

Proof. If $\mathrm{Char}\,(K) = 0$, then α is the element of $\widetilde{\mathrm{UD}}$ and $a - \alpha I$ is nilpotent, and thus 0.

If $\mathrm{Char}\,(K) = k$, then α^{k^l} is the element of $\widetilde{\mathrm{UD}}$, therefore $a^{k^l} - \alpha^{k^\ell} I$ is nilpotent, and thus 0. Thus a is k^l-scalar. This is impossible if k is not the devisor of the size of the matrices n. ☐

Lemma 5. *The multiplicity of any eigenvalue of an element a of $\widetilde{\mathrm{UD}}$ must divide n. In particular, when n is odd, a cannot have an eigenvalue of multiplicity 2.*

Proof. Recall [Row06, Remark 4.106] that for any element a in a division algebra, represented as a matrix, the eigenvalues of a occur with the same multiplicity, which thus must divide n. □

We need a slight modification of Amitsur's Theorem, which also is well known (see [KBMR12, Proposition 1] for the details).

Proposition 1. *The algebra of generic matrices with traces is a domain which can be embedded in the division algebra* UD *of central fractions of Amitsur's algebra of generic matrices. Likewise, all of the functions in Donkin's theorem can be embedded in* UD.

For $n > 2$, we also have an easy consequence of the theory of division algebras.

Lemma 6. *Suppose for some polynomial p and some number $q < n$, that p^q takes on only scalar values in $M_n(K)$, over an infinite field K, for n prime. Then p takes on only scalar values in $M_n(K)$.*

Proof. We can view p as an element of the generic division algebra $\widetilde{\mathrm{UD}}$ of degree n, and we adjoin a q-root of 1 to K if necessary. Then p generates a subfield of dimension 1 or n of $\widetilde{\mathrm{UD}}$. The latter is impossible, so the dimension is 1; i.e., p is already central. □

2.1. **The case $M_3(K)$.** Now we turn specifically to the case $n = 3$.

Lemma 7. *We define functions $\omega_k : M_3(K) \to K$ as follows: Given a matrix a, let $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues of a, and denote*

$$\omega_k := \omega_k(a) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le 3} \lambda_{i_1} \dots \lambda_{i_k}.$$

Let $p(x_1, \dots, x_m)$ be a semi-homogeneous, trace-vanishing polynomial.

Consider the rational function $H(x_1, \dots, x_m) = \frac{\omega_2(p(x_1,\dots,x_m))^3}{\omega_3(p(x_1,\dots,x_m))^2}$ (taking values in $K \cup \{\infty\}$). If $\operatorname{Im} H$ is dense in K, then $\operatorname{Im} p$ is dense in sl_3.

Proof. Note that $\omega_2(p)^3$ and $\omega_3(p)^2$ are semi-homogeneous. Thus, $\operatorname{Im} H$ is dense in K iff the image of the pair $(\omega_2(p)^3, \omega_3(p)^2)$ is dense in K^2. But since ω_2 and ω_3 are algebraically independent, so are $\omega_2(p)^3$ and $\omega_3(p)^2$, so we conclude that the image of the pair $(\omega_2(p)^3, \omega_3(p)^2)$ is dense in K^2. Thus, the set of characteristic polynomials of evaluations of p is dense in the space of all possible characteristic polynomials of trace zero matrices. Therefore, the set of all triples $(\lambda_1, \lambda_2, -\lambda_1 - \lambda_2)$ of eigenvalues of matrices from $\operatorname{Im} p$ is dense in the plane $x + y + z = 0$ defined in K^3, implying that $\operatorname{Im} p$ is dense in sl_3. □

Let K be an algebraically closed field. We say that a polynomial p is **trace-vanishing** if each of its evaluations has trace 0; i.e., $\mathrm{tr}(p)$ is a trace identity of p. Also, for $\mathrm{Char}(K) \neq 3$ we fix a primitive cube root $\varepsilon \neq 1$ of 1; when $\mathrm{Char}(K) = 3$ we take $\varepsilon = 1$.

Theorem 3. *Let $p(x_1, \dots, x_m)$ be a semi-homogeneous polynomial which is trace-vanishing on 3×3 matrices. Then $\operatorname{Im} p$ is one of the following:*

- $\{0\}$,
- *the set of scalar matrices (which can occur only if $\operatorname{Char}(K) = 3$),*
- *a dense subset of $\mathrm{sl}_3(K)$, or*
- *the set of 3-scalar matrices, i.e., the set of matrices with eigenvalues $(\gamma, \gamma\varepsilon, \gamma\varepsilon^2)$, where ε is our cube root of 1.*

Proof of Theorem 3. We define the functions $\omega_k : M_n(K) \to K$ as in Lemma 7, and consider the rational function $H = \frac{\omega_2(p(x_1,\dots,x_m))^3}{\omega_3(p(x_1,\dots,x_m))^2}$ (taking values in $K \cup \{\infty\}$).

If $\omega_2(p) = \omega_3(p) = 0$, then each evaluation of p is a nilpotent matrix, contradicting Amitsur's Theorem. Thus, either $\operatorname{Im} H$ is dense in K, or H must be constant.

If $\operatorname{Im} H$ is dense in K, then $\operatorname{Im} p$ is dense in sl_3 by Lemma 7.

So we may assume that H is a constant, i.e., $\alpha\omega_2^3(p) + \beta\omega_3^2(p) = 0$ for some $\alpha, \beta \in K$ not both 0. Fix generic matrices $Y_1, \dots, Y_m$. We claim that the eigenvalues $\lambda_1, \lambda_2, -\lambda_1 - \lambda_2$ of $q := p(Y_1, \dots, Y_m)$ are pairwise distinct. Otherwise either they are all equal, or two of them are equal and the third is not, each of which is impossible by Lemmas 4 and 5 since $q \in \widetilde{\mathrm{UD}}$.

Let $\lambda_1', \lambda_2', -\lambda_1' - \lambda_2'$ be the eigenvalues of another matrix $r \in \operatorname{Im} p$. Thus we have the following:

$$\alpha\omega_2^3(r) + \beta\omega_3^2(r) = \alpha\omega_2^3(q) + \beta\omega_3^2(q) = 0.$$

Therefore we have homogeneous equations on the eigenvalues. Dividing by λ_2^6 and $\lambda_2'^6$ respectively, we have the same two polynomial equations of degree 6 on $\frac{\lambda_1}{\lambda_2}$ and $\frac{\lambda_1'}{\lambda_2'}$, yielding six possibilities for $\frac{\lambda_1'}{\lambda_2'}$. The six permutations of λ_1, λ_2, and $\lambda_3 = -\lambda_1 - \lambda_2$ define six pairwise different $\frac{\lambda_1'}{\lambda_2'}$ unless $(\lambda_1, \lambda_2, \lambda_3)$ is a permutation (multiplied by a scalar) of one of the following triples: $(1, 1, -2)$, $(1, -1, 0)$, $(1, \varepsilon, \varepsilon^2)$. The first case is impossible since the eigenvalues must be pairwise distinct. The second case gives us an element of Amitsur's algebra $\widetilde{\mathrm{UD}}$ with eigenvalue 0 and thus determinant 0, contradicting Amitsur's Theorem. In the third case the polynomial p is 3-scalar. Thus, either p is a 3-scalar polynomial, or each matrix from $\operatorname{Im} p$ will have the same eigenvalues up to permutation and scalar multiple. Note for p being 3-scalar this is true also.

Assume for some $i \in \{2, 3\}$ that tr p^i is not identically zero. Then λ_1^i, λ_2^i, and λ_3^i are three linear functions on tr p^i. Hence we have the PI (polynomial identity) $(p^i - \lambda_1^i)(p^i - \lambda_2^i)(p^i - \lambda_3^i)$. Thus by Amitsur's Theorem, one of the factors is a PI. Hence p^i is a scalar matrix. However $i \neq 2$ by Lemma 5. Hence $i = 3$. In this case the image of p is the set of matrices with eigenvalues $\{(\gamma, \gamma\varepsilon, \gamma\varepsilon^2) : \gamma \in K\}$.

Thus, we may assume that p satisfies $\operatorname{tr}(p^i) = 0$ for $i = 1$, 2, and 3. Now $\omega_1(p) = \operatorname{tr}(p) = 0$ and $2\omega_2(p) = (\operatorname{tr}(p))^2 - \operatorname{tr}(p^2) = 0$.

Hence $\omega_1 = \omega_2 = 0$ if $\operatorname{Char}(K) \neq 2$; in this case ω_3 is either 0 (and hence p is PI) or not 0 (and hence p is 3-scalar).

So assume that $\operatorname{Char}(K) = 2$. Recall that

$$0 = \operatorname{tr}(p^3) = \lambda_1^3 + \lambda_2^3 + \lambda_3^3 = \lambda_1^3 + \lambda_2^3 + \lambda_3^3 - 3\lambda_1\lambda_2\lambda_3 + 3\lambda_1\lambda_2\lambda_3.$$

But $\lambda_1^3 + \lambda_2^3 + \lambda_3^3 - 3\lambda_1\lambda_2\lambda_3$ is a multiple of $\lambda_1 + \lambda_2 + \lambda_3$ (seen by substituting $-(\lambda_1 + \lambda_2)$ for λ_3) and thus equals 0. Thus, $0 = 3\lambda_1\lambda_2\lambda_3 = \lambda_1\lambda_2\lambda_3 = \omega_3(p)$, and the Hamilton-Cayley equation yields $p^3 + \omega_2 p = 0$. Therefore, $p(p^2 + \omega_2) = 0$ and by Amitsur's Theorem either p is PI, or $p^2 = -\omega_2$ (which is central), implying by Lemma 6 that p is central. □

Example 1. The element $[x,[y,x]x[y,x]^{-1}]$ of $\widetilde{\mathrm{UD}}$ takes on only 3-scalar values (see [Row80, Theorem 3.2.21, p. 180]) and thus gives rise to a homogeneous polynomial taking on only 3-scalar values.

Now we consider the possible image sets of multilinear trace-vanishing polynomials.

Lemma 8. *If p is a multilinear polynomial, not PI nor central, then there exists a collection of matrix units $(E_1, E_2, \dots, E_m)$ such that $p(E_1, E_2, \dots, E_m)$ is a diagonal but not scalar matrix.*

Proof. By Lemmas 1 and 2, the linear span of all $p(E_1, E_2, \dots, E_m)$ for any matrix units E_i such that $p(E_1, E_2, \dots, E_m)$ is diagonal includes all $\mathrm{Diag}\{x, y, -x-y\}$. In particular there exists a collection of matrix units $(E_1, E_2, \dots, E_m)$ such that $p(E_1, E_2, \dots, E_m)$ is a diagonal but not scalar matrix. □

Theorem 4. *Let p be a multilinear polynomial which is trace-vanishing on 3×3 matrices over a field K of arbitrary characteristic. Then $\mathrm{Im}\, p$ is one of the following:*

- $\{0\}$,
- *the set of scalar matrices,*
- *the set of 3-scalar matrices, or*
- *for each triple $\lambda_1 + \lambda_2 + \lambda_3 = 0$ there exists a matrix $M \in \mathrm{Im}\, p$ with eigenvalues λ_1, λ_2, and λ_3.*

Proof. If the polynomial $\omega_2(p)$ (defined in the proof of Theorem 3) is identically zero, then the characteristic polynomial is $p^3 - \omega_3(p) = 0$, implying p is either scalar (which can happen only if $\mathrm{Char}\,(K) = 3$) or 3-scalar. Therefore we may assume that the polynomial $\omega_2(p)$ is not identically zero. Let

$$f_{\alpha,\beta}(M) = \alpha\omega_2(M)^3 + \beta\omega_3(M)^2.$$

It is enough to show that for any $\alpha, \beta \in K$ there exists a nonnilpotent matrix $M = p(a_1, \dots, a_m)$ such that $f_{\alpha,\beta}(p(a_1, \dots, a_m)) = 0$, since this will imply that the image of H (defined in Lemma 7) contains all $-\frac{\beta}{\alpha}$ and thus $K \cup \{\infty\}$. (For example, if $\alpha = 0$ and $\beta \neq 0$, then $\omega_3(M) = 0$, implying $\omega_2(M) \neq 0$ since $\omega_1(M) = 0$ and M is nonnilpotent, and thus $H = \infty$.) Therefore, for any trace-vanishing polynomial (i.e., a polynomial $x^3 + \gamma_1 x + \gamma_0$) there is a matrix in $\mathrm{Im}\, p$ for which this is the characteristic polynomial. Hence whenever $\lambda_1 + \lambda_2 + \lambda_3 = 0$ there is a matrix with eigenvalues λ_i.

Without loss of generality we may assume that $a = p(Y_1, \dots, Y_m)$ and $b = p(\tilde{Y}_1, Y_2 \dots, Y_m)$ are not proportional for generic matrices $\tilde{Y}_1, Y_1, \dots, Y_m$; cf. [BMR2, Lemma 2]. Consider the polynomial $\varphi_{\alpha,\beta}(t) = f_{\alpha,\beta}(a + tb)$. There are three cases to consider:

Case I. $\varphi_{\alpha,\beta} = 0$ identically. Then $f_{\alpha,\beta}(a) = 0$, and a is not nilpotent by Proposition 1.

Case II. $\varphi_{\alpha,\beta}$ is a constant $\tilde{\beta} \neq 0$. Then $f_{\alpha,\beta}(b + ta) = t^6\varphi_{\alpha,\beta}(t^{-1}) = \tilde{\beta}t^6$; thus $f_{\alpha,\beta}(b) = 0$, and b is not nilpotent by Proposition 1.

Case III. $\varphi_{\alpha,\beta}$ is not constant. Then it has finitely many roots. Assume that for each substitution t the matrix $a+tb$ is nilpotent; in particular, $\omega_2(a+tb)=0$. Note that $\omega_2(a+tb)$ equals the sum of principal 2×2 minors and thus is a quadratic polynomial (for otherwise $\omega_2(b)=0$ which means that $\omega_2(p)$ is identically zero, a contradiction). Hence $\omega_2(a+tb)$ has two roots, which we denote as t_1 and t_2. If $t_1=t_2$, then t_1 is uniquely defined and thus, in view of Remark 2, is a rational function in the entries of a and b, and $a+t_1b$ is a nilpotent rational function (because we assumed that one of $a+t_1b$ or $a+t_2b$ is nilpotent, but here they are equal). At least one of t_1 or t_2 is a root of $\varphi_{\alpha,\beta}$.

If only t_1 is a root, then t_1 is uniquely defined and thus, by Remark 2, is a rational function; hence, $a+t_1b$ is a nilpotent polynomial, contradicting Proposition 1. Thus, we may assume that both t_1 and t_2 are roots of $\varphi_{\alpha,\beta}$. But $\varphi_{\alpha,\beta}(t_i)$ is nilpotent, and in particular $\omega_3(a+t_ib)=0$. Thus there exists exactly one more root t_3 of $\omega_3(a+tb)$, which is uniquely defined and thus, by Remark 2, is rational. Hence we may consider the polynomial $q(x_1,\dots,x_m,\tilde{x}_1)=a+t_3b$, which must satisfy the condition $\mathrm{tr}(q)=\det(q)=0$. This is impossible for homogeneous q by Theorem 3, and also impossible for nonhomogeneous q since the leading homogenous component q_d would satisfy $\mathrm{tr}(q_d)=\det(q_d)=0$, a contradiction. $\square$

Remark 4. Assume that $\mathrm{Char}(K)=3$ and p is a multilinear polynomial, which is neither PI nor central. Then, according to Lemma 8 there exists a collection of matrix units E_i such that

$$p(E_1,\dots,E_m)=\mathrm{Diag}\{\alpha,\beta,\gamma\}$$

is diagonal but not scalar. Without loss of generality, $\alpha\neq\beta$. Hence $p^3(E_1,\dots,E_m)$ $=\mathrm{Diag}\{a^3,\beta^3,\gamma^3\}$ and $\alpha^3\neq\beta^3$ because $\mathrm{Char}(K)=3$. Therefore p is not 3-scalar.

Theorem 5. *If there exist α, β, and γ in K such that $\alpha+\beta+\gamma$, $\alpha+\beta\varepsilon+\gamma\varepsilon^2$, and $\alpha+\beta\varepsilon^2+\gamma\varepsilon$ are nonzero, together with matrix units $E_1,E_2,\dots,E_m$ such that $p(E_1,E_2,\dots,E_m)$ has eigenvalues α, β, and γ, then* $\mathrm{Im}\,p$ *is dense in* M_3.

Proof. Define χ to be the permutation of the set of matrix units, sending the indices $1\to 2$, $2\to 3$, and $3\to 1$. For example, $\chi(e_{12})=e_{23}$. For triples $T_1,\dots,T_m$ (each $T_i=(t_{i,1},t_{i,2},t_{i,3})$) consider the function

$$\begin{aligned}(1)\quad f(T_1,\dots,T_m)=p(&t_{1,1}x_1+t_{1,2}\chi(x_1)+t_{1,3}\chi^{-1}(x_1),t_{2,1}x_2+t_{2,2}\chi(x_2)\\&+t_{2,3}\chi^{-1}(x_2),\dots,t_{m,1}x_m+t_{m,2}\chi(x_m)+t_{m,3}\chi^{-1}(x_m)).\end{aligned}$$

Opening the brackets, we have many terms, each of which we claim is a diagonal matrix. Each term is a monomial with coefficient of the type

$$\chi^{k_{\pi(1)}}\chi^{k_{\pi(2)}}\cdots\chi^{k_{\pi(m)}}x_{\pi(1)}x_{\pi(2)}\cdots x_{\pi(m)},$$

where k_i is $-1,0$, or 1, and π is a permutation. Since we substitute only matrix units in p, by Lemma 1 the image is either diagonal or a matrix unit with some coefficient. For each of the three vertices v_1,v_2,v_3 in our graph define the index ι_ℓ, for $1\le\ell\le 3$, to be the number of incoming edges to v_ℓ minus the number of outgoing edges from v_ℓ. Thus, at the outset, when the image is diagonal, we have $\iota_1=\iota_2=\iota_3=0$.

We claim that after applying χ to any matrix unit the new ι'_ℓ will all still be congruent modulo 3. Indeed, if the edge $\vec{12}$ is changed to $\vec{23}$, then $\iota'_1=\iota+1$ and $\iota'_3=\iota_3+1$, whereas $\iota'_2=\iota_2-2\equiv\iota_2+1$. The same with changing $\vec{23}$ to $\vec{31}$ and

$\vec{31}$ to $\vec{12}$. If we make the opposite change $\vec{21}$ to $\vec{13}$, then (modulo 3) we subtract 1 throughout. If we make a change of the type $\vec{ii} \mapsto \vec{jj}$, then $\iota'_\ell = \iota_\ell$ for each ℓ.

If $p(\chi^{k_1}x_1, \chi^{k_2}x_2, \dots, \chi^{k_m}x_m) = e_{ij}$, this means that the number of incoming edges minus the number of outgoing edges of the vertex i is $-1 \pmod 3$ and the number of incoming edges minus the number of outgoing edges of j is $1 \pmod 3$, which are not congruent modulo 3. Thus the values of the mapping f defined in (1) are diagonal matrices. Now fix $3m$ algebraically independent triples $T_1, \dots, T_m, \Theta_1, \dots, \Theta_m, \Upsilon_1, \dots, \Upsilon_m$. Assume that $\operatorname{Im} f$ is 2-dimensional. Then $\operatorname{Im} df$ must also be 2-dimensional at any point. Consider the differential df at the point $(\Theta_1, T_2, \dots, T_m)$. Thus,

$$f(\Theta_1, T_2, \dots, T_m),\ f(T_1, T_2, \dots, T_m),\ f(\Theta_1, \Theta_2, \dots, T_m)$$

belong to $\operatorname{Im}\ df$. Thus these three matrices must span a linear space of dimension not more than 2. Hence they lie in some plane P. Now take

$$f(\Theta_1, \Theta_2, T_3, \dots, T_m),\ f(\Theta_1, T_2, T_3, \dots, T_m),\ f(\Theta_1, \Theta_2, \Theta_3, T_4, \dots, T_m).$$

For the same reason they lie in a plane, which is the plane P because it has two vectors from P. By the same argument, we conclude that all the matrices of type $f(\Theta_1, \dots, \Theta_k, T_{k+1}, \dots, T_m)$ lie in P. Now we see that

$$f(\Theta_1, \dots, \Theta_{m-1}, T_m),\ f(\Theta_1, \dots, \Theta_m),\ f(\Upsilon_1, \Theta_2, \dots, \Theta_m)$$

also lie in P. Analogously we obtain that also

$$f(\Upsilon_1, \dots, \Upsilon_k, \Theta_{k+1}, \dots, \Theta_m) \in P$$

for any k.

Hence for $3m$ algebraically independent triples

$$T_1, \dots, T_m; \Theta_1, \dots, \Theta_m; \Upsilon_1, \dots, \Upsilon_m,$$

we have obtained that $f(T_1, \dots, T_M)$, $f(\Theta_1, \dots, \Theta_m)$, and $f(\Upsilon_1, \dots, \Upsilon_m)$ lie in one plane. Thus any three values of f, in particular $\operatorname{Diag}\{\alpha, \beta, \gamma\}$, $\operatorname{Diag}\{\beta, \gamma, \alpha\}$, and $\operatorname{Diag}\{\gamma, \alpha, \beta\}$, must lie in one plane. We claim that this can happen only if

$$\alpha + \beta + \gamma = 0, \quad \alpha + \beta\varepsilon + \gamma\varepsilon^2 = 0, \quad \text{or} \quad \alpha + \beta\varepsilon^2 + \gamma\varepsilon = 0.$$

Indeed, $\operatorname{Diag}\{\alpha, \beta, \gamma\}$, $\operatorname{Diag}\{\beta, \gamma, \alpha\}$, and $\operatorname{Diag}\{\gamma, \alpha, \beta\}$ are dependent if and only if the matrix

$$\begin{pmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{pmatrix}$$

is singular, i.e., its determinant $3\alpha\beta\gamma - (\alpha^3 + \beta^3 + \gamma^3) = 0$. But this has the desired three roots when viewed as a cubic equation in γ.

We have a contradiction to our hypothesis. □

Remark 5. If there exist α, β, and γ such that $\alpha + \beta + \gamma = 0$ but (α, β, γ) is not proportional to $(1, \varepsilon, \varepsilon^2)$ or $(1, \varepsilon^2, \varepsilon)$, with matrices $E_1, E_2, \dots, E_m$ such that $p(E_1, E_2, \dots, E_m)$ has eigenvalues α, β, and γ, then either all diagonalizable trace zero matrices lie in $\operatorname{Im} p$, or $\operatorname{Im} p$ is dense in $M_3(K)$. If $\alpha + \beta\varepsilon + \gamma\varepsilon^2 = 0$ but (α, β, γ) is not proportional to $(1, \varepsilon, \varepsilon^2)$ or $(1, 1, 1)$, then all diagonalizable matrices with eigenvalues $\alpha + \beta$, $\alpha + \beta\varepsilon$, and $\alpha + \beta\varepsilon^2$ lie in $\operatorname{Im} p$ or $\operatorname{Im} p$ is dense in $M_3(K)$.

Remark 6. The proof of Theorem 5 works also for any field K of characteristic 3. In this case $\varepsilon = 1$. Hence, if there are α, β, and γ in K such that

$$\alpha + \beta + \gamma \neq 0,$$

together with matrix units $E_1, E_2, \dots, E_m$ such that $p(E_1, E_2, \dots, E_m)$ has eigenvalues α, β, and γ, then $\operatorname{Im} p$ is dense in M_3. Therefore, for $\operatorname{Char}(K) = 3$, any multilinear polynomial p is either trace-vanishing or $\operatorname{Im} p$ is dense in $M_3(K)$.

Theorem 6. *If p is a multilinear polynomial such that $\operatorname{Im} p$ does not satisfy the equation $\gamma\omega_1(p)^2 = \omega_2(p)$ for $\gamma = 0$ or $\gamma = \frac{1}{4}$, then $\operatorname{Im} p$ contains a matrix with two equal eigenvalues that is not diagonalizable and of determinant not zero. If $\operatorname{Im} p$ does not satisfy any equation of the form $\gamma\omega_1(p)^2 = \omega_2(p)$ for any γ, then the set of nondiagonalizable matrices of $\operatorname{Im} p$ is Zariski dense in the set of all nondiagonalizable matrices, and $\operatorname{Im} p$ is dense.*

Proof. If not, then by [BMR2, Lemma 2] there is at least one variable (say, x_1) such that $a = p(x_1, x_2, \dots, x_m)$ does not commute with $b = p(\tilde{x}_1, x_2, \dots, x_m)$. Consider the matrix $a + tb = p(x_1 + t\tilde{x}_1, x_2, \dots, x_m)$, viewed as a polynomial in t.

Recall that the discriminant of a 3×3 matrix with eigenvalues $\lambda_1, \lambda_2, \lambda_3$ is defined as $\prod_{1\leq i<j\leq 3}(\lambda_i - \lambda_j)^2$. Thus, the discriminant of $a + tb$ is a polynomial $f(t)$ of degree 6. If $f(t)$ has only one root t_0, then this root is defined in terms of the entries of $\tilde{x}_1, x_1, x_2, \dots, x_m$, and invariant under the action of the symmetric group, and thus is in Amitsur's division algebra $\widetilde{\mathrm{UD}}$. By Lemma 4, $a + t_0 b$ is scalar, and the uniqueness of t_0 implies that a and b are scalar, contrary to assumption.

Thus, $f(t)$ has at least two roots–say, $t_1 \neq t_2$–and the matrices $a+t_1 b$ and $a+t_2 b$ each must have multiple eigenvalues. If both of these matrices are diagonalizable, then each of $a + t_i b$ have a 2-dimensional plane of eigenvectors. Therefore we have two 2-dimensional planes in 3-dimensional linear space, which must intersect. Hence there is a common eigenvector of both $a + t_i b$ and this is a common eigenvector of a and b. If a and b have a common eigenspace of dimension 1 or 2, then there is at least one eigenvector (and thus eigenvalue) of a that is uniquely defined, implying $a \in \widetilde{\mathrm{UD}}$ by Remark 2, contradicting Lemma 4. If a and b have a common eigenspace of dimension 3, then a and b commute, a contradiction.

We claim that there cannot be a diagonalizable matrix with equal eigenvalues on the line $a + tb$. Indeed, if there were such a matrix, then either it would be unique (and thus an element of $\widetilde{\mathrm{UD}}$, which cannot happen), or there would be at least two such diagonalizable matrices, which also cannot happen, as shown above.

Assume that all matrices on the line $a+tb$ of discriminant zero have determinant zero. Then either all of them are of the type $\operatorname{Diag}\{\lambda, \lambda, 0\} + e_{12}$ or all of them are of the type $\operatorname{Diag}\{0, 0, \mu\} + e_{12}$. (Indeed, there are three roots of the determinant equation $\det(a + tb) = 0$, which are pairwise distinct, and all of them give a matrix with two equal eigenvalues, all belonging to one of these types, since otherwise one eigenvalue is uniquely defined and thus yields an element of $\widetilde{\mathrm{UD}}$, which cannot happen.

In the first case, all three roots of the determinant equation $\det(a+tb) = 0$ satisfy the equation $(\omega_1(a+tb))^2 = 4\omega_2(a+tb)$. Hence, we have three pairwise distinct roots of the polynomial of maximal degree 2, which can occur only if the polynomial is identically zero. It follows that also $(\omega_1(a))^2 - 4\omega_2(a) = 0$, so $(\omega_1(p))^2 - 4\omega_2(p) = 0$ is identically zero, which by hypothesis cannot happen.

In the second case we have the analogous situation, but $\omega_2(p)$ will be identically zero, a contradiction.

Thus on the line $a+tb$ we have at least one matrix of the type $\mathrm{Diag}\{\lambda,\lambda,\mu\}+e_{12}$ and $\lambda\mu \neq 0$. Consider the algebraic expression $\mu\lambda^{-1}$. If not constant, then it takes on almost all values, so assume that it is a constant δ. Then $\delta \neq -2$, since otherwise this matrix will be the unique matrix of trace 0 on the line $a+tb$ and thus an element of $\widetilde{\mathrm{UD}}$, contrary to Lemmas 4 and 5. Consider the polynomial $q = p - \frac{\mathrm{tr} p}{\delta+2}$. At the same point t it takes on the value $\mathrm{Diag}\{0,0,(\delta-1)\lambda\}+e_{12}$. Hence all three pairwise distinct roots of the equation $\det q(x_1+t\tilde{x}_1, x_2, \dots, x_m) = 0$ will give us a matrix of the form $\mathrm{Diag}\{0,0,*\}+e_{12}$ (otherwise we have uniqueness and thus an element of $\widetilde{\mathrm{UD}}$), contradicting Lemma 5. Therefore q satisfies an equation $\omega_2(q) = 0$. Hence, p satisfies an equation $\omega_1(p)^2 - c\omega_2(p) = 0$, for some constant c, a contradiction. Hence almost all nondiagonalizable matrices belong to the image of p, and they are almost all matrices of discriminant 0 (a subvariety of $M_3(K)$ of codimension 1). By Amitsur's Theorem, $\mathrm{Im}\, p$ cannot be a subset of the discriminant surface. Thus, $\mathrm{Im}\, p$ is dense in $M_3(K)$. □

Remark 7. Note that if $\omega_1(p)$ is identically zero, and $\omega_2(p)$ is not identically zero, then $\mathrm{Im}\, p$ contains a matrix similar to $\mathrm{Diag}\{1,1,-2\}+e_{12}$. Hence $\mathrm{Im}\, p$ contains all diagonalizable trace zero matrices (perhaps with the exception of the diagonalizable matrices of discriminant 0, i.e., matrices similar to $\mathrm{Diag}\{c,c,-2c\}$), all nondiagonalizable nonnilpotent trace zero matrices, and all matrices N for which $N^2 = 0$. Nilpotent matrices of order 3 also belong to the image of p, as we shall see in Lemma 10.

3. Proof of the main theorem

Lemma 9. *A matrix is* 3*-scalar iff its eigenvalues are in* $\{\gamma, \gamma\varepsilon, \gamma\varepsilon^2 : \gamma \in K\}$*, where* $\gamma^3 \in K$ *is its determinant. The variety* V_3 *of* 3*-scalar matrices has dimension* 7*.*

Proof. The first assertion is immediate since the characteristic polynomial is $x^3 - \gamma^3$. Hence V_3 is a variety. The second assertion follows since the invertible elements of V_3 are defined by two equations: $\mathrm{tr}(x) = 0$ and $\mathrm{tr}(x^{-1}) = 0$ and thus V_3 is a variety of codimension 2. □

Lemma 10. *Assume* $\mathrm{Char}\,(K) \neq 3$*. If* p *is neither PI nor central, then the variety* V_3 *is contained in* $\mathrm{Im}\, p$*.*

Proof. According to Lemma 2 there exist matrix units $E_1, E_2, \dots, E_m$ such that $p(E_1, E_2, \dots, E_m) = e_{1,2}$. Consider the mapping χ described in the proof of Theorem 5. For any triples $T_i = (t_{1,i}, t_{2,i}, t_{3,i})$, let

$$f(T_1, T_2, \dots, T_m) = p(\dots, t_{1,i}E_i + t_{2,i}\chi(E_i) + t_{3,i}\chi^2(E_i), \dots).$$

$\mathrm{Im}\, f$ (a subset of $\mathrm{Im}\, p$) is a subset of the 3-dimensional linear space

$$L = \{\alpha e_{12} + \beta e_{23} + \gamma e_{31},\ \alpha, \beta, \gamma \in K\}.$$

Since e_{12}, e_{23}, and e_{31} belong to $\mathrm{Im}\, f$, we see that $\mathrm{Im}\, f$ is dense in L, and hence at least one matrix $a = \alpha e_{12} + \beta e_{23} + \gamma e_{31}$ for $\alpha\beta\gamma \neq 0$ belongs to $\mathrm{Im}\, p$. Note that this matrix is 3-central. Thus the variety V_3, excluding the nilpotent matrices, is contained in $\mathrm{Im}\, p$. The nilpotent matrices of order 2 also belong to the image of p since they are similar to e_{12}.

Let us show that all nilpotent matrices of order 3 (i.e., matrices similar to $e_{12}+e_{23}$), also belong to $\operatorname{Im} p$. We have the multilinear polynomial

$$\begin{aligned} f(T_1,T_2,\dots,T_m) = q(T_1,T_2,\dots,T_m)e_{12} + r(T_1,T_2,\dots,T_m)e_{23} \\ + s(T_1,T_2,\dots,T_m)e_{31}, \end{aligned}$$

and therefore q, r, and s are three scalar multilinear polynomials. Assume there is no nilpotent matrix of order 3 in $\operatorname{Im} p$. Then we have the following: if $q = 0$, then $rs = 0$; if $r = 0$, then $sq = 0$; and if $s = 0$, then $qr = 0$. Assume q_1 is the greatest common divisor of q and r and $q_2 = \frac{q}{q_1}$. Note that both q_i are multilinear polynomials defined on disjoint sets of variables. If $q_1 = 0$, then $r = 0$, and if $q_2 = 0$, then $s = 0$. Note there are no double efficients, and thus $r = q_1 r'$ is a multiple of q_1 and $s = q_2 s'$ is a multiple of q_2. The polynomial r' cannot have common divisors with q_2, therefore if we consider any generic point $(T_1,\dots,T_m)$ on the surface $r' = 0$, then $r(T_1,\dots,T_m) = 0$ and $q(T_1,\dots,T_m) \neq 0$. Hence $s(T_1,\dots,T_m) = 0$ for any generic $(T_1,\dots,T_m)$ from the surface $r' = 0$. Therefore r' is the divisor of s. Recall that both q_1 and q_2 are multilinear polynomials defined on disjoint subsets of $\{T_1,T_2,\dots,T_m\}$. Without loss of generality $q_1 = q_1(T_1,\dots,T_k)$ and $q_2 = q_2(T_{k+1},\dots,T_m)$. Therefore $r' = r'(T_{k+1},\dots,T_m)$ and it is divisor of s. Also recall that $s = s'q_2$ so $q_2(T_{k+1},\dots,T_m)$ is also a divisor of s. Hence $r' = cq_2$ where c is constant. Thus $r = q_1 r' = cq_1q_2 = cq$. However there exist $(T_{k+1},\dots,T_m)$ such that $q = 0$ and $r = 1$ (i.e., such that $f(T_{k+1},\dots,T_m) = e_{23}$). A contradiction. $\square$

Remark 8. When $\operatorname{Char}(K) = 3$, then V_3 is the space of the matrices with equal eigenvalues (including also scalar matrices). The same proof shows that all nilpotent matrices belong to the image of p, as well as all matrices similar to $cI + e_{12} + e_{23}$. But we do not know how to show that scalar matrices and matrices similar to $cI + e_{12}$ belong to the image of p.

Proof of Theorem 2. First assume that $\operatorname{Char}(K) \neq 3$. According to Lemma 10 the variety V_3 is contained in $\operatorname{Im} p$. Therefore $\operatorname{Im} p$ is either the set of 3-scalar matrices, or some 8-dimensional variety (with 3-scalar subvariety), or is 9-dimensional (and thus dense).

It remains to classify the possible 8-dimensional images. Let us consider all matrices $p(E_1,\dots,E_m)$ where E_i are matrix units. If all such matrices have trace 0, then $\operatorname{Im} p$ is dense in $\mathrm{sl}_3(K)$, by Theorem 4. Therefore we may assume that at least one such matrix a has eigenvalues α, β, and γ such that $\alpha+\beta+\gamma \neq 0$. By Theorem 5 we cannot have $\alpha+\beta+\gamma$, $\alpha+\beta\varepsilon+\gamma\varepsilon^2$, and $\alpha+\beta\varepsilon^2+\gamma\varepsilon$ all nonzero. Hence a either is scalar, or a linear combination (with nonzero coefficients) of a scalar matrix and $\mathrm{Diag}\{1,\varepsilon,\varepsilon^2\}$ (or with $\mathrm{Diag}\{1,\varepsilon^2,\varepsilon\}$, without loss of generality–with $\mathrm{Diag}\{1,\varepsilon,\varepsilon^2\}$). By Theorem 6, if $\operatorname{Im} p$ is not dense, then p satisfies an equation of the type $(\mathrm{tr}(p))^2 = \gamma \mathrm{tr}(p^2)$ for some $\gamma \in K$. Therefore, if a scalar matrix belongs to $\operatorname{Im} p$, then $\gamma = \frac{1}{3}$ and $\operatorname{Im} p$ is the set of 3-scalar plus scalar matrices. If the matrix a is not scalar, then it is a linear combination of a scalar matrix and $\mathrm{Diag}\{1,\varepsilon,\varepsilon^2\}$. Hence, by Remark 5, $\operatorname{Im} p$ is also the set of 3-scalar plus scalar matrices. In any case, we have shown that $\operatorname{Im} p$ is either $\{0\}$, K, the set of 3-scalar matrices, the set of 3-scalar plus scalar matrices (matrices with eigenvalues $(\alpha+\beta, \alpha+\beta\varepsilon, \alpha+\beta\varepsilon^2)$), $\mathrm{sl}_3(K)$ (perhaps lacking nilpotent matrices of order 3), or is dense in $M_3(K)$.

If $\operatorname{Char}(K) = 3$, then by Remark 6 the multilinear polynomial p is either trace-vanishing or $\operatorname{Im} p$ is dense in $M_3(K)$. If p is trace-vanishing, then, by Theorem 4,

$\operatorname{Im} p$ is one of the following: $\{0\}$, the set of scalar matrices, the set of 3-scalar matrices, or for each triple $\lambda_1 + \lambda_2 + \lambda_3 = 0$ there exists a matrix $M \in \operatorname{Im} p$ with eigenvalues $\lambda_1,\ \lambda_2$, and λ_3. □

4. Open problems

Problem 1. Does there actually exist a multilinear polynomial whose image evaluated on 3×3 matrices consists of 3-scalar matrices?

Problem 2. Does there actually exist a multilinear polynomial whose image evaluated on 3×3 matrices is the set of scalars plus 3-scalar matrices?

Remark 9. Problems 1 and 2 both have the same answer. If they both have affirmative answers, such a polynomial would be a counter-example to Kaplansky's problem.

Problem 3. Is it possible that the image of a multilinear polynomial evaluated on 3×3 matrices is dense but not all of $M_3(K)$?

Problem 4. Is it possible that the image of a multilinear polynomial evaluated on 3×3 matrices is the set of all trace vanishing matrices without discriminant vanishing diagonalizable matrices?

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