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CONJUGACY CLASS CONDITIONS IN LOCALLY COMPACT SECOND COUNTABLE GROUPS

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ABSTRACT. Many non-locally compact second countable groups admit a comeagre conjugacy class. For example, this is the case for S_∞, $Aut(\mathbb{Q},<)$, and, less trivially, $Aut(\mathcal{R})$ for $\mathcal{R}$ the random graph. A. Kechris and C. Rosendal ask if a non-trivial locally compact second countable group can admit a comeagre conjugacy class. We answer the question in the negative via an analysis of locally compact second countable groups with topological conditions on a conjugacy class.

1. Introduction

Our goal is to answer a question of Kechris and Rosendal [7]: Can a non-trivial locally compact second countable group admit a comeagre conjugacy class? By an unpublished argument due to K.H. Hofmann, such a group cannot be connected. This, of course, leaves the problematic totally disconnected locally compact case. Here the theory is not nearly as well developed. We therefore begin with an analysis of *totally disconnected locally compact second countable* (t.d.l.c.s.c.) groups.

Theorem 1.1. *Suppose U is a second countable profinite group and $g \in U$. The following are equivalent:*

(1) $g^U := \{ugu^{-1} \mid u \in U\}$ is non-meagre.
(2) g^U is open.
(3) $\mu(g^U) > 0$ where μ is the normalized Haar measure on U.
(4) There is an integer $M > 0$ such that $|C_{U/V}(gV)| \leqslant M$ for every open normal subgroup $V \trianglelefteq U$.

Recall $C_{U/V}(gV)$ denotes the centralizer of gV in U/V. We extend Theorem 1.1 to the non-compact case.

Corollary 1.2. *Suppose G is a t.d.l.c.s.c. group and $g \in G$ is such that $cl(\langle g\rangle)$ is compact. The following are equivalent:*

(1) g^G is open.
(2) g^G is non-meagre.
(3) g^G is non-null.

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Applying the above results and two deep theorems in profinite group theory, we eliminate t.d.l.c.s.c. groups as possible candidates for a positive answer to the motivating question.

Theorem 1.3. *If G is a non-trivial t.d.l.c.s.c. group and g^G is dense, then g^G is meagre and Haar null.*

We conclude by presenting Hofmann's proof in the connected case to give a complete answer to the question of Kechris and Rosendal.

Theorem 1.4 (Hofmann)**.** *A non-trivial connected locally compact group cannot have a dense conjugacy class.*

Theorem 1.5. *A non-trivial locally compact second countable group does not admit a comeagre conjugacy class.*

Along the way, we present an example due to Rosendal showing the existence of an infinite profinite group with a non-meagre conjugacy class, and we sketch the example built by E. Akin, E. Glasner, and B. Weiss [1] of a non-trivial t.d.l.c.s.c. group with a dense conjugacy class.

2. Preliminaries

We fix a few conventions and notations for this paper. All topological groups are assumed to be Hausdorff, and all subgroups are taken to be closed unless otherwise noted. To indicate O is an open subgroup of a topological group G, we write $O \leqslant_o G$. For $g \in G$ and $A \subseteq G$, $g^A := \{aga^{-1} \mid a \in A\}$. We use the abbreviations l.c., t.d., and s.c., for "locally compact", "totally disconnected", and "second countable", respectively.

2.1. **L.c.s.c. groups.** L.c.s.c. groups are K_σ and metrizable by classical results [6]. These groups have a canonical left invariant Borel measure which is unique up to constant multiples called the *Haar measure*. When a subset of a group is said to be *non-null*, we mean with respect to the Haar measure. A familiarity with the Haar measure and basic properties thereof is assumed; [3] contains a nice, brief introduction.

L.c.s.c. groups are connected-by-totally disconnected. Indeed, for G a l.c.s.c. group, let G° denote the connected component of the identity. It is easy to see G° is a closed normal subgroup and there is a short exact sequence of topological groups

$$1 \to G^\circ \to G \to G/G^\circ \to 1$$

where G° is connected and G/G° is totally disconnected. The study of l.c.s.c. groups thus reduces to the study of connected l.c.s.c. groups and t.d.l.c.s.c. groups.

The study of connected locally compact groups reduces to the study of inverse limits of Lie groups by the celebrated solution to Hilbert's fifth problem.

Theorem 2.1 (Gleason, Montgomery, Yamabe, Zippin; see [10])**.** *A connected locally compact group is pro-Lie.*

A group G is a *Lie group* if G has an analytic $\mathbb{K}$-manifold structure such that the map $(g,h) \mapsto gh^{-1}$ is analytic where $\mathbb{K}$ is either $\mathbb{R}, \mathbb{C}$ or some non-discrete ultrametric field. If a Lie group G is connected and locally compact, then $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$, and G is finite dimensional. Associated to a Lie group G is the

Lie algebra, $\mathfrak{g}$, where $\mathfrak{g}$ is the tangent space at the identity along with a bracket operation. The self action of G by conjugation induces an action of G on $\mathfrak{g}$ by vector space isomorphisms. When G is connected and locally compact, this action is given by the map $Ad : G \to GL(\mathfrak{g})$ where $GL(\mathfrak{g}) = GL_n(\mathbb{K})$ for some finite n and $\mathbb{K}$ is equal to $\mathbb{R}$ or $\mathbb{C}$.

Fact 2.2 ([2, III.6.4 Corollary 4])**.** If G is a connected locally compact Lie group, then $Ad : G/Z(G) \to im(Ad)$ is an isomorphism of Lie groups.

Proofs of the aforementioned properties of Lie groups may also be found in [2].

For t.d.l.c. groups, a central theorem is an old result of D. van Dantzig.

Theorem 2.3 (van Dantzig; see [4, II.7.7])**.** *A t.d.l.c. group admits a basis at the identity of compact open subgroups.*

Elements of a t.d.l.c. group that lie in a compact open subgroup are called *periodic*. Note that periodic elements do not necessarily have finite order; we call an element *torsion* if it has finite order. For a t.d.l.c. group G, we denote the collection of periodic elements of G by $P_1(G)$. It is easy to see

$$P_1(G) = \{g \in G \mid cl(\langle g \rangle) \text{ is compact}\}.$$

2.2. **Profinite groups.** The compact open subgroups of a t.d.l.c. group given by van Dantzig's theorem are *profinite*, i.e. they are inverse limits of finite groups. An introduction to the theory of profinite groups may be found in the texts [11] and [14]. We assume a familiarity with profinite groups and merely recall a few relevant definitions and theorems.

Profinite groups have a basis at 1 consisting of open normal subgroups. For a second countable profinite group U, we say $(N_i)_{i\in\mathbb{N}}$ is a *normal basis at* 1 for U if each N_i is open and normal in U, $\bigcap_{i\in\mathbb{N}} N_i = \{1\}$, and $(N_i)_{i\in\mathbb{N}}$ is $\subseteq$-decreasing.

The *Frattini subgroup* of a profinite group U, denoted $\Phi(U)$, is the intersection of all maximal proper open subgroups. The Frattini subgroup is the collection of *non-generators* of U: $x \in U$ is a non-generator if whenever $U = \overline{\langle X \cup \{x\}\rangle}$ for $X \subseteq U$, then $U = \overline{\langle X \rangle}$. This implies a useful observation: If $H \leqslant U$ is closed and $H\Phi(U) = U$, then $H = U$. We note an easy and illuminating proof: Suppose $H \leqslant U$ is closed and $H\Phi(U) = U$. If $H \neq U$, then there is some maximal proper open $W \lneq U$ such that $H \leqslant W$. However, then $H\Phi(U) \leqslant W$ contradicting that $H\Phi(U) = U$. We thus conclude $H = U$.

When U is pro-p, an inverse limit of p-groups, $\Phi(U)$ has a well-understood structure.

Fact 2.4 ([11, Lemma 2.8.7])**.** If U is pro-p, then $\Phi(U) = U^p[U, U]$ where $[U, U]$ is the closure of the commutator subgroup and U^p is the closed subgroup generated by all p powers.

We also note two deep results in profinite group theory.

Theorem 2.5 (Zel'manov [15])**.** *Every torsion pro-p group is locally finite.*

Theorem 2.6 (Wilson [13])**.** *Let U be a compact Hausdorff torsion group. Then U has a finite series*

$$\{1\} = U_0 \leqslant U_1 \leqslant \ldots \leqslant U_n = U$$

*of closed characteristic subgroups in which each factor U_i/U_{i-1} is either **(1)** pro-p for some prime p or **(2)** isomorphic to a Cartesian product of isomorphic finite simple groups.*

3. Non-meagre conjugacy classes

We first consider t.d.l.c.s.c. groups with a non-meagre conjugacy class. The key step is to initially consider profinite groups with a *non-null* conjugacy class.

Lemma 3.1. *Let U be a profinite group with normalized Haar measure μ. If $h \in U$ is such that $\mu(h^U) > 0$, then*

(1) For all $N \trianglelefteq_o U$, $|C_{U/N}(hN)| \leqslant \frac{1}{\mu(h^U)}$.

(2) $|C_U(h)| \leqslant \frac{1}{\mu(h^U)}$, and in particular, h is torsion.

Proof. Let $h \in U$ have a non-null conjugacy class and take $N \trianglelefteq_o U$. Take a minimal set of coset representatives $h, k_1hk_1^{-1}, \dots, k_nhk_n^{-1}$ for $(hN)^{U/N}$ in U/N. Certainly,

$$h^U \subseteq hN \cup \dots \cup k_nhk_n^{-1}N,$$

so

$$0 < \mu(h^U) \leqslant |(hN)^{U/N}|\mu(N).$$

Since $1 = \mu(U) = |U/N|\mu(N)$, we have

$$\begin{aligned} |(hN)^{U/N}|\mu(N) &= |U/N : C_{U/N}(hN)|\mu(N) \\ &= \tfrac{|U/N|\mu(N)}{|C_{U/N}(hN)|} \\ &= \tfrac{1}{|C_{U/N}(hN)|}. \end{aligned}$$

Hence, $|C_{U/N}(hN)| \leqslant \frac{1}{\mu(h^U)}$, and it follows that $|C_U(h)| \leqslant \frac{1}{\mu(h^U)}$. □

Via Lemma 3.1, category and measure theoretic notions of size for conjugacy classes in second countable profinite groups agree.

Theorem 3.2. *Suppose U is a second countable profinite group and $g \in U$. The following are equivalent:*

(1) g^U is non-meagre.

(2) g^U is open.

(3) $\mu(g^U) > 0$ where μ is the normalized Haar measure on U.

(4) There is $M > 0$ such that $|C_{U/N}(gN)| \leqslant M$ for all $N \trianglelefteq_o U$.

Proof. (1) ⇒ (2) Since g^U is non-meagre, it is somewhere dense. However, g^U is closed since the continuous image of a compact set, and therefore g^U contains a non-empty open set. Say $O \subseteq g^U$. So

$$g^U = \bigcup_{u \in U} uOu^{-1},$$

and g^G is open.

(2) ⇒ (3) is immediate from properties of the Haar measure.

For (3) ⇒ (4), take $N \trianglelefteq_o U$. By Lemma 3.1,

$$|C_{U/N}(gN)| \leqslant \frac{1}{\mu(g^U)}.$$

Fixing $M \geqslant \frac{1}{\mu(g^U)}$, we have (4).

(4) ⇒ (1) Fix $(N_i)_{i \in \mathbb{N}}$ a normal basis at 1 for U and take $M > 0$ which witnesses (4) for g. For each i, let $A_i \subseteq g^U$ be a minimal set of coset representatives for $(gN_i)^{U/N_i}$ in U/N_i.

For each $i \in \mathbb{N}$,

$$g^U \subseteq A_i N_i := \bigcup_{a \in A_i} aN_i,$$

and it is easy to check

$$\bigcap_{i \in \mathbb{N}} A_i N_i = g^U.$$

Since $(A_i N_i)_{i \in \mathbb{N}}$ is an $\subseteq$-decreasing sequence,

$$\mu(A_i N_i) \to \mu(g^U)$$

by continuity from above for μ.

On the other hand, $|C_{U/N_i}(gN_i)| \leqslant M$ for any $i \in \mathbb{N}$, and therefore, $|C_{U/N_i}(gN_i)|$ only takes on finitely many values as i varies. By passing to a subsequence, we may assume $|C_{U/N_i}(gN_i)| = k \leqslant M$ for all i. So

$$\mu(A_i N_i) = |(gN_i)^{U/N}|\mu(N_i) = \frac{1}{|C_{U/N_i}(gN_i)|} = \frac{1}{k}$$

for each i, and it follows that $\mu(A_i N_i) = \mu(g^U)$ for all i.

We now fix an i and consider

$$E := (A_i N_i) \setminus g^U.$$

Certainly E is open and $\mu(E) = 0$. As the only Haar null open set is $\emptyset$, $E = \emptyset$, and $A_i N_i = g^U$. We conclude g^U is open and, a fortiori, non-meagre. □

We remark that (4) of the above theorem is an algebraic characterization of a profinite group having an open conjugacy class.

We now apply our results for second countable profinite groups to obtain a result for all t.d.l.c.s.c. groups.

Lemma 3.3. *Let G be a t.d.l.c.s.c. group. If g^G is non-null, then g^U is non-null for any compact open subgroup U of G. If g is also periodic, then g is torsion and g^U is open for any compact open subgroup U containing g.*

Proof. Fix $U \leqslant_o G$ compact. Since G is second countable, there is a countable set $(h_i)_{i \in \mathbb{N}}$ such that $G = \bigcup_{i \in \mathbb{N}} h_i U$. So $g^G = \bigcup_{i \in \mathbb{N}} g^{h_i U}$, and for some $i \in \mathbb{N}$, $\mu(g^{h_i U}) > 0$. Fix such an i; now

$$0 < \mu(g^{h_i U}) = \mu(h_i g^U h_i^{-1}) = \mu(g^U)\Delta(h_i^{-1})$$

where Δ is the modular function. Since the modular function is strictly positive, $\mu(g^U) > 0$ as desired.

If g is also periodic, take W a compact open subgroup containing g. By the uniqueness of the Haar measure, g^W is non-null in W with respect to the normalized Haar measure on W. Lemma 3.1 then implies g is torsion, and Theorem 3.2 implies g^W is open. □

Corollary 3.4. *Suppose G is a t.d.l.c.s.c. group and g is periodic. The following are equivalent:*

(1) g^G is open.
(2) g^G is non-meagre.
(3) g^G is non-null.

Proof. (1) ⇒ (2) This follows by the Baire category theorem.

(2) ⇒ (3) Suppose g^G is non-meagre. Since G is K_σ, we may write $G = \bigcup_{i\in\mathbb{N}} K_i$ with the K_i compact, so $g^G = \bigcup_{i\in\mathbb{N}} g^{K_i}$. The Baire category theorem implies g^{K_i} is non-meagre for some i; fix such an i. We have that g^{K_i} is also closed and, thereby, has non-empty interior. It follows that $\mu(g^G) > 0$ proving (3).

(3) ⇒ (1) This is immediate from Lemma 3.3. □

It is not clear there are non-discrete examples of the groups discussed in this section. For completeness, we present an example due to C. Rosendal of a non-discrete second countable profinite group with a non-null conjugacy class. The author wishes to express his thanks to Rosendal for allowing this example to be included in the present work.

3.1. **Rosendal's example.** Let D_6 be the dihedral group of the triangle. Recall

$$D_6 = \langle s, r \mid r^3 = 1,\ s^2 = 1,\ \text{and } sr = r^{-1}s\rangle.$$

Every element of D_6 is of the form r^i or sr^i for $i = 0, 1, 2$. Form $D_6^{\mathbb{N}}$ and define $G \leqslant D_6^{\mathbb{N}}$ to be the collection of α such that all coordinates of α are of the form sr^i or all coordinates are of the form r^i. Certainly, G is a closed subset of $D_6^{\mathbb{N}}$ and is closed under inverses. It is easy to check G is also closed under multiplication.

Let $H \trianglelefteq G$ be the collection of $\alpha \in G$ for which all coordinates are of the form r^i. So H is closed and index two in G, whereby H is also open. Now consider the element $\beta \in G$ which is constantly equal to s.

Claim. Every $\gamma \in G \setminus H$ is conjugate to β by an element of H.

Proof. Let $\gamma \in G \setminus H$ and say $\gamma(i) = sr^{j(i)}$ where $j(i) \in \{0, 1, 2\}$. Define $\eta \in H$ as follows:

$$\eta(i) = \begin{cases} 1 & \text{if } j(i) = 0, \\ r^2 & \text{if } j(i) = 1, \\ r & \text{if } j(i) = 2. \end{cases}$$

One checks $\eta\gamma\eta^{-1} = \beta$. □

By the claim, $G \setminus H$ is the conjugacy class of β, so β^G is open and non-null.

Remark 3.5. We conclude this section with two remarks.

(1) Theorem 3.2 gives a measure and category equivalence which may be independently useful. For example, Theorem 3.2 seems potentially useful to answer the following open question asked by L. Lévai and L. Pyber in [9]: Let U be profinite and put $T_n := \{x \in U \mid x^n = 1\}$ for $n \geqslant 2$. If $\mu(T_n) > 0$, then is T_n non-meagre? For $n = 2$, the question is known to have a positive answer [9].

(2) Rosendal's example is solvable. It is unknown if all such examples must be virtually solvable. The following partial results are known to the author: (*i*) Second countable pronilpotent groups with an open conjugacy class are solvable. (*ii*) Second countable profinite groups U such that $g \in U$ has an open conjugacy class and $|g|$ is prime are virtually solvable.

4. Dense conjugacy classes

In this section, we consider a much different topological condition on a conjugacy class: density.

Lemma 4.1. *A torsion pro-p group with an open conjugacy class is finite.*

Proof. Suppose U is a torsion pro-p group with an open conjugacy class h^U. Since $h^{-1}h^U \subseteq [U,U]$, we have that $[U,U]$ is open, and $\Phi(U)$ is open by Fact 2.4. Let $u_1,\dots,u_k$ be coset representatives for $\Phi(U)$ in U. Plainly,

$$U = cl(\langle u_1,\dots,u_n\rangle)\Phi(U),$$

so $U = cl(\langle u_1,\dots,u_n\rangle)$ since $\Phi(U)$ is the collection of non-generators. Zel'manov's theorem, Theorem 2.5, now implies U is finite. □

Theorem 4.2. *If G is a non-trivial t.d.l.c.s.c. group and g^G is dense, then g^G is meagre and null.*

Proof. Suppose toward a contradiction G is a non-trivial t.d.l.c.s.c. group and g^G is dense and either non-meagre or non-null. Since g^G is dense, $g \in P_1(G)$ by Theorem 2.3. Corollary 3.4 now implies g^G is open, and by Lemma 3.3, g is torsion. Say $|g| = n$ and observe G must have exponent n.

Consider U a compact open subgroup of G. Since U is torsion, there is a series of closed characteristic subgroups

$$\{1\} = U_0 \leqslant U_1 \leqslant \dots \leqslant U_n = U$$

given by Wilson's theorem, Theorem 2.6. Let $k < n$ be greatest such that U_k is not open in U. Since U_{k+1} is open, g^G meets U_{k+1}; without loss of generality, $g \in U_{k+1}$. So Lemma 3.3 implies $g^{U_{k+1}}$ is open, and therefore, U_{k+1}/U_k also has an open conjugacy class.

By Lemma 4.1, U_{k+1}/U_k cannot be pro-p since else U_k is a finite index and, therefore, open. We conclude U_{k+1}/U_k must be isomorphic to a Cartesian product of isomorphic finite simple groups. Say $U_{k+1}/U_k \simeq \prod_{i\in I} S_i =: S$ and let $(s_i)_{i\in I} = s \in S$ have an open conjugacy class. Lemma 3.1 implies s has a finite centralizer. However,

$$C_S(s) = \prod_{i\in I} C_{S_i}(s_i),$$

and each $C_{S_i}(s_i)$ contains at least two elements. So S must be a finite product, and U_k is again open. We have thus contradicted the choice of k. □

Corollary 4.3. *If G is a non-trivial, t.d.l.c.s.c. group, then G does not admit a comeagre or co-null conjugacy class.*

Remark 4.4. Corollary 4.3 shows that a non-discrete t.d.l.c.s.c. analogue of an infinite discrete group with two conjugacy classes, e.g. [5], is impossible. The main result of this paper shows indeed there is no such non-trivial l.c.s.c. group.

We note the hypotheses of Theorem 4.2 are not vacuous. Indeed, Akin, Glasner, and Weiss have built an example of a non-trivial t.d.l.c.s.c. group with a dense conjugacy class in [1]. We include a sketch of their example for completeness.

4.1. **The example of Akin, Glasner, and Weiss.** Let $\mathcal{J} = \{J_i \mid i \in \mathbb{N}\}$ be a sequence of non-empty finite subsets of $\mathbb{N}$ which partition $\mathbb{N}$ and have strictly increasing cardinality. Let $J^k := \bigcup_{i=0}^k J_i$ and let K_n be the collection of permutations in $Sym(\mathbb{N})$ which setwise stabilize each of $J^n, J_{n+1}, J_{n+2}, \dots$; K_n is the collection of permutations of $\mathbb{N}$ which preserve the partition beyond the n-th part. It is easy to see K_n is compact as a subset of $Sym(\mathbb{N})$ and $K_n \leqslant_o K_{n+1}$. Put $G := \bigcup_{i\in\mathbb{N}} K_i$ and give G the inductive topology: $A \subseteq G$ is open if and only if $A \cap K_i$ is open in K_i for all i. Akin, Glasner, and Weiss show G is a t.d.l.c.s.c. group. Further, the sets

$$G(\pi) := \{g \in K_n \mid g \restriction_{J^n} = \pi\}$$

where $n \in \mathbb{N}$ and $\pi \in Sym(J^n)$ vary form a basis for the topology on G.

Claim. G has a dense conjugacy class.

Proof. It is enough to show G is topologically transitive, i.e. for all basic open sets $G(\pi), G(\xi) \subseteq G$ there is $k \in G$ such that $kG(\pi)k^{-1} \cap G(\xi) \neq \emptyset$. Without loss of generality, we may assume $\pi, \xi \in Sym(J^n)$ for some n. Choose $k > n$ such that $|J_k| > |J^n|$; this is possible since the J_i are strictly increasing in cardinality. Fix an injective map $\beta : J^n \to J_k$ and extend β to an element b of G by

$$b(i) = \begin{cases} \beta(i), & i \in J^n, \\ \beta^{-1}(i), & i \in \beta(J^n), \\ id, & \text{else.} \end{cases}$$

It is easy to check that there is $a \in G(\pi)$ such that $ab \restriction_{J^n} = b\xi \restriction_{J^n}$. We thus have $b^{-1}ab \in b^{-1}G(\pi)b \cap G(\xi)$ as desired. ☐

Remark 4.5. It is worth noting the above example has *ample dense elements*: the diagonal action by conjugation of G on the n-th Cartesian power of G has a dense orbit for every $n \geqslant 1$.

5. The non-existence of a comeagre conjugacy class

By Corollary 4.3, a non-trivial t.d.l.c.s.c. group does not admit a comeagre conjugacy class. We now eliminate connected groups as candidates for admitting a comeagre class. The connected case is a previously unpublished result of Professor Hofmann. The author wishes to express his thanks to Professor Hofmann for permitting this result to be included in the present work.

To present Hofmann's result, an old theorem due to W. Burnside is required.

Fact 5.1 (Burnside; see [8, XVII.3 Corollary 3.3])**.** Let E be a finite dimensional vector space over an algebraically closed field k and R a subalgebra of $End_k(E)$. If E has no non-trivial proper R-invariant subspaces, then $R = End_k(E)$. We say E is R-simple in such a case.

Lemma 5.2. *If $G \leqslant GL_n(\mathbb{C})$ is a closed subgroup with a dense conjugacy class, then $G = \{Id\}$.*

Proof. We consider general linear groups to be written as matrix groups in the standard basis. Suppose $G \leqslant GL_n(\mathbb{C})$ and say $A \in G$ has a dense conjugacy class.

Since A^G is dense, we may find $B_iAB_i^{-1} \to Id$ with $B_i \in G$. The determinant $\det : M_n(\mathbb{C}) \to \mathbb{C}$ is continuous, and therefore,

$$\det(tId - B_iAB_i^{-1}) \to \det(tId - Id) = (t-1)^n.$$

Since det is invariant under conjugation by elements of $GL_n(\mathbb{C})$, $\det(tId - B_iAB_i) = \det(tId - A)$ for all i, so $\det(tId - A) = (t-1)^n$. It follows that every $B \in G$ has characteristic polynomial $(t-1)^n$. By the Cayley−Hamilton theorem, $(B - Id)^n = 0$ for all $B \in G$.

Consider $B, C \in G$ and let Tr be the usual trace function. Then,

$$Tr(C(B - Id)) = Tr(CB - Id) - Tr(C - Id) = 0 \tag{1}$$

since the trace is linear and Tr vanishes on elements for which some power is zero. Note (1) holds for any linear group over $\mathbb{C}$ with a dense conjugacy class.

We now consider E a least dimension non-trivial G-invariant subspace of $\mathbb{C}^n$. Let $r : G \to GL(E)$ be the induced map. Certainly, $\tilde{G} := cl(r(G))$ again has a dense conjugacy class, and E is a $\tilde{G}$-simple vector space. Set $R \subseteq End_{\mathbb{C}}(E)$ to be the algebra generated by $\tilde{G}$. Since R contains $\tilde{G}$, we see that E is R-simple and $R = End_{\mathbb{C}}(E)$ by Burnside's theorem, Fact 5.1.

Fix $B \in \tilde{G}$ and take $M \in End_{\mathbb{C}}(E)$. Since $M = \sum_{i=1}^{m} \alpha_i A_i$ where $A_i \in \tilde{G}$ and $\alpha_i \in \mathbb{C}$, we have that $Tr(M(B - Id)) = 0$ by (1). Consider M_{ij} the matrix which is 1 on the (i,j)-th entry and 0 else. So $M_{ij}(B - Id)$ is the matrix with diagonal consisting of zeros and the (j,i)-th entry of $(B - Id)$. In view of (1), the (j,i)-th entry must be zero. We conclude $\tilde{G}$ is trivial, every element of G fixes E, and E is one dimensional. Since $E =: E_1$ is fixed by G, the action of G on $\mathbb{C}^n$ gives an action on $\mathbb{C}^n/E_1$. We may thus repeat the argument above to obtain $E_1 < E_2 < \cdots < E_n = \mathbb{C}^n$, a strictly increasing sequence of G-invariant vector subspaces such that E_{i+1}/E_i is one dimensional and G acts trivially on E_{i+1}/E_i.

Form $K_i := \ker(G \curvearrowright E_i)$ for each $1 \leqslant i \leqslant n$. Certainly, $K_1 = G$. For K_2, E_2 has two dimensions; say $E_2 = M \oplus E_1$. Fix $m + e \in E_2$, take $A, B \in G$, and consider their action on $m+e$. Since G acts trivially on E_1 and E_2/E_1, $A(m+e) = m+\alpha e+e$ and $B(m+e) = m+\beta e+e$ for some $\alpha, \beta \in \mathbb{C}$. Additionally, $A^{-1}(m+e) = m-\alpha e+e$ and $B^{-1}(m+e) = m - \beta e + e$. These observations yield

$$\begin{aligned} A^{-1}B^{-1}AB(m+e) &= A^{-1}B^{-1}A(m+\beta e+e) \\ &= A^{-1}B^{-1}(m+\alpha e+\beta e+e) \\ &= A^{-1}(m-\beta e+\alpha e+\beta e+e) \\ &= (m-\alpha e-\beta e+\alpha e+\beta e+e) \\ &= m+e. \end{aligned}$$

Thus, $A^{-1}B^{-1}AB \in K_2$, G/K_2 is abelian, and $K_2 = G$ since G/K_2 has a dense conjugacy class. Continuing in this fashion, $G = K_n$, and G acts trivially on $\mathbb{C}^n$. We have thus demonstrated $G = \{1\}$. □

Theorem 5.3 (Hofmann). *A non-trivial connected locally compact group cannot have a dense conjugacy class.*

Proof. For contradiction, suppose G is a non-trivial connected locally compact group with a dense conjugacy class. By Theorem 2.1, G is pro-Lie. We may thereby find a proper closed subgroup $N \triangleleft G$ such that G/N is Lie. Since N is proper, G/N has a dense conjugacy class.

Let $\tilde{G} := Ad(G/N) \leqslant GL_n(\mathbb{K})$ where $\mathbb{K}$ is either the real or complex field; under the natural inclusion $GL_n(\mathbb{R}) \leqslant GL_n(\mathbb{C})$, we may assume $\mathbb{K} = \mathbb{C}$. Fact 2.2 implies $\tilde{G}$ has a dense conjugacy class, and therefore, $\tilde{G} = \{1\}$ by Lemma 5.2. So G/N is trivial since it is abelian with a dense conjugacy class, contradicting the choice of N. □

Combining Corollary 4.3 and Theorem 5.3, we answer Kechris' and Rosendal's question.

Theorem 5.4. *A non-trivial l.c.s.c. group does not admit a comeagre conjugacy class.*

Proof. Suppose for contradiction G is an l.c.s.c. group and g^G is comeagre. Since G is K_σ, we may write $G = \bigcup_{i\in\mathbb{N}} K_i$ with the K_i compact sets. So $g^G = \bigcup_{i\in\mathbb{N}} g^{K_i}$, and there is $i \in \mathbb{N}$ such that g^{K_i} is non-meagre by the Baire category theorem. For such an i, we see that g^{K_i} is closed and, therefore, must have non-empty interior. It follows that g^G is open as well as dense.

If G is connected, we contradict Theorem 5.3, so $\tilde{G} := G/G^\circ$ must be non-trivial where G° is the connected component of the identity. However, the image of g^G in G/G° is now an open dense conjugacy class in the non-trivial t.d.l.c.s.c. group G/G°. This contradicts Corollary 4.3, and we conclude the theorem. □

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References

[1] Ethan Akin, Eli Glasner, and Benjamin Weiss, *Generically there is but one self homeomorphism of the Cantor set*, Trans. Amer. Math. Soc. **360** (2008), no. 7, 3613–3630, DOI 10.1090/S0002-9947-08-04450-4. MR2386239 (2008m:22009)

[2] N. Bourbaki, *Lie groups and Lie algebras. Chapters 1–3*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 1998, Translated from the French, Reprint of the 1989 English translation. MR1728312 (2001g:17006)

[3] Anton Deitmar and Siegfried Echterhoff, *Principles of harmonic analysis*, Universitext, Springer, New York, 2009. MR2457798 (2010g:43001)

[4] Edwin Hewitt and Kenneth A. Ross, *Abstract harmonic analysis. Vol. I*, 2nd ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 115, Springer-Verlag, Berlin-New York, 1979. Structure of topological groups, integration theory, group representations. MR551496 (81k:43001)

[5] Graham Higman, B. H. Neumann, and Hanna Neumann, *Embedding theorems for groups*, J. London Math. Soc. **24** (1949), 247–254. MR0032641 (11,322d)

[6] Alexander S. Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics, vol. 156, Springer-Verlag, New York, 1995. MR1321597 (96e:03057)

[7] Alexander S. Kechris and Christian Rosendal, *Turbulence, amalgamation, and generic automorphisms of homogeneous structures*, Proc. Lond. Math. Soc. (3) **94** (2007), no. 2, 302–350, DOI 10.1112/plms/pdl007. MR2308230 (2008a:03079)

[8] Serge Lang, *Algebra*, 3rd ed., Graduate Texts in Mathematics, vol. 211, Springer-Verlag, New York, 2002. MR1878556 (2003e:00003)

[9] L. Lévai and L. Pyber, *Profinite groups with many commuting pairs or involutions*, Arch. Math. (Basel) **75** (2000), no. 1, 1–7, DOI 10.1007/s000130050466. MR1764885 (2001i:20059)

[10] Deane Montgomery and Leo Zippin, *Topological transformation groups*, Interscience Publishers, New York-London, 1955. MR0073104 (17,383b)

[11] Luis Ribes and Pavel Zalesskii, *Profinite groups*, 2nd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 40, Springer-Verlag, Berlin, 2010. MR2599132 (2011a:20058)

[12] J. K. Truss, *Generic automorphisms of homogeneous structures*, Proc. London Math. Soc. (3) **65** (1992), no. 1, 121–141, DOI 10.1112/plms/s3-65.1.121. MR1162490 (93f:20008)

[13] John S. Wilson, *On the structure of compact torsion groups*, Monatsh. Math. **96** (1983), no. 1, 57–66, DOI 10.1007/BF01298934. MR721596 (85a:22007)

[14] John S. Wilson, *Profinite groups*, London Mathematical Society Monographs. New Series, vol. 19, The Clarendon Press, Oxford University Press, New York, 1998. MR1691054 (2000j:20048)

[15] E. I. Zel′manov, *On periodic compact groups*, Israel J. Math. **77** (1992), no. 1-2, 83–95, DOI 10.1007/BF02808012. MR1194787 (94e:20055)

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