PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 144, Number 1, January 2016, Pages 87–101
http://dx.doi.org/10.1090/proc/12640
Article electronically published on September 4, 2015

CHARACTERIZATIONS OF ULTRASPHERICAL POLYNOMIALS AND THEIR q-ANALOGUES

STEFAN KAHLER

(Communicated by Walter Van Assche)

ABSTRACT. We investigate symmetric, suitably normalized orthogonal polynomial sequences $(P_n(x))_{n\in\mathbb{N}_0}$ and characterize the class of ultraspherical polynomials in terms of certain constancy properties of the Fourier coefficients which belong to $(P'_{2n-1}(x))_{n\in\mathbb{N}}$. Similar characterizations are obtained for the discrete, resp. continuous, q-ultraspherical polynomials after replacing the derivative $\frac{\mathrm{d}}{\mathrm{d}x}$ by the q-difference operator $D_{q^{-1}}$, resp. Askey–Wilson operator $\mathcal{D}_q$. Our results extend earlier work of Lasser–Obermaier and Ismail–Obermaier where the whole sequences $(P'_n(x))_{n\in\mathbb{N}}$, $(D_{q^{-1}}P_n(x))_{n\in\mathbb{N}}$ and $(\mathcal{D}_qP_n(x))_{n\in\mathbb{N}}$ had to be taken into account; we shall see that the characterizing properties concerning the even indices turn out to be redundant. We also characterize a large subclass of the continuous q-ultraspherical polynomials via the averaging operator $\mathcal{A}_q$, and we show that this characterization cannot be extended to the whole class.

1. Introduction

Characterizing families of orthogonal polynomials by specific properties has a long history, and the corresponding literature is extensive; [1] provides a valuable survey up to 1990. In this paper, we find new characterizations for the class of ultraspherical polynomials and two q-analogues. Given some fixed $A > 0$, we consider sequences $(P_n(x))_{n\in\mathbb{N}_0} \subseteq \mathbb{R}[x]$ of polynomials which satisfy a recurrence relation of the form $P_0(x) = 1$,

$$xP_n(x) = a_nP_{n+1}(x) + c_nP_{n-1}(x) \quad (n \in \mathbb{N}_0), \tag{1.1}$$

where $c_0 := 0$, $(c_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$, $a_n := A - c_n$ $(n \in \mathbb{N}_0)$ and $c_na_{n-1} > 0$ $(n \in \mathbb{N})$.[1] Due to Favard's theorem (see standard literature such as [3]), this is equivalent to requiring the normalization $P_n(A) = 1$ $(n \in \mathbb{N}_0)$ and orthogonality w.r.t. a

Received by the editors October 15, 2013 and, in revised form, August 20, 2014.
2010 *Mathematics Subject Classification.* Primary 33C45, 33D45; Secondary 42C05, 42C10.
Key words and phrases. Ultraspherical polynomials, discrete q-ultraspherical polynomials, continuous q-ultraspherical polynomials, q-difference operator, Askey–Wilson operator, averaging operator, expansions of orthogonal polynomials, Fourier coefficients.
The author gratefully acknowledges support from the graduate program TopMath of the ENB (Elite Network of Bavaria) and the TopMath Graduate Center of TUM Graduate School at Technische Universität München. During large parts of his research the author was partially supported by a scholarship from the Max Weber-Programm within the ENB and by a scholarship from the Studienstiftung des deutschen Volkes.

[1] As is widely common, we make the convention that $c_0 = 0$ times something undefined shall be 0.

symmetric probability (Borel) measure μ on $\mathbb{R}$ with $|\mathsf{supp}\ \mu| = \infty$. If $(P_n(x))_{n\in\mathbb{N}_0} \subseteq \mathbb{R}[x]$ is even a (symmetric) random walk polynomial sequence (RWPS), which means that

$$\text{(RW)} \qquad (c_n)_{n\in\mathbb{N}} \subseteq (0, A)$$

holds,[2] then $\mathsf{supp}\ \mu \subseteq [-A, A]$ (because $P_n(x) \geq P_{n-1}(x) \geq 1$ for all $x \geq A$ and $n \in \mathbb{N}$, which follows from (1.1) and (RW) via induction and implies that all zeros are located in $(-A, A)$; in fact, $\mathsf{supp}\ \mu \subseteq [-A, A]$ even characterizes (RW), cf. [12] for the reverse direction) and μ is unique. Therefore, $\mathrm{d}\mu^*(x) := (A^2 - x^2)\,\mathrm{d}\mu(x)$ defines another measure, and we obtain a corresponding RWPS $(P_n^*(x))_{n\in\mathbb{N}_0}$ with $P_n^*(A) = 1$ $(n \in \mathbb{N}_0)$. Defining $h : \mathbb{N}_0 \to (0, \infty)$ by

$$h(n) := \frac{1}{\int_{\mathbb{R}} P_n^2(x)\,\mathrm{d}\mu(x)} = \begin{cases} 1, & n = 0, \\ \prod_{j=1}^{n} \frac{a_{j-1}}{c_j}, & \text{else} \end{cases}$$

[8], it is easy to see that

$$P_n^*(x) = \frac{\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} h(n-2k) P_{n-2k}(x)}{\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} h(n-2k)} \quad (n \in \mathbb{N}_0). \tag{1.2}$$

Since h remains well-defined if (RW) is dropped again (note that, although μ need no longer be unique, the values of $\int_{\mathbb{R}} P(x)\,\mathrm{d}\mu(x)$, $P(x) \in \mathbb{R}[x]$, are still uniquely determined), (1.2) enables us to extend the definition of $(P_n^*(x))_{n\in\mathbb{N}_0}$ to the general case. It is not difficult to see that, for each $n \in \mathbb{N}$, the polynomial $(A^2 - x^2)P_n^*(x)$ is orthogonal to $P_0(x), \ldots, P_{n-1}(x)$ (and, due to symmetry, also to $P_{n+1}(x)$): while the special case (RW) is obvious, the general case follows by computing the expansion of $(A^2 - x^2)P_n^*(x)$ in the basis $\{P_0(x), \ldots, P_{n+2}(x)\}$ via (1.1), or, alternatively, by a twofold application of the Christoffel–Darboux formula (cf. [3]) to the quotient $\frac{P_{n+2}(x) - P_n(x)}{A^2 - x^2}$. In the following, we do not require $(P_n(x))_{n\in\mathbb{N}_0}$ to be an RWPS, i.e., (RW), unless explicitly stated otherwise.

As usual, we denote by $(T_n(x))_{n\in\mathbb{N}_0}$, resp. $(U_n(x))_{n\in\mathbb{N}_0}$, the Chebyshev polynomials of the first, resp. second, kind, i.e., $T_n(\cos\theta) = \cos(n\theta)$, resp. $U_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (2 - \delta_{n-2k,0}) T_{n-2k}(x)$ $(n \in \mathbb{N}_0)$. Moreover, we put $U_{-1}(x) := 0$. Recall that

$$\frac{\mathrm{d}}{\mathrm{d}x} T_n(x) = n U_{n-1}(x) \quad (n \in \mathbb{N}_0). \tag{1.3}$$

Hence, if $P_n(x) = T_n(x)$ $(n \in \mathbb{N}_0)$, then $A = 1$ and $P_n'(x) = P_n'(1) P_{n-1}^*(x)$ $(n \in \mathbb{N})$. The problem which symmetric RWPS share this property has been solved in [12, Lemma 1, Theorem 1]:

Theorem 1.1 (Lasser–Obermaier 2008)**.** *If* (RW) *holds, then the following are equivalent:*

(i) $P_n(x) = P_n^{\left(\frac{1}{2c_1} - \frac{3}{2}\right)}(x)$ $(n \in \mathbb{N}_0)$,
(ii) $A = 1$ *and* $P_n'(x) = P_n'(1) P_{n-1}^*(x)$ $(n \in \mathbb{N})$.

[2]Some authors require that $A = 1$ when using the expression RWPS; we shall not generally do this.

Here, $(P_n^{(\alpha)}(x))_{n\in\mathbb{N}_0}$ denotes the sequence of ultraspherical polynomials which corresponds to $\alpha > -1$, normalized so that $P_n^{(\alpha)}(1) = 1$ $(n \in \mathbb{N}_0)$. It is explicitly given by $A = 1$ and

$$c_n = \frac{n}{2n + 2\alpha + 1} \tag{1.4}$$

$(n \in \mathbb{N})$ or, equivalently, $\mathrm{d}\mu(x) = \frac{\Gamma(2\alpha+2)}{2^{2\alpha+1}\Gamma(\alpha+1)^2}(1-x^2)^\alpha \chi_{(-1,1)}(x)\,\mathrm{d}x$ [12]. The latter shows that if $(P_n(x))_{n\in\mathbb{N}_0}$ belongs to the class of ultraspherical polynomials, then so does $(P_n^*(x))_{n\in\mathbb{N}_0}$. The interesting direction "(ii) $\Rightarrow$ (i)" in Theorem 1.1 can also be obtained from older contributions to the theory of orthogonal polynomials: for instance, one could apply a well-known characterization result of W. Hahn which states that only the family of classical orthogonal polynomials possesses orthogonal derivatives [5]; another alternative would be to use a related (but independent) characterization of the classical orthogonal polynomials given by W. A. Al-Salam and T. S. Chihara [2]. It is one of the purposes of this paper to improve Theorem 1.1 in such a way that these older results no longer apply.

Another purpose is to similarly improve results of M. E. H. Ismail and J. Obermaier on q-ultraspherical polynomials. Given $q \in (0,1)$ and $\alpha, \beta \in \left(0, \frac{1}{\sqrt{q}}\right)$, the corresponding – suitably normalized – sequences of discrete q-ultraspherical (or symmetric big q-Jacobi) polynomials $(P_n(x;\alpha : q))_{n\in\mathbb{N}_0}$, resp. continuous q-ultraspherical polynomials (or Rogers polynomials; in essence, symmetric continuous q-Jacobi polynomials) $(P_n(x;\beta|q))_{n\in\mathbb{N}_0}$, are given by $A = \alpha\sqrt{q}$, $c_n = \alpha\sqrt{q}\frac{1-q^n}{1-\alpha^2 q^{2n}}$, resp. $A = \frac{\sqrt{\beta}}{2} + \frac{1}{2\sqrt{\beta}}$,

$$c_n = \frac{\sqrt{\beta}}{2}\frac{1-q^n}{1-\beta q^n} \tag{1.5}$$

$(n \in \mathbb{N})$ [8]. The ultraspherical polynomials appear as limiting cases because $\lim_{q\to 1} P_n(x; q^{\alpha+\frac{1}{2}} : q) = \lim_{q\to 1} P_n(x; q^{\alpha+\frac{1}{2}}|q) = P_n^{(\alpha)}(x)$ for each $\alpha > -1$, $n \in \mathbb{N}_0$ and $x \in \mathbb{R}$; moreover, $T_n(x) = P_n(x;1|q)$ and $U_n(x) = U_n\left(\frac{\sqrt{q}}{2} + \frac{1}{2\sqrt{q}}\right) P_n(x;q|q)$ $(n \in \mathbb{N}_0)$ [10]. We shall not make use of the explicit formulas for the orthogonalization measures. Concerning $(P_n(x;\alpha : q))_{n\in\mathbb{N}_0}$, we just note that μ is purely discrete and that $\max \mathsf{supp}\,\mu = A$ [8]. Concerning $(P_n(x;\beta|q))_{n\in\mathbb{N}_0}$, we refer to Remark 4.1 below. Further information about q-ultraspherical polynomials, in particular representations as basic hypergeometric series, can be found in [7, 8, 10]. The latter reference contains the full (q-)Askey scheme. We consider two q-generalizations of the classical derivative: on the one hand, the (linear) 'Askey–Wilson operator' $\mathcal{D}_q : \mathbb{R}[x] \to \mathbb{R}[x]$ is defined by a q-extension of (1.3), namely via

$$\mathcal{D}_q T_n(x) = \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{\sqrt{q} - \frac{1}{\sqrt{q}}} U_{n-1}(x) \ \ (n \in \mathbb{N}_0);$$

hence, if $P_n(x) = T_n(x)$ $(n \in \mathbb{N}_0)$, then $\mathcal{D}_q P_n(x) = \mathcal{D}_q P_n(1) P_{n-1}^*(x)$ $(n \in \mathbb{N})$. On the other hand, one has the 'q-difference operator' $D_q : \mathbb{R}[x] \to \mathbb{R}[x]$ given by $D_q P(x) = \frac{P(x)-P(qx)}{x-qx}$ $(x \neq 0)$, where $q \in (0,\infty)\backslash\{1\}$ and $\frac{\mathrm{d}}{\mathrm{d}x}$ is contained as limiting case $q \to 1$ again. These basics and further information about D_q and $\mathcal{D}_q$ can be found in [7] or [8].

Theorem 1.2 (Ismail–Obermaier 2011). *Let* $q \in (0,1)$, $\alpha \in \left(0, \frac{1}{\sqrt{q}}\right)$, $A = \alpha\sqrt{q}$; *moreover, let* $c_1 = \alpha\sqrt{q}\frac{1-q}{1-\alpha^2 q^2}$. *Then the following are equivalent:*

(i) $P_n(x) = P_n(x;\alpha:q)$ $(n \in \mathbb{N}_0)$,
(ii) $D_{q^{-1}}P_n(x) = D_{q^{-1}}P_n(A)P^*_{n-1}(x)$ $(n \in \mathbb{N})$.

Theorem 1.3 (Ismail–Obermaier 2011). *Let* $q \in (0,1)$, $\beta \in \left(0, \frac{1}{\sqrt{q}}\right)$ *and* $A = \frac{\sqrt{\beta}}{2} + \frac{1}{2\sqrt{\beta}}$; *assume* $c_1 = \frac{\sqrt{\beta}}{2}\frac{1-q}{1-\beta q}$. *Then the following are equivalent:*

(i) $P_n(x) = P_n(x;\beta|q)$ $(n \in \mathbb{N}_0)$,
(ii) $\mathcal{D}_q P_n(x) = \mathcal{D}_q P_n(A)P^*_{n-1}(x)$ $(n \in \mathbb{N})$.

Theorem 1.2, resp. Theorem 1.3, is contained in [8, Theorem 4.1], resp. [8, Theorem 5.2].[3] With regard to the results of Hahn, resp. Al-Salam–Chihara, mentioned above, we note that there are analogues for the q-difference operator D_q ([6], resp. [4]); however, it seems to be open whether there are also analogues for the Askey–Wilson operator $\mathcal{D}_q$ [7, Conjecture 24.7.10, resp. Conjecture 24.7.8]. In view of the latter, Theorem 1.3 "(ii) $\Rightarrow$ (i)" is of particular interest. In [9], Ismail and Simeonov obtained extensions to other classes, including symmetric Al-Salam–Chihara, symmetric Askey–Wilson and symmetric Meixner–Pollaczek polynomials (with suitably chosen corresponding operators). Our aim is to sharpen the Lasser–Obermaier and Ismail–Obermaier results in the following way: in Section 2, we will show that the directions "(ii) $\Rightarrow$ (i)" remain fully true if (ii) is just assumed to be satisfied for all *odd* positive integers – as well in Theorem 1.1 as in Theorem 1.2 and in Theorem 1.3. We shall also give a characterization of (a subclass of) the continuous q-ultraspherical polynomials in terms of the averaging operator $\mathcal{A}_q$, whose definition will be recalled below. The proofs will be given in Section 4, with some preliminaries in Section 3.

2. Statement of the main results

Let A and $(P_n(x))_{n\in\mathbb{N}_0}$ be as in Section 1 – we do not require property (RW). Let $q \in (0,1)$. The 'averaging operator' $\mathcal{A}_q : \mathbb{R}[x] \to \mathbb{R}[x]$ is given by linearity and

$$\mathcal{A}_q T_n(x) = \frac{q^{\frac{n}{2}} + q^{-\frac{n}{2}}}{2} T_n(x) \quad (n \in \mathbb{N}_0)$$

[8]. Following [8,12], we consider the Fourier coefficients associated with $\frac{\mathrm{d}}{\mathrm{d}x}$, $D_{q^{-1}}$, $\mathcal{D}_q$ and $\mathcal{A}_q$: for each $n \in \mathbb{N}_0$, we define four functions $\kappa_n, \kappa_n(.:q), \kappa_n(.|q), \alpha_n(.|q) : \mathbb{N}_0 \to \mathbb{R}$ via

$$\begin{aligned}\kappa_n(k) &:= \int_{\mathbb{R}} P_n'(x)P_k(x)\,\mathrm{d}\mu(x),\\ \kappa_n(k:q) &:= \int_{\mathbb{R}} D_{q^{-1}}P_n(x)P_k(x)\,\mathrm{d}\mu(x),\\ \kappa_n(k|q) &:= \int_{\mathbb{R}} \mathcal{D}_q P_n(x)P_k(x)\,\mathrm{d}\mu(x),\ \alpha_n(k|q) := \int_{\mathbb{R}} \mathcal{A}_q P_n(x)P_k(x)\,\mathrm{d}\mu(x).\end{aligned}$$

[3] The special case $\beta = 1$ (Chebyshev polynomials of the first kind), which occurs very naturally in our normalization but only as a limiting case in the "standard" normalization of the continuous q-ultraspherical polynomials [10], is excluded in [8, Theorem 5.2]. However, the proof given in [8] is easily seen to extend to this case.

Note that $\kappa_n(k) = \kappa_n(k : q) = \kappa_n(k|q) = 0$ if $k \geq n$ or if $n-k$ is even, and that $\alpha_n(k|q) = 0$ if $k \geq n+1$ or if $n-k$ is odd. For brevity, we write $\sigma, \sigma(. : q), \sigma(.|q) : \mathbb{N} \to \mathbb{R}\backslash\{0\}$, $\sigma(n) := \kappa_n(n-1)$, $\sigma(n : q) := \kappa_n(n-1 : q)$, $\sigma(n|q) := \kappa_n(n-1|q)$. The following three theorems are the announced sharper results. In each of them, (vi) is the apparently weakest condition (and hence provides the strongest characterization); in (vi), the functions κ_n, $\kappa_n(. : q)$ and $\kappa_n(.|q)$ have to be considered only for odd indices and only at some carefully chosen points. The characterizations (iv) and (v) have the advantage of being "stable" w.r.t. renormalization of $(P_n(x))_{n\in\mathbb{N}_0}$.

Theorem 2.1. *Let $\alpha > -1$, $A = 1$ and $c_1 = \frac{1}{2\alpha+3}$. Then the following are equivalent:*

(i) $P_n(x) = P_n^{(\alpha)}(x)$ $(n \in \mathbb{N}_0)$,
(ii) $P_n'(x) = P_n'(1)P_{n-1}^*(x)$ $(n \in \mathbb{N})$,
(iii) $P_{2n-1}'(x) = P_{2n-1}'(1)P_{2n-2}^*(x)$ $(n \in \mathbb{N})$,
(iv) $(1-x^2)P_n'(x)$ *is orthogonal to* $P_0(x), \dots, P_{n-2}(x)$ $(n \geq 2)$,
(v) $(1-x^2)P_{2n-1}'(x)$ *is orthogonal to* $P_0(x), P_2(x), \dots, P_{2n-4}(x)$ $(n \geq 2)$,
(vi) *one has*

$$\kappa_{2n+1}(2n-2) = \sigma(2n+1) \quad (n \in \mathbb{N}),$$

and for every $n \in \mathbb{N}$ there is a $k \in \{0, \dots, n-1\}$ such that

$$\kappa_{2n+3}(2k) = \sigma(2n+3).$$

Theorem 2.2. *Under the conditions of Theorem* 1.2, *the following are equivalent:*

(i) $P_n(x) = P_n(x;\alpha : q)$ $(n \in \mathbb{N}_0)$,
(ii) $D_{q^{-1}}P_n(x) = D_{q^{-1}}P_n(A)P_{n-1}^*(x)$ $(n \in \mathbb{N})$,
(iii) $D_{q^{-1}}P_{2n-1}(x) = D_{q^{-1}}P_{2n-1}(A)P_{2n-2}^*(x)$ $(n \in \mathbb{N})$,
(iv) $(A^2-x^2)D_{q^{-1}}P_n(x)$ *is orthogonal to* $P_0(x), \dots, P_{n-2}(x)$ $(n \geq 2)$,
(v) $(A^2-x^2)D_{q^{-1}}P_{2n-1}(x)$ *is orthogonal to* $P_0(x), P_2(x), \dots, P_{2n-4}(x)$ $(n \geq 2)$,
(vi) *one has*

$$\kappa_{2n+1}(2n-2 : q) = \sigma(2n+1 : q) \quad (n \in \mathbb{N}),$$

and for every $n \in \mathbb{N}$ there is a $k \in \{0, \dots, n-1\}$ such that

$$\kappa_{2n+3}(2k : q) = \sigma(2n+3 : q).$$

Theorem 2.3. *Under the conditions of Theorem* 1.3, *the following are equivalent:*

(i) $P_n(x) = P_n(x;\beta|q)$ $(n \in \mathbb{N}_0)$,
(ii) $\mathcal{D}_qP_n(x) = \mathcal{D}_qP_n(A)P_{n-1}^*(x)$ $(n \in \mathbb{N})$,
(iii) $\mathcal{D}_qP_{2n-1}(x) = \mathcal{D}_qP_{2n-1}(A)P_{2n-2}^*(x)$ $(n \in \mathbb{N})$,
(iv) $(A^2-x^2)\mathcal{D}_qP_n(x)$ *is orthogonal to* $P_0(x), \dots, P_{n-2}(x)$ $(n \geq 2)$,
(v) $(A^2-x^2)\mathcal{D}_qP_{2n-1}(x)$ *is orthogonal to* $P_0(x), P_2(x), \dots, P_{2n-4}(x)$ $(n \geq 2)$,
(vi) *one has*

$$\kappa_{2n+1}(2n-2|q) = \sigma(2n+1|q) \quad (n \in \mathbb{N}),$$

and for every $n \in \mathbb{N}$ there is a $k \in \{0, \dots, n-1\}$ such that

$$\kappa_{2n+3}(2k|q) = \sigma(2n+3|q).$$

We now come to the announced characterization which involves $\mathcal{A}_q$:

Theorem 2.4. *Under the conditions of Theorem* 1.3 *and the additional assumption that* $\beta \le 1$*, the following are equivalent:*

(i) $P_n(x) = P_n(x;\beta|q)$ $(n \in \mathbb{N}_0)$,

(ii) *the quotient* $\frac{\alpha_{n+1}(n-1|q)}{\sigma(n+1|q)}$ $\left(= \frac{\int_{\mathbb{R}} \mathcal{A}_q P_{n+1}(x) P_{n-1}(x)\,\mathrm{d}\mu(x)}{\int_{\mathbb{R}} \mathcal{D}_q P_{n+1}(x) P_n(x)\,\mathrm{d}\mu(x)}\right)$ *is independent of* $n \in \mathbb{N}$.

If the condition $\beta \le 1$ *is dropped, then only "*(i) $\Rightarrow$ (ii)*" remains valid.*

Remark 2.1. (i) Since the ultraspherical polynomials are limiting cases of both the discrete and the continuous q-ultraspherical polynomials, it is a natural question to ask whether Theorem 2.2 or Theorem 2.3 implies Theorem 2.1 by $q \to 1$. However, starting with Theorem 2.2 or Theorem 2.3 there is no reason why the (in each case most non-trivial) implication "(vi) $\Rightarrow$ (i)" should hold true if one "passes to the limit". A similar situation already arises concerning the original results by Ismail and Obermaier (Theorem 1.2 and Theorem 1.3) on the one hand and by Lasser and Obermaier (Theorem 1.1) on the other hand. Another interesting observation in this context is the following: since $\mathcal{A}_q$ becomes the identity as $q \to 1$, the "limiting case" of (ii) of Theorem 2.4 is a trivial property which is always true. Therefore, one has no analogous characterization for the class of ultraspherical polynomials.

(ii) Using similar methods as in this paper and some additional ideas to overcome the lack of symmetry, we were able to generalize Theorem 1.1 (and parts of Theorem 2.1) to the class of Jacobi polynomials. This will be presented in another paper.

3. Preliminaries

The proofs of our main results need some preparation. In the following, we maintain the general assumptions made in the previous section. We collect some identities which shall be used frequently: one has

$$\sigma(n) = \frac{n}{c_n h(n)},\ \frac{\sigma(n)}{\sigma(n+1)} = \frac{n a_n}{(n+1)c_n}\quad (n \in \mathbb{N}), \tag{3.1}$$

$$\sigma(n:q) = q^{1-n}\frac{1-q^n}{1-q}\frac{1}{c_n h(n)},\ \frac{\sigma(n:q)}{\sigma(n+1:q)} = q\frac{1-q^n}{1-q^{n+1}}\frac{a_n}{c_n}\quad (n \in \mathbb{N}), \tag{3.2}$$

$$\sigma(n|q) = q^{\frac{1-n}{2}}\frac{1-q^n}{1-q}\frac{1}{c_n h(n)},\ \frac{\sigma(n|q)}{\sigma(n+1|q)} = \sqrt{q}\frac{1-q^n}{1-q^{n+1}}\frac{a_n}{c_n}\quad (n \in \mathbb{N}), \tag{3.3}$$

$$\alpha_n(n|q) = \frac{q^{\frac{n}{2}} + q^{-\frac{n}{2}}}{2h(n)}\quad (n \in \mathbb{N}_0), \tag{3.4}$$

$$\alpha_n(n-2|q) = \underbrace{\frac{(1-q)\left(q^{\frac{n-2}{2}} - q^{-\frac{n}{2}}\right)}{2}}_{=:D_n}\frac{\frac{n}{4} - \sum_{k=1}^{n-1} a_{k-1}c_k}{h(n-2)a_{n-2}a_{n-1}}\quad (n \ge 2). \tag{3.5}$$

Under stronger assumptions, (3.1) has been established in [12, proof of Theorem 1] – the idea presented there can easily be transferred to our setting. However, we provide a shorter argument by observing that (3.1) follows immediately by integrating the confluent version of the Christoffel–Darboux formula, which reads

$$\frac{1}{c_n h(n)} \sum_{k=0}^{n-1} h(k) P_k^2(x) = P_n'(x) P_{n-1}(x) - P_{n-1}'(x) P_n(x) \quad (n \in \mathbb{N}, x \in \mathbb{R})$$

in our normalization (cf. [11, equation (12)]), w.r.t. μ. Concerning the q-generalizations (3.2), resp. (3.3), we refer to [8, proof of Theorem 4.1], resp. [8, proof of Theorem 5.2]. (3.4) and (3.5) are taken from [8, Lemma 5.1].

Next, we recall the recurrence relations for $(\kappa_n)_{n\in\mathbb{N}_0}$, resp. $(\kappa_n(.:q))_{n\in\mathbb{N}_0}$, resp. $(\kappa_n(.|q))_{n\in\mathbb{N}_0}$, which rely on the product rules for $\frac{\mathrm{d}}{\mathrm{d}x}$, resp. $D_{q^{-1}}$, resp. $\mathcal{D}_q$ (cf. [12, equation (9)], resp. [8, proof of Theorem 4.1], resp. [8, proof of Theorem 5.2]):

$$\kappa_0 = \kappa_0(.:q) = \kappa_0(.|q) = 0,$$

$$a_n \kappa_{n+1}(k) + c_n \kappa_{n-1}(k) = a_k \kappa_n(k+1) + c_k \kappa_n(k-1) + \frac{\delta_{n,k}}{h(n)} \quad (n, k \in \mathbb{N}_0), \tag{3.6}$$

$$\begin{aligned} &a_n \kappa_{n+1}(k:q) + c_n \kappa_{n-1}(k:q) \\ &\quad = \frac{1}{q}[a_k \kappa_n(k+1:q) + c_k \kappa_n(k-1:q)] + \frac{\delta_{n,k}}{h(n)} \quad (n, k \in \mathbb{N}_0), \end{aligned} \tag{3.7}$$

$$\begin{aligned} &a_n \kappa_{n+1}(k|q) + c_n \kappa_{n-1}(k|q) \\ &\quad = \left(\frac{\sqrt{q}}{2} + \frac{1}{2\sqrt{q}}\right)[a_k \kappa_n(k+1|q) + c_k \kappa_n(k-1|q)] + \alpha_n(k|q) \quad (n, k \in \mathbb{N}_0). \end{aligned} \tag{3.8}$$

Compared to (3.6) and (3.7), which have essentially the same structure, the recurrence relation for $(\kappa_n(.|q))_{n\in\mathbb{N}_0}$ is more complicated because it simultaneously involves the Fourier coefficients associated with the averaging operator $\mathcal{A}_q$, i.e., $(\alpha_n(.|q))_{n\in\mathbb{N}_0}$. This important difference can be traced back to the more complicated product rule for the Askey–Wilson operator, namely

$$\mathcal{D}_q[PQ](x) = \mathcal{D}_q P(x) \mathcal{A}_q Q(x) + \mathcal{A}_q P(x) \mathcal{D}_q Q(x) \quad (P(x), Q(x) \in \mathbb{R}[x]) \tag{3.9}$$

[7]. An important tool for overcoming the resulting problems will be the following result concerning the auxiliary functions $\beta_n(.|q) : \{0, \ldots, n-2\} \to \mathbb{R}$,

$$\beta_n(k|q) := \int_{\mathbb{R}} \mathcal{A}_q[x P_n(x)] P_k(x) \,\mathrm{d}\mu(x)$$

$(n \geq 2)$.

Lemma 3.1. *For each $n \in \mathbb{N}$, the recursion coefficients $c_1, \ldots, c_n$ determine $\alpha_{n+1}(.|q)|_{\{0,\ldots,n\}}$ and $\beta_{n+1}(.|q)$ uniquely.*

Proof. Let $c_1, \ldots, c_n$ be fixed. Then $P_0(x), \ldots, P_{n+1}(x)$ are uniquely determined, and there exist unique $A_0, \ldots, A_{n+1} \in \mathbb{R}$ such that $\mathcal{A}_q P_{n+1}(x) = \sum_{k=0}^{n+1} A_k P_k(x)$. Since $\alpha_{n+1}(k|q) = \frac{A_k}{h(k)}$ $(k \in \{0, \ldots, n+1\})$, and since $h(k)$ is uniquely determined by $c_1, \ldots, c_n$ provided $k \leq n$, the first assertion follows. The second part of the lemma is more interesting and based on the following idea: if $c_1, \ldots, c_n$ are still fixed, there are unique $B_0, \dots, B_{n+2} \in \mathbb{R}$ such that

$$\begin{aligned}\mathcal{A}_q[xP_{n+1}(x)] &= B_0 + x\sum_{k=0}^{n+1} B_{k+1}P_k(x) \\ &= B_0 + B_2c_1 + \sum_{k=1}^{n}[B_k a_{k-1} + B_{k+2}c_{k+1}]P_k(x) \\ &\qquad + B_{n+1}a_nP_{n+1}(x) + B_{n+2}a_{n+1}P_{n+2}(x).\end{aligned}$$

Hence, there are unique $C_0, \dots, C_{n-1} \in \mathbb{R}$ such that

$$\begin{aligned}\mathcal{A}_q[xP_{n+1}(x)] = \sum_{k=0}^{n-1} C_kP_k(x) &+ [B_na_{n-1} + B_{n+2}c_{n+1}]P_n(x) \\ &+ B_{n+1}a_nP_{n+1}(x) + B_{n+2}a_{n+1}P_{n+2}(x).\end{aligned}$$

Now if $k \in \{0, \dots, n-1\}$, we can conclude that $\beta_{n+1}(k|q) = \frac{C_k}{h(k)}$, and the right hand side of the latter is uniquely determined by $c_1, \dots, c_n$. □

Note that $\beta_n(k|q) = 0$ if $n - k$ is even. Finally, we need the following ingredient (its special case $k = 1$ is already contained in [8, proof of Theorem 5.2]):

Lemma 3.2. *Let* $\beta \in \left(0, \frac{1}{\sqrt{q}}\right)$ *and* $P_n(x) = P_n(x; \beta|q)$ $(n \in \mathbb{N}_0)$. *Then*

$$\frac{\alpha_{n+1}(n+1-2k|q)}{\sigma(n+1|q)} = \frac{(\beta-1)(1-q)}{4\sqrt{\beta q}}$$

for each $n \in \mathbb{N}$ *and* $k \in \left\{1, \dots, \left\lfloor \frac{n+1}{2} \right\rfloor\right\}$.

Proof. Let $n \in \mathbb{N}$. Applying the product rule (3.9) to $xP_{n+1}(x)$ and using that $\mathcal{D}_q(x) = 1$, $\mathcal{A}_q(x) = \left(\frac{\sqrt{q}}{2} + \frac{1}{2\sqrt{q}}\right)x$, we get

$$\mathcal{D}_q[xP_{n+1}(x)] = \mathcal{A}_qP_{n+1}(x) + \left(\frac{\sqrt{q}}{2} + \frac{1}{2\sqrt{q}}\right)x\mathcal{D}_qP_{n+1}(x).$$

Together with (1.1), this implies

$$\mathcal{A}_qP_{n+1}(x) = a_{n+1}\mathcal{D}_qP_{n+2}(x) + c_{n+1}\mathcal{D}_qP_n(x) - \left(\frac{\sqrt{q}}{2} + \frac{1}{2\sqrt{q}}\right)x\mathcal{D}_qP_{n+1}(x).$$

Now Theorem 1.3 yields

$$\begin{aligned}&a_{n+1}\mathcal{D}_qP_{n+2}(x) + c_{n+1}\mathcal{D}_qP_n(x) \\ &\quad = a_{n+1}h(n+1)\sigma(n+2|q)P_{n+1}(x) \\ &\qquad + [a_{n+1}\sigma(n+2|q) + c_{n+1}\sigma(n|q)]\sum_{k=1}^{\left\lfloor \frac{n+1}{2}\right\rfloor} h(n+1-2k)P_{n+1-2k}(x)\end{aligned}$$

and

$$\begin{aligned}x\mathcal{D}_qP_{n+1}(x) &= c_{n+1}h(n+1)\sigma(n+1|q)P_{n+1}(x) \\ &\qquad + A\sigma(n+1|q)\sum_{k=1}^{\left\lfloor \frac{n+1}{2}\right\rfloor} h(n+1-2k)P_{n+1-2k}(x).\end{aligned}$$

Thus

$$\frac{\alpha_{n+1}(n+1-2k|q)}{\sigma(n+1|q)} = a_{n+1}\frac{\sigma(n+2|q)}{\sigma(n+1|q)} + c_{n+1}\frac{\sigma(n|q)}{\sigma(n+1|q)} - \left(\frac{\sqrt{q}}{2} + \frac{1}{2\sqrt{q}}\right)A,$$

$k \in \left\{1, \ldots, \left\lfloor \frac{n+1}{2} \right\rfloor\right\}$. Using (1.5) and (3.3), one sees that the right hand side of the latter equation equals $\frac{(\beta-1)(1-q)}{4\sqrt{\beta q}}$. □

4. Proof of the main results

Proof (*Theorem* 2.1). The implication "(i) ⇒ (ii)" is contained in Theorem 1.1, "(ii) ⇒ (iii)" and "(iii) ⇒ (vi)" are trivial. Furthermore, (iv), resp. (v), is an obvious reformulation of (ii), resp. (iii). Hence, we only have to prove the implication "(vi) ⇒ (i)"; to this end, we use induction to show that (1.4) is satisfied for all $n \in \mathbb{N}$ if (vi) is assumed to hold. By the assumptions of the theorem, (1.4) is valid for $n = 1$. Moreover, (3.6) yields $a_2\sigma(3) + c_2\sigma(1) = \sigma(2)$, which, in turn, implies $1 = \frac{3}{2}c_2 + \frac{a_1}{2c_1}c_2 = \frac{2\alpha+5}{2}c_2$ because of (3.1). Thus (1.4) is established for $n = 2$. Let $n \in \mathbb{N}$ be arbitrary but fixed from now on, and assume that $1, \ldots, 2n$ fulfill (1.4). Let $k \in \{0, \ldots, n-1\}$ be such that $\kappa_{2n+3}(2k) = \sigma(2n+3)$. Since κ_{2n} and κ_{2n+1} are uniquely determined by $c_1, \ldots, c_{2n}$, Theorem 1.1 implies that $\kappa_{2n}(2j-1) = \sigma(2n)$ ($j \in \{1, \ldots, n\}$), $\kappa_{2n+1}(2j-2) = \sigma(2n+1)$ ($j \in \{1, \ldots, n+1\}$), and then (3.6) yields

$$a_{2n+1}\kappa_{2n+2}(2k+1) + c_{2n+1}\sigma(2n) = \sigma(2n+1), \tag{4.1}$$

$$a_{2n+1}\kappa_{2n+2}(2n-1) + c_{2n+1}\sigma(2n) = \sigma(2n+1) \tag{4.2}$$

and

$$a_{2n+2}\sigma(2n+3) + c_{2n+2}\sigma(2n+1) = a_{2k}\kappa_{2n+2}(2k+1) + c_{2k}\kappa_{2n+2}(2k-1), \tag{4.3}$$

$$a_{2n+2}\sigma(2n+3) + c_{2n+2}\sigma(2n+1) = a_{2n}\sigma(2n+2) + c_{2n}\kappa_{2n+2}(2n-1). \tag{4.4}$$

We now distinguish two cases:

Case 1 ($k \neq 0$). Using (3.6), we get

$$a_{2n+1}\kappa_{2n+2}(2k-1) + c_{2n+1}\sigma(2n) = \sigma(2n+1),$$

and combining this with (4.1) and (4.2), we obtain that

$$\kappa_{2n+2}(2k+1) = \kappa_{2n+2}(2k-1) = \kappa_{2n+2}(2n-1).$$

Therefore, equation (4.3) reduces to

$$a_{2n+2}\sigma(2n+3) + c_{2n+2}\sigma(2n+1) = \kappa_{2n+2}(2n-1). \tag{4.5}$$

Case 2 ($k = 0$). In this case, (4.3) reads $a_{2n+2}\sigma(2n+3) + c_{2n+2}\sigma(2n+1) = \kappa_{2n+2}(1)$, which becomes (4.5), too, because (4.1) and (4.2) imply that

$$\kappa_{2n+2}(1) = \kappa_{2n+2}(2n-1).$$

It is remarkable that – in both Case 1 and Case 2 – the former dependence of (4.3) on k has vanished. By combining (4.4) and (4.5), we deduce that $\kappa_{2n+2}(2n-1) = \sigma(2n+2)$. Then (4.2) and (4.5) simplify to

$$a_{2n+1}\sigma(2n+2) + c_{2n+1}\sigma(2n) = \sigma(2n+1), \tag{4.6}$$

$$a_{2n+2}\sigma(2n+3) + c_{2n+2}\sigma(2n+1) = \sigma(2n+2). \tag{4.7}$$

Now (3.1) and (4.6) yield

$$\left[\frac{2n+2}{2n+1}+\frac{2na_{2n}}{(2n+1)c_{2n}}\right]c_{2n+1}=1,$$

which shows that (1.4) holds true for $2n+1$. After that, in the same way we obtain from (4.7) that (1.4) is also satisfied for $2n+2$. This finishes the proof. □

Proof (*Theorem* 2.2). Since the recurrence relations (3.6) and (3.7) essentially have the same structure, the proof of Theorem 2.1 can be copied; we hence omit the details. □

The proof of Theorem 2.3 requires considerably more work because unknown Fourier coefficients of $\mathcal{A}_q$ occur in the course of the induction.

Proof (*Theorem* 2.3). Again, the non-obvious part is the direction "(vi) ⇒ (i)". Assuming (vi) to be valid, we use induction to establish (1.5) for all $n \in \mathbb{N}$. (1.5) is satisfied for $n=1$; so $\alpha_2(0|q)=\frac{(\beta-1)(1-q)}{4\sqrt{\beta q}}\sigma(2|q)$ by Lemma 3.1 and Lemma 3.2, and (3.8) yields

$$a_2\sigma(3|q)+c_2\sigma(1|q)=\left(\frac{\sqrt{q}}{2}+\frac{1}{2\sqrt{q}}\right)A\sigma(2|q)+\alpha_2(0|q)=\frac{1}{2}\left(\sqrt{\frac{\beta}{q}}+\sqrt{\frac{q}{\beta}}\right)\sigma(2|q).$$

Applying (3.3), we get

$$\frac{1}{2}\left(\sqrt{\frac{\beta}{q}}+\sqrt{\frac{q}{\beta}}\right)=\frac{1-q^3}{\sqrt{q}(1-q^2)}c_2+\frac{\sqrt{q}}{1+q}\frac{a_1}{c_1}c_2=\left(\sqrt{\frac{\beta}{q}}+\sqrt{\frac{q}{\beta}}\right)\frac{1-\beta q^2}{\sqrt{\beta}(1-q^2)}c_2$$

and therefore the validity of (1.5) for $n=2$. Now let $n \in \mathbb{N}$ be arbitrary but fixed, and suppose that (1.5) holds true for $1,\dots,2n$. Moreover, let $k \in \{0,\dots,n-1\}$ be such that $\kappa_{2n+3}(2k|q)=\sigma(2n+3|q)$. We proceed as in the proof of Theorem 2.1 (but use Theorem 1.3 instead of Theorem 1.1 and (3.8) instead of (3.6)) and take into account Lemma 3.1 and Lemma 3.2 which yield

$$\alpha_{2n+1}(2k+1|q)=\alpha_{2n+1}(2n-1|q)=\frac{(\beta-1)(1-q)}{4\sqrt{\beta q}}\sigma(2n+1|q). \tag{4.8}$$

In doing so, we obtain

$$a_{2n+1}\kappa_{2n+2}(2k+1|q)+c_{2n+1}\sigma(2n|q)=\frac{1}{2}\left(\sqrt{\frac{\beta}{q}}+\sqrt{\frac{q}{\beta}}\right)\sigma(2n+1|q), \tag{4.9}$$

$$a_{2n+1}\kappa_{2n+2}(2n-1|q)+c_{2n+1}\sigma(2n|q)=\frac{1}{2}\left(\sqrt{\frac{\beta}{q}}+\sqrt{\frac{q}{\beta}}\right)\sigma(2n+1|q), \tag{4.10}$$

$$\begin{aligned} &a_{2n+2}\sigma(2n+3|q)+c_{2n+2}\sigma(2n+1|q)\\ &\quad=\left(\frac{\sqrt{q}}{2}+\frac{1}{2\sqrt{q}}\right)[a_{2k}\kappa_{2n+2}(2k+1|q)+c_{2k}\kappa_{2n+2}(2k-1|q)]+\alpha_{2n+2}(2k|q), \end{aligned} \tag{4.11}$$

(4.12)

$$\begin{aligned} &a_{2n+2}\sigma(2n+3|q)+c_{2n+2}\sigma(2n+1|q)\\ &\quad=\left(\frac{\sqrt{q}}{2}+\frac{1}{2\sqrt{q}}\right)[a_{2n}\sigma(2n+2|q)+c_{2n}\kappa_{2n+2}(2n-1|q)]+\alpha_{2n+2}(2n|q)\end{aligned}$$

as analogues of (4.1), (4.2), (4.3) and (4.4). Again we distinguish two cases at this stage:

Case 1 ($k\neq 0$). Using (3.8), Lemma 3.1 and Lemma 3.2, we see that

$$\alpha_{2n+1}(2k-1|q)=\frac{(\beta-1)(1-q)}{4\sqrt{\beta q}}\sigma(2n+1|q)$$

and

$$a_{2n+1}\kappa_{2n+2}(2k-1|q)+c_{2n+1}\sigma(2n|q)=\frac{1}{2}\left(\sqrt{\frac{\beta}{q}}+\sqrt{\frac{q}{\beta}}\right)\sigma(2n+1|q).$$

The interplay with (4.9) and (4.10) then yields $\kappa_{2n+2}(2k+1|q)=\kappa_{2n+2}(2k-1|q)=\kappa_{2n+2}(2n-1|q)$, and (4.11) becomes

(4.13)

$$a_{2n+2}\sigma(2n+3|q)+c_{2n+2}\sigma(2n+1|q)=\left(\frac{\sqrt{q}}{2}+\frac{1}{2\sqrt{q}}\right)A\kappa_{2n+2}(2n-1|q)+\alpha_{2n+2}(2k|q).$$

Case 2 ($k=0$). Here (4.11) reads

$$a_{2n+2}\sigma(2n+3|q)+c_{2n+2}\sigma(2n+1|q)=\left(\frac{\sqrt{q}}{2}+\frac{1}{2\sqrt{q}}\right)A\kappa_{2n+2}(1|q)+\alpha_{2n+2}(0|q).$$

Since $\kappa_{2n+2}(1|q)=\kappa_{2n+2}(2n-1|q)$ due to (4.9) and (4.10), this reduces to the same result as in Case 1, i.e., to (4.13).

Equation (4.13) is the analogue to equation (4.5) above. Comparing the proof of Theorem 2.1 and the present proof, we observe that (4.5) was no longer dependent on k, whereas this dependence has *not* completely vanished in (4.13). This is the actual difficulty; moreover, note that $\alpha_{2n+2}(2k|q)$ is *not* determined by the induction hypothesis (i.e. $c_1,\ldots,c_{2n}$), and the same is the case with $\alpha_{2n+2}(2n|q)$ in (4.12). Our key idea to overcome these problems is to consider the auxiliary function $\beta_{2n+1}(.|q)$ defined in Section 3. From now on, we write an additional tilde when explicitly referring to the sequence $(P_n(x;\beta|q))_{n\in\mathbb{N}_0}$ (for instance $\widetilde{c}_n\left(=\frac{\sqrt{\beta}}{2}\frac{1-q^n}{1-\beta q^n}\right)$, $\widetilde{\sigma}(n|q)$ and so on). We first tackle $\alpha_{2n+2}(2n|q)$, then $\alpha_{2n+2}(2k|q)$. As a consequence of (3.5), we have

$$\begin{aligned} a_{2n+1}\alpha_{2n+2}(2n|q)&=D_{2n+2}\frac{\frac{2n+2}{4}-\sum_{k=1}^{2n+1}a_{k-1}c_k}{a_{2n}h(2n)}\\ &=D_{2n+2}\left[\frac{1-4a_{2n}c_{2n+1}}{4a_{2n}h(2n)}+\frac{\frac{2n+1}{4}-\sum_{k=1}^{2n}a_{k-1}c_k}{a_{2n}h(2n)}\right]\\ &=D_{2n+2}\left[\frac{1-4a_{2n}c_{2n+1}}{4a_{2n}h(2n)}+\frac{c_{2n}}{D_{2n+1}}\alpha_{2n+1}(2n-1|q)\right];\end{aligned}$$

using (3.3) and (4.8), we obtain

(4.14)

$$\begin{aligned}&\frac{a_{2n+1}\alpha_{2n+2}(2n|q)}{\sigma(2n+1|q)}\\ &\quad=-\frac{q^n-q^{n+1}}{1-q^{2n+1}}D_{2n+2}a_{2n}c_{2n+1}+\frac{(\beta-1)(1-q)}{4\sqrt{\beta q}}\frac{D_{2n+2}}{D_{2n+1}}c_{2n}+\frac{1}{4}\frac{q^n-q^{n+1}}{1-q^{2n+1}}D_{2n+2}\\ &\quad=\underbrace{\frac{(1-q)^2}{2q}a_{2n}}_{=:A_1(n)}c_{2n+1}+\underbrace{\frac{(\beta-1)(1-q)}{4\sqrt{\beta q}}\frac{D_{2n+2}}{D_{2n+1}}c_{2n}-\frac{(1-q)^2}{8q}}_{=:A_2(n)}.\end{aligned}$$

Moreover, we have

$$\frac{a_{2n+1}\alpha_{2n+2}(2k|q)}{\sigma(2n+1|q)}=\frac{\beta_{2n+1}(2k|q)-c_{2n+1}\alpha_{2n}(2k|q)}{\sigma(2n+1|q)}$$

and know from Lemma 3.1 that $\beta_{2n+1}(2k|q)$ is fixed by the induction hypothesis. Therefore, Lemma 3.2 implies

$$\begin{aligned}&\frac{a_{2n+1}\alpha_{2n+2}(2k|q)}{\sigma(2n+1|q)}\\ &\quad=\frac{\widetilde{a}_{2n+1}\widetilde{\alpha}_{2n+2}(2k|q)+\widetilde{c}_{2n+1}\alpha_{2n}(2k|q)-c_{2n+1}\alpha_{2n}(2k|q)}{\sigma(2n+1|q)}\\ &\quad=\frac{(\beta-1)(1-q)}{4\sqrt{\beta q}}\left[\widetilde{a}_{2n+1}\frac{\widetilde{\sigma}(2n+2|q)}{\sigma(2n+1|q)}+\widetilde{c}_{2n+1}\frac{\sigma(2n|q)}{\sigma(2n+1|q)}-c_{2n+1}\frac{\sigma(2n|q)}{\sigma(2n+1|q)}\right].\end{aligned}$$

Using (3.3), the latter becomes

(4.15)

$$\begin{aligned}\frac{a_{2n+1}\alpha_{2n+2}(2k|q)}{\sigma(2n+1|q)}=&\underbrace{-\frac{(\beta-1)(1-q)}{4\sqrt{\beta}}\frac{1-q^{2n}}{1-q^{2n+1}}\frac{a_{2n}}{c_{2n}}}_{=:A_3(n)}c_{2n+1}\\ &+\underbrace{\frac{(\beta-1)(1-q)}{4\sqrt{\beta}(q-q^{2n+2})}\left[(1-q^{2n+2})+(q-q^{2n+1})\frac{a_{2n}}{c_{2n}}\right]}_{=:A_4(n)}\widetilde{c}_{2n+1}.\end{aligned}$$

We now take the difference of (4.12) and (4.13); this yields

$$\begin{aligned}&\frac{a_{2n+1}\alpha_{2n+2}(2n|q)}{\sigma(2n+1|q)}-\frac{a_{2n+1}\alpha_{2n+2}(2k|q)}{\sigma(2n+1|q)}\\ &\quad=\left(\frac{\sqrt{q}}{2}+\frac{1}{2\sqrt{q}}\right)a_{2n}\left[\frac{a_{2n+1}\kappa_{2n+2}(2n-1|q)}{\sigma(2n+1|q)}-\frac{a_{2n+1}\sigma(2n+2|q)}{\sigma(2n+1|q)}\right].\end{aligned}\tag{4.16}$$

Using (4.10) and applying (3.3), we compute

$$\frac{a_{2n+1}\kappa_{2n+2}(2n-1|q)}{\sigma(2n+1|q)}=\frac{1}{2}\left(\sqrt{\frac{\beta}{q}}+\sqrt{\frac{q}{\beta}}\right)-\sqrt{q}\frac{1-q^{2n}}{1-q^{2n+1}}\frac{a_{2n}}{c_{2n}}c_{2n+1},$$

and as another consequence of (3.3) we get

$$\frac{a_{2n+1}\sigma(2n+2|q)}{\sigma(2n+1|q)}=\frac{1}{\sqrt{q}}\frac{1-q^{2n+2}}{1-q^{2n+1}}c_{2n+1}.$$

Thus (4.16) simplifies to

(4.17)
$$\frac{a_{2n+1}\alpha_{2n+2}(2n|q)}{\sigma(2n+1|q)} - \frac{a_{2n+1}\alpha_{2n+2}(2k|q)}{\sigma(2n+1|q)}$$
$$= \underbrace{-\frac{q+1}{2(q-q^{2n+2})}a_{2n}\left[(1-q^{2n+2})+(q-q^{2n+1})\frac{a_{2n}}{c_{2n}}\right]}_{=:A_5(n)} c_{2n+1} + \underbrace{\frac{(\beta+q)(q+1)}{4q\sqrt{\beta}}a_{2n}}_{=:A_6(n)}.$$

Combining (4.14), (4.15) and (4.17), we obtain

$$[A_1(n)-A_3(n)-A_5(n)]c_{2n+1} = -[A_2(n)-A_4(n)-A_6(n)]. \tag{4.18}$$

Since $A_1(n),\ldots,A_6(n)$ are determined by the induction hypothesis, we could compute c_{2n+1} from the latter equation by explicitly calculating and dividing the (complicated) terms on the left and right hand side. We choose a much shorter argument, however, and observe that (4.18) remains valid if c_{2n+1} is replaced by $\widetilde{c}_{2n+1}$. Considering the argument up to this point again, this is an obvious consequence of Theorem 1.3. We hence have $[A_1(n)-A_3(n)-A_5(n)](c_{2n+1}-\widetilde{c}_{2n+1}) = 0$, which yields $c_{2n+1} = \widetilde{c}_{2n+1}$ because

$$\frac{2\beta q(1-q^{2n+1})}{a_{2n}}[A_1(n)-A_3(n)-A_5(n)]$$
$$= (1-q)^2(1-q^{2n+1})\beta + (\beta-1)(1-q)q(1-\beta q^{2n}) + (\beta+q)(q+1)(1-\beta q^{2n+1})$$
$$> 0 - 1\cdot(1-q)q\cdot 1 + q\cdot 1\cdot(1-q^{2n+\frac{1}{2}}) > 0$$

(short calculation). This means that (1.5) is valid for $2n+1$. To finish the induction, it remains to establish (1.5) for $2n+2$. Since besides knowing $c_1,\ldots,c_{2n}$ the value of c_{2n+1} is also known now, Lemma 3.1 and Lemma 3.2 imply that both (4.12) and (4.13) reduce to

$$a_{2n+2}\sigma(2n+3|q) + c_{2n+2}\sigma(2n+1|q) = \frac{1}{2}\left(\sqrt{\frac{\beta}{q}}+\sqrt{\frac{q}{\beta}}\right)\sigma(2n+2|q).$$

A division by $\sigma(2n+2|q)$ and another application of (3.3) yield the desired result for c_{2n+2}. □

Proof (*Theorem* 2.4). We first do not impose the additional condition $\beta \leq 1$. The direction "(i) ⇒ (ii)" is immediate from Lemma 3.2. We now reformulate condition (ii) in a suitable way: given an arbitrary $n \in \mathbb{N}$, (3.3) combined with (3.5) yields

$$\frac{\alpha_{n+1}(n-1|q)}{\sigma(n+1|q)} = -\frac{(1-q)^2}{2\sqrt{q}}\frac{1-q^n}{1-q^{n+1}}\frac{\frac{n+1}{4}-\sum_{k=1}^{n}a_{k-1}c_k}{c_n}, \tag{4.19}$$

and in particular

$$\frac{\alpha_2(0|q)}{\sigma(2|q)} = -\frac{(1-q)^2}{2\sqrt{q}}\frac{1}{1+q}\frac{\frac{1}{2}-Ac_1}{c_1} = -\frac{(1-q)^2}{2\sqrt{q}}\frac{1-\beta}{2\sqrt{\beta}(1-q)}. \tag{4.20}$$

In view of (4.19) and (4.20), (ii) is equivalent to

$$\sum_{k=1}^{n} a_{k-1}c_k = \frac{n+1}{4} - \frac{1-\beta}{2\sqrt{\beta}(1-q)}\frac{1-q^{n+1}}{1-q^n}c_n \quad (n \in \mathbb{N}).$$

Moreover, the latter can be reformulated as

$$\left[a_n + \frac{1-\beta}{2\sqrt{\beta}(1-q)}\frac{1-q^{n+2}}{1-q^{n+1}}\right]c_{n+1} = \frac{1}{4} + \frac{1-\beta}{2\sqrt{\beta}(1-q)}\frac{1-q^{n+1}}{1-q^n}c_n \quad (n \in \mathbb{N}). \tag{4.21}$$

Let (ii) be satisfied, and assume that $\beta \leq 1$ now. We show that (1.5) is valid for each $n \in \mathbb{N}$: as this is trivial for $n = 1$, let $n \in \mathbb{N}$ be arbitrary but fixed and assume (1.5) to hold true for n. We compute

$$\begin{aligned}
&a_n + \frac{1-\beta}{2\sqrt{\beta}(1-q)}\frac{1-q^{n+2}}{1-q^{n+1}}\\
&\quad= \frac{2-q-\beta-q^{n+1}-\beta q^n+\beta q^{n+2}+\beta^2 q^{n+1}+\beta q^{2n+2}+\beta^2 q^{2n+1}-2\beta^2 q^{2n+2}}{2\sqrt{\beta}(1-q)(1-q^{n+1})(1-\beta q^n)}\\
&\quad= \frac{1-\beta q^{n+1}}{2\sqrt{\beta}(1-q^{n+1})}\frac{2-q-q^{n+1}-(1+q^n-2q^{n+1})\beta}{(1-q)(1-\beta q^n)}
\end{aligned}$$

and

$$\frac{1}{4} + \frac{1-\beta}{2\sqrt{\beta}(1-q)}\frac{1-q^{n+1}}{1-q^n}c_n = \frac{2-q-q^{n+1}-(1+q^n-2q^{n+1})\beta}{4(1-q)(1-\beta q^n)};$$

since $\beta \leq 1$, we have $2-q-q^{n+1}-(1+q^n-2q^{n+1})\beta \geq (1-q)(1-q^n) > 0$. Putting this all together, we can conclude from (4.21) that $c_{n+1} = \frac{\sqrt{\beta}}{2}\frac{1-q^{n+1}}{1-\beta q^{n+1}}$, i.e., that (1.5) is also satisfied for $n+1$. Thus (i) is established.

It is easy to see that the preceding argument concerning the direction "(ii) $\Rightarrow$ (i)" does not work if the assumption $\beta \leq 1$ is dropped (for $2-q-q^{n+1}-(1+q^n-2q^{n+1})\beta$ may become zero then). To see that not only the proof, but also the direction itself does not remain valid, we provide an explicit counterexample. We first use induction to establish that the recursion $C_1 := \frac{\sqrt{5}}{3}$, $C_2 := \frac{1}{2}$,

$$C_{n+1} := \frac{\sqrt{5} - \frac{2^{n+1}-1}{2^n-1}C_n}{9 - 4\sqrt{5}C_n - \frac{2^{n+2}-1}{2^{n+1}-1}} \quad (n \geq 2)$$

defines a sequence $(C_n)_{n\in\mathbb{N}}$ with $0 < C_n \leq \frac{1}{2}$ $(n \geq 2)$. As the initial step is clear, let $n \geq 2$ be arbitrary but fixed and assume that $C_n \in \left(0, \frac{1}{2}\right]$ is well defined. Since $9-4\sqrt{5}C_n - \frac{2^{n+2}-1}{2^{n+1}-1} > 0$, C_{n+1} is well defined. Moreover, since $\sqrt{5}-\frac{2^{n+1}-1}{2^n-1}C_n > 0$, we have $C_{n+1} > 0$, and it is just left to show that $C_{n+1} \leq \frac{1}{2}$. The latter is equivalent to $C_n \leq \frac{9-2\sqrt{5}-\frac{2^{n+2}-1}{2^{n+1}-1}}{4\sqrt{5}-\frac{2^{n+2}-2}{2^n-1}}$; it therefore suffices to observe that $\frac{1}{2} \leq \frac{9-2\sqrt{5}-\frac{2^{n+2}-1}{2^{n+1}-1}}{4\sqrt{5}-\frac{2^{n+2}-2}{2^n-1}}$. Now the preceding yields that $(c_n)_{n\in\mathbb{N}} := (C_n)_{n\in\mathbb{N}}$ and $A := \frac{9}{4\sqrt{5}}$ define a symmetric RWPS $(P_n(x))_{n\in\mathbb{N}_0}$ as in Section 1; putting $q := \frac{1}{2}$ and $\beta := \frac{5}{4}$, the conditions of Theorem 1.3 are fulfilled. However: obviously $(P_n(x))_{n\in\mathbb{N}_0} \neq (P_n(x;\beta|q))_{n\in\mathbb{N}_0}$ (for otherwise one would have $c_2 = \frac{3\sqrt{5}}{11}$), so (i) of Theorem 2.4 is not satisfied – nevertheless, (the reformulation (4.21) of) (ii) is valid, which follows from the construction of the sequence $(C_n)_{n\in\mathbb{N}}$. □

Remark 4.1. The proof of Theorem 2.4 "(ii) $\Rightarrow$ (i)" shows that the condition $\beta \leq 1$ cannot be dropped but weakened: in fact, it would be enough to require that $c_{n+1} = \frac{\sqrt{\beta}}{2}\frac{1-q^{n+1}}{1-\beta q^{n+1}}$ if $\frac{2-q-q^{n+1}}{1+q^n-2q^{n+1}} = \beta$ ($n \in \mathbb{N}$; there exists at most one such n, and if $\beta \leq 1$, then there is none). In view of the original condition, we recall its meaning for the sequence $(P_n(x;\beta|q))_{n\in\mathbb{N}_0}$: if $\beta \leq 1$, then the orthogonalization measure μ is absolutely continuous (w.r.t. the Lebesgue–Borel measure on $\mathbb{R}$) and one has $\operatorname{supp} \mu = [-1, 1]$, whereas if $\beta > 1$, then point measures appear at $\pm A$ and $\max \operatorname{supp} \mu = A > 1$ [8].

References

[1] W. A. Al-Salam, *Characterization theorems for orthogonal polynomials*, Orthogonal polynomials (Columbus, OH, 1989), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 294, Kluwer Acad. Publ., Dordrecht, 1990, pp. 1–24, DOI 10.1007/978-94-009-0501-6_1. MR1100286 (92g:42011)

[2] W. A. Al-Salam and T. S. Chihara, *Another characterization of the classical orthogonal polynomials*, SIAM J. Math. Anal. **3** (1972), 65–70. MR0316772 (47 #5320)

[3] T. S. Chihara, *An introduction to orthogonal polynomials*, Mathematics and its Applications, Vol. 13, Gordon and Breach Science Publishers, New York-London-Paris, 1978. MR0481884 (58 #1979)

[4] Somjit Datta and James Griffin, *A characterization of some q-orthogonal polynomials*, Ramanujan J. **12** (2006), no. 3, 425–437, DOI 10.1007/s11139-006-0152-5. MR2293799 (2008i:33029)

[5] W. Hahn, *Über die Jacobischen Polynome und zwei verwandte Polynomklassen*, Math. Z. **39** (1935), no. 1, 634–638. MR1545524

[6] Wolfgang Hahn, *Über Orthogonalpolynome, die q-Differenzengleichungen genügen* (German), Math. Nachr. **2** (1949), 4–34. MR0030647 (11,29b)

[7] Mourad E. H. Ismail, *Classical and quantum orthogonal polynomials in one variable*, Encyclopedia of Mathematics and its Applications, vol. 98, Cambridge University Press, Cambridge, 2009. With two chapters by Walter Van Assche; With a foreword by Richard A. Askey; Reprint of the 2005 original. MR2542683 (2010i:33001)

[8] Mourad E. H. Ismail and Josef Obermaier, *Characterizations of continuous and discrete q-ultraspherical polynomials*, Canad. J. Math. **63** (2011), no. 1, 181–199, DOI 10.4153/CJM-2010-080-0. MR2779137 (2011m:42052)

[9] Mourad E. H. Ismail and Plamen Simeonov, *Connection relations and characterizations of orthogonal polynomials*, Adv. in Appl. Math. **49** (2012), no. 2, 134–164, DOI 10.1016/j.aam.2012.04.004. MR2946429

[10] Roelof Koekoek, Peter A. Lesky, and René F. Swarttouw, *Hypergeometric orthogonal polynomials and their q-analogues*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2010. With a foreword by Tom H. Koornwinder. MR2656096 (2011e:33029)

[11] Rupert Lasser, *Amenability and weak amenability of l^1-algebras of polynomial hypergroups*, Studia Math. **182** (2007), no. 2, 183–196, DOI 10.4064/sm182-2-6. MR2338484 (2008k:43012)

[12] R. Lasser and J. Obermaier, *A new characterization of ultraspherical polynomials*, Proc. Amer. Math. Soc. **136** (2008), no. 7, 2493–2498, DOI 10.1090/S0002-9939-08-09378-7. MR2390518 (2009d:33025)

Department of Mathematics, Chair for Mathematical Modelling, Chair for Mathematical Modeling of Biological Systems, Technische Universität München, Boltzmannstr. 3, 85747 Garcing b. München, Germany

E-mail address: kahler@ma.tum.de