PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 144, Number 1, January 2016, Pages 41–54
http://dx.doi.org/10.1090/proc/12691
Article electronically published on September 24, 2015

ON THE p'-EXTENSIONS OF INERTIAL BLOCKS

YUANYANG ZHOU

(Communicated by Pham Huu Tiep)

ABSTRACT. Let p be a prime number, G a finite group, H a normal subgroup of G, and b a p-block of H. Assuming that the index of H in G is coprime to p, we prove that any p-block of G covering b is inertial if and only if the block b is inertial.

1. Introduction

Throughout this paper, p is a prime number and $\mathcal{O}$ is a complete discrete valuation ring with an algebraically closed residue field k of characteristic p.

Let G be a finite group and H a normal subgroup of G. Let b be a G-stable block of H over $\mathcal{O}$, a primitive idempotent in the center of the group algebra $\mathcal{O}H$. The ideal $\mathcal{O}Hb$ of $\mathcal{O}H$ generated by b is the block algebra associated to b. Let Q be a defect group of the block b. Set

$$G' = N_G(Q), H' = N_H(Q) \quad \text{and} \quad K = (H \times H')\Delta(G'),$$

where $\Delta(G')$ is the diagonal subgroup of $G' \times G'$. Let b' be the Brauer correspondent of b in H'.

Assume that Q is abelian and that the index of H in G is coprime to p. It is conjectured that there is a complex C of $\mathcal{O}K$-modules, whose restriction to $\mathcal{O}(H \times H')$ induces a splendid Rickard equivalence between $\mathcal{O}Hb$ and $\mathcal{O}H'b'$ (see [14]). There is a more general formulation in [5]. When $\mathcal{O}Hb$ and $\mathcal{O}H'b'$ are Puig equivalent, by [3, Lemma 3.6] the conjectural complex C exists. The proof of [3, Lemma 3.6] relies on Dade's criterion on the extendibility of modules.

According to [11], the block b is inertial if there is a Morita equivalence between the block algebras $\mathcal{O}Hb$ and $\mathcal{O}H'b'$, induced by a bimodule with an endopermutation source. In the sense of [10], such a Morita equivalence between $\mathcal{O}Hb$ and $\mathcal{O}H'b'$ is basic. In this paper, we investigate whether there is an $\mathcal{O}K$-module such that its restriction to $\mathcal{O}(H \times H')$ has an endopermutation source and induces a Morita equivalence between $\mathcal{O}Hb$ and $\mathcal{O}H'b'$, when the block b is inertial and the index of H in G is coprime to p.

Denote by $\mathcal{O}Gb$ the ideal of $\mathcal{O}G$ generated by b, which is an algebra with b as the identity element. The algebra $\mathcal{O}Gb$ is a ring extension of the block algebra $\mathcal{O}Hb$ and thus is called an extension of the block b. The extension $\mathcal{O}Gb$ clearly has a G/H-graded agebra structure. Assume that the block b is nilpotent (see [1]). In [4] the structure of the extension $\mathcal{O}Gb$ was described in terms of a new finite group L

Received by the editors January 9, 2014 and, in revised form, December 5, 2014.
2010 *Mathematics Subject Classification.* Primary 20C20.
The author was supported by self-determined research funds of CCNU from the colleges' basic research and operation of MOE (No. 20205140052) and by NSFC (No. 11071091).

and its central extension by k^*, the multiplicative group of k. The main results of [4] are Theorems 1.8 and 1.12. Later, we revisited [4], took into account the graded algebra structure of $\mathcal{O}Gb$, and refined [4, Theorem 1.12] to get [12, Corollary 3.15]. By this corollary, as in the proof of [12, Theorem 1.6], one can prove that there is an $\mathcal{O}K$-module such that its restriction to $\mathcal{O}(H \times H')$ has an endopermutation source and induces a Morita equivalence between $\mathcal{O}Hb$ and $\mathcal{O}H'b'$.

We use the same idea as the one in [12] to investigate the problem in the third paragraph above. We characterize the graded algebra structure of the extension $\mathcal{O}Gb$ by a central extension of a group by k^* (see Proposition 3.3 below), and then prove the following theorem, comparing the graded algebra structures of two block extensions in question.

Theorem. *Keep the above notation. Assume that the block b is inertial and that the index of H in G is coprime to p. Then there is an $\mathcal{O}K$-module M such that $\mathrm{Res}^K_{H\times H'}(M)$ induces a basic Morita equivalence between $\mathcal{O}Hb$ and $\mathcal{O}H'b'$.*

Remark. It is not clear whether Dade's criterion in [2] on the extendibilities of modules can be used to find the module M in the Theorem.

Under the setting of the Theorem and when Q is abelian, people may want to know whether there is a complex C of p-permutation $\mathcal{O}K$-modules, whose restriction to $H \times H'$ induces a splendid Rickard equivalence between $\mathcal{O}Hb$ and $\mathcal{O}H'b'$. But this is not necessary. The purpose of finding such a complex C is to know which blocks e of G cover b, whereas we have the following Corollary. Moreover, if e' denotes the Brauer correspondent of e in $N_G(Q)$, there is a complex of p-permutation modules, which induces a splendid Rickard equivalence between $\mathcal{O}Ge$ and $\mathcal{O}G'e'$ and whose cohomology at some degree induces a basic Morita equivalence between $\mathcal{O}Ge$ and $\mathcal{O}G'e'$ (see [13] and [17]).

Corollary. *Keep the above notation. Assume that the index of H in G is coprime to p. Then a block e of G covering b is inertial if and only if the block b is inertial.*

Notation, quoted terminology and conventions. Let $\mathcal{B}$ be a unitary ring. We denote by $1_{\mathcal{B}}$, $J(\mathcal{B})$ and $\mathcal{B}^*$ the identity element of $\mathcal{B}$, the Jacobson radical of $\mathcal{B}$ and the multiplicative group of $\mathcal{B}$, respectively. Let $\mathcal{C}$ be a finitely generated $\mathcal{O}$-free local algebra. Then $\mathcal{C}/J(\mathcal{C}) \cong k$ and there is a canonical decomposition $\mathcal{C}^* \cong k^* \times (1_{\mathcal{C}} + J(\mathcal{C}))$ (see [15, Chapter II, Proposition 8]). We always identify k^* with a subgroup of $\mathcal{C}^*$ through such a decomposition. Note that $\mathcal{O}^* \subset \mathcal{C}^*$ and k^* can be identified with a subgroup of $\mathcal{C}^*$ through another canonical decomposition $\mathcal{O}^* \cong k^* \times (1_{\mathcal{O}} + J(\mathcal{O}))$. The two identified subgroups k^* of $\mathcal{C}^*$ actually are the same. We use the k^*-groups and their isomorphisms in [8, §5]. If a group $\hat{G}$ with an injective group homomorphism $k^* \to \hat{G}$ becomes a k^*-group with k^*-quotient G, we always identify k^* and its image in $\hat{G}$ through this group homomorphism. Also, we often use G-interior algebras, their homomorphisms and embeddings, and pointed groups on them; these can be found in [6]. Let $\mathcal{A}$ be a G-interior algebra with the structural homomorphism $\rho : G \to \mathcal{A}^*$. For any $x, y \in G$ and any $a \in \mathcal{A}$, we write the product $\rho(x)a\rho(y)$ as xay.

2. A k^*-group isomorphism

The injective k^*-group homomorphism in [8, Proposition 6.12] becomes a k^*-group isomorphism in the case of group algebras. The purpose of this section is to extend this k^*-group isomorphism.

2.1. Let G be a finite group and H a normal subgroup of G. Let b be a G-stable block of H over $\mathcal{O}$. Set $\mathcal{A} = \mathcal{O}Gb$, $\mathcal{B} = \mathcal{O}Hb$ and $\dot{G} = G/H$. For any $x \in G$, $\dot{x}$ denotes the image of x in $\dot{G}$. Clearly $\mathcal{A}$ is a $\dot{G}$-graded algebra with the $\dot{x}$-component $\mathcal{O}Hxb$. Let P_γ be a defect pointed group of the block b. Take $i \in \gamma$ and set

$$\mathcal{A}_\gamma = i\mathcal{A}i \text{ and } \mathcal{B}_\gamma = i\mathcal{B}i.$$

Then $\mathcal{A}_\gamma$ is both a $\dot{G}$-graded algebra and a P-interior algebra with the group homomorphism $P \to \mathcal{A}_\gamma^*$ sending u onto ui for any $u \in P$. The subalgebra $\mathcal{B}_\gamma$ with the same group homomorphism is also a P-interior algebra, which is a source algebra of the block b.

2.2. Denote by $N_G(P_\gamma)$ the stabilizer of P_γ under the G-conjugation and by $\mathcal{B}(P_\gamma)$ the simple factor of $\mathcal{B}^P$ such that the image of γ through the canonical homomorphism $s_\gamma : \mathcal{B}^P \to \mathcal{B}(P_\gamma)$ is nonzero. Clearly $N_G(P_\gamma)$ acts on $\mathcal{B}(P_\gamma)$ by conjugation, so that $\mathcal{B}(P_\gamma)$ is an $N_G(P_\gamma)$-algebra. Since any k-algebra automorphism on $\mathcal{B}(P_\gamma)$ is inner, the $N_G(P_\gamma)$-action on $\mathcal{B}(P_\gamma)$ can be lifted to a group homomorphism $\theta : N_G(P_\gamma) \to \mathrm{Aut}(\mathcal{B}(P_\gamma)) \cong (\mathcal{B}(P_\gamma))^*/k^*$. Denote by $\hat{N}_G(P_\gamma)$ the set of all pairs $(x, s_\gamma(a))$ in $N_G(P_\gamma) \times (\mathcal{B}(P_\gamma))^*$ such that $\theta(x)$ is equal to the image of $s_\gamma(a)$ in $(\mathcal{B}(P_\gamma))^*/k^*$. The set $\hat{N}_G(P_\gamma)$ becomes a group with the multiplication determined by the multiplications of G and $\mathcal{B}$. The map $k^* \to \hat{N}_G(P_\gamma)$, $\lambda \mapsto (1, s_\gamma(\lambda))$ is an injective group homomorphism. Clearly the image of k^* lies in the center of $\hat{N}_G(P_\gamma)$ and the quotient of $\hat{N}_G(P_\gamma)$ by the image of k^* is isomorphic to $N_G(P_\gamma)$. In particular, $\hat{N}_G(P_\gamma)$ is a k^*-group with k^*-quotient $N_G(P_\gamma)$.

2.3. There is another injective group homomorphism $PC_H(P) \to \hat{N}_G(P_\gamma)$ mapping u onto $(u, 1)$ and y onto $(y, s_\gamma(yb))$, where $u \in P$ and $y \in C_H(P)$. It is easily checked that the image of $PC_H(P)$ in $\hat{N}_G(P_\gamma)$ is normal and intersects k^* trivially. We identify $PC_H(P)$ and its image in $\hat{N}_G(P_\gamma)$. Set

$$\hat{E}_{G,\dot{G}}(P_\gamma) = \hat{N}_G(P_\gamma)/PC_H(P) \text{ and } E_{G,\dot{G}}(P_\gamma) = N_G(P_\gamma)/PC_H(P).$$

Denote by $\overline{(x, s_\gamma(a))}$ the image in $\hat{E}_{G,\dot{G}}(P_\gamma)$ of a pair $(x, s_\gamma(a))$ in $\hat{N}_G(P_\gamma)$. The group $\hat{E}_{G,\dot{G}}(P_\gamma)$ becomes a k^*-group with k^*-quotient $E_{G,\dot{G}}(P_\gamma)$ with the group homomorphism

$$k^* \to \hat{E}_{G,\dot{G}}(P_\gamma),\ \lambda \mapsto \overline{(1, s_\gamma(\lambda))}. \tag{2.3.1}$$

Let $\hat{N}_H(P_\gamma)$ be the inverse image of $N_H(P_\gamma)$ in $\hat{N}_G(P_\gamma)$ and set

$$\hat{E}_H(P_\gamma) = \hat{N}_H(P_\gamma)/PC_H(P) \text{ and } E_H(P_\gamma) = N_H(P_\gamma)/PC_H(P).$$

Clearly $\hat{E}_H(P_\gamma)$ is a normal k^*-subgroup of $\hat{E}_{G,\dot{G}}(P_\gamma)$ with k^*-quotient $E_H(P_\gamma)$. For the constructions of the k^*-groups above, readers can also refer to [16].

2.4. For any $\dot{x} \in \dot{G}$, we denote by $\mathcal{N}^{\dot{x}}_{\mathcal{A}^*_\gamma}(P)$ the set of all invertible elements in the $\dot{x}$-component of $\mathcal{A}_\gamma$ normalizing Pi. Set $\mathcal{N}_{\mathcal{A}^*_\gamma}(P) = \bigcup_{\dot{x}\in\dot{G}} \mathcal{N}^{\dot{x}}_{\mathcal{A}^*_\gamma}(P)$. It is easily checked that $\mathcal{N}_{\mathcal{A}^*_\gamma}(P)$ is a group with the multiplication of $\mathcal{A}_\gamma$, and that $P(\mathcal{B}^P_\gamma)^*$ and $P(i + J(\mathcal{B}^P_\gamma))$ are normal subgroups of $\mathcal{N}_{\mathcal{A}^*_\gamma}(P)$. Set

$$\hat{\mathcal{F}}_{\mathcal{A},\,\dot{G}}(P_\gamma) = \mathcal{N}_{\mathcal{A}^*_\gamma}(P)/P(i + J(\mathcal{B}^P_\gamma)) \text{ and } \mathcal{F}_{\mathcal{A},\,\dot{G}}(P_\gamma) = \mathcal{N}_{\mathcal{A}^*_\gamma}(P)/P(\mathcal{B}^P_\gamma)^*.$$

The inclusion $k^* \subset (\mathcal{B}^P_\gamma)^*$ induces an injective group homomorphism from k^* to the center of $\hat{\mathcal{F}}_{\mathcal{A},\,\dot{G}}(P_\gamma)$, and the quotient of $\hat{\mathcal{F}}_{\mathcal{A},\,\dot{G}}(P_\gamma)$ by the image of k^* is isomorphic to $\mathcal{F}_{\mathcal{A},\,\dot{G}}(P_\gamma)$. Thus $\hat{\mathcal{F}}_{\mathcal{A},\,\dot{G}}(P_\gamma)$ is a k^*-group with k^*-quotient $\mathcal{F}_{\mathcal{A},\,\dot{G}}(P_\gamma)$. For any $d \in \mathcal{N}_{\mathcal{A}^*_\gamma}(P)$, we denote by $\bar{d}$ its image in $\hat{\mathcal{F}}_{\mathcal{A}}(P_\gamma)$.

2.5. For an element $\overline{(x,\, s_\varepsilon(a))}$ in $\hat{E}_{G,\,\dot{G}}(P_\gamma)^\circ$, we have $s_\gamma(i^{xa^{-1}}) = s_\gamma(i)$ and $s_\epsilon(i^{xa^{-1}}) = s_\epsilon(i)$ for any other point ϵ of P on $\mathcal{B}$. Then by [8, Lemma 6.3] there is a suitable element c of $b + J(\mathcal{B}^P)$ such that $xa^{-1}c^{-1}$ and i commute and such that $i(xa^{-1})i = (xa^{-1}c^{-1})i$. In particular, $i(xa^{-1})i$ belongs to $\mathcal{N}^{\dot{x}}_{\mathcal{A}^*_\gamma}(P)$. We claim that the correspondence

$$\hat{\theta}_\gamma : \hat{E}_{G,\,\dot{G}}(P_\gamma)^\circ \to \hat{\mathcal{F}}_{\mathcal{A},\,\dot{G}}(P_\gamma),\ \overline{(x,\, s_\gamma(a))} \mapsto \overline{i(xa^{-1})i} \tag{2.5.1}$$

is a k^*-group isomorphism.

2.6. Let a' be another invertible element of $\mathcal{B}^P$ such that $s_\gamma(a') = s_\gamma(a)$. By [8, Lemma 6.3] again, there is a suitable element c' of $b + J(\mathcal{B}^P)$ such that $xa'^{-1}c'^{-1}$ and i commute and $i(xa'^{-1})i = (xa'^{-1}c'^{-1})i$. Clearly $((xa'^{-1}c'^{-1})i)^{-1}(xa^{-1}c^{-1})i = c'a'a^{-1}c^{-1}i$ and $c'a'a^{-1}c^{-1}i$ belongs to $i + J(\mathcal{B}^P_\gamma)$. Thus $\overline{i(xa^{-1})i} = \overline{xa^{-1}c^{-1}i} = \overline{xa'^{-1}c'^{-1}i} = \overline{i(xa'^{-1})i}$. Suppose $\overline{(y,\, s_\gamma(d))} = \overline{(x,\, s_\gamma(a))}$. There are $u \in P$ and $z \in C_H(P)$ such that $s_\gamma(d) = s_\gamma(az)$ and $y = xuz$. The proof above shows that we choose d to be az without changing $\overline{i(yd^{-1})i}$. Then $\overline{i(xa^{-1})i} = \overline{i(yd^{-1})i}$. So $\hat{\theta}_\gamma$ is a well-defined map.

2.7. Now take two elements $(x,\, s_\gamma(a))$ and $(y,\, s_\gamma(d))$ in $\hat{N}_G(P_\gamma)$ such that xa^{-1} and yd^{-1} commute with i. Take an element c'' of $b + J(\mathcal{B}^P)$ such that $xy(ad)^{-1}c''^{-1}$ and i commute and such that $ixy(ad)^{-1}i = ixy(ad)^{-1}c''^{-1}$. Since $s_\gamma(a^{yd^{-1}}) = s_\gamma(c''a) = s_\gamma(a)$, we have

$$\begin{aligned}
\hat{\theta}_\gamma(\overline{(x,\, s_\gamma(a))}\,\overline{(y,\, s_\gamma(d))}) &= \hat{\theta}_\gamma\overline{(xy,\, s_\gamma(ad))} \\
&= \overline{xy(ad)^{-1}c''^{-1}i} \\
&= \overline{xa^{-1}i}\;\overline{yd^{-1}a^{yd^{-1}}a^{-1}c''^{-1}i} \\
&= \overline{xa^{-1}i}\,\overline{yd^{-1}i} \\
&= \hat{\theta}_\gamma\overline{(x,\, s_\gamma(a))}\hat{\theta}_\gamma\overline{(y,\, s_\gamma(d))}.
\end{aligned}$$

Hence $\hat{\theta}_\gamma$ is a group homomorphism. Moreover, $\hat{\theta}_\gamma$ maps $\overline{(x,\, s_\gamma(\lambda a))}$ onto $\overline{i(x\lambda a^{-1})i}$ for any $\lambda \in k^*$ and so it is a k^*-group homomorphism.

2.8. Suppose that $\hat{\theta}_\gamma$ maps $\overline{(x,\, s_\gamma(a))}$ to the identity element of $\hat{F}_{\mathcal{A},\,\dot{G}}(R_\varepsilon)$. Then $i(xa^{-1})i$ has to be in $P(i + J(\mathcal{B}^P_\gamma))$. Consequently, $x \in H$ and there is $u \in P$ such that

$$(xvx^{-1})i = (i(xa^{-1})i)(vi)(i(xa^{-1})i)^{-1} = (uvu^{-1})i$$

for any $v \in P$. Since the map $P \to Pi$, $v \mapsto vi$ is a group isomorphism, $xvx^{-1} = uvu^{-1}$ for any $v \in P$ and so x belongs to $PC_H(P)$. This shows that $\overline{(x,\, s_\gamma(a))}$ is contained in k^*, the subgroup of $\hat{E}_{G,\,\dot{G}}(P_\gamma)^\circ$. But $\hat{\theta}_\gamma$ is a k^*-group homomorphism and its restriction to k^* is injective. Therefore $\overline{(x,\, s_\gamma(a))}$ is the identity element of $\hat{E}_{G,\,\dot{G}}(P_\gamma)^\circ$ and $\hat{\theta}_\gamma$ itself is injective. Clearly $\hat{\theta}_\gamma$ induces an injective group homomorphism

$$\theta_\gamma : E_{G,\,\dot{G}}(P_\gamma) \to F_{\mathcal{A},\,\dot{G}}(P_\gamma), \tag{2.8.1}$$

which maps the image of $x \in N_G(P_\gamma)$ in $E_{G,\,\dot{G}}(P_\gamma)$ onto the image of $i(xa^{-1})i$ in $F_{\mathcal{A},\,\dot{G}}(P_\gamma)$.

2.9. Let $d \in \mathcal{N}^{\dot{x}}_{\mathcal{A}^*_\gamma}(P)$. Since d normalizes Pi, there is an automorphism φ of P such that

$$d(vi)d^{-1} = \varphi(v)i \tag{2.9.1}$$

for any $v \in P$. In addition, there are $y \in N_G(P_\gamma)$ and $d_y \in \mathcal{N}^{\dot{x}}_{\mathcal{A}^*_\gamma}(P)$ such that $\dot{x} = \dot{y}$ and $d_y v d_y^{-1} = (yvy^{-1})i$ for any $v \in P$. Therefore

$$(y\varphi^{-1}(v)y^{-1})i = d_y d^{-1}(vi)(d_y d^{-1})^{-1} \tag{2.9.2}$$

for any $v \in P$. Since $d_y d^{-1}$ is an invertible element of $\mathcal{B}_\gamma$, there is $w \in N_H(P_\gamma)$ such that

$$d_y d^{-1}(vi)(d_y d^{-1})^{-1} = (wvw^{-1})i \tag{2.9.3}$$

for any $v \in P$. By the equalities (2.9.1)-(2.9.3), we get

$$d(vi)d^{-1} = ((w^{-1}y)v(w^{-1}y)^{-1})i$$

for any $v \in P$. It is easy to see that θ_γ maps the image of $w^{-1}y$ in $E_{G,\,\dot{G}}(P_\gamma)$ onto the image of d in $F_{\mathcal{A},\,\dot{G}}(P_\gamma)$. Thus θ_γ is surjective and so is $\hat{\theta}_\gamma$. The claim in section 2.5 is proved.

3. On the p'-extensions of inertial blocks

In this section, we continue to use the notation in section 2.1. We will describe the algebraic structure of $\mathcal{A}_\gamma$ when the index of H in G is coprime to p. Denote by $\mathrm{Aut}(P)$ the automorphism group of P and by $\mathrm{Inn}(P)$ the inner automorphism group of P. Set $\mathrm{Out}(P) = \mathrm{Aut}(P)/\mathrm{Inn}(P)$. The $N_G(P_\gamma)$-conjugation induces a group homomorphism $E_{G,\,\dot{G}}(P_\gamma) \to \mathrm{Out}(P)$.

Lemma 3.1. *Keep the above notation. Assume that the index of H in G is coprime to p. Then the group homomorphism $E_{G,\,\dot{G}}(P_\gamma) \to \mathrm{Out}(P)$ can be lifted to a group homomorphism*

$$E_{G,\,\dot{G}}(P_\gamma) \to \mathrm{Aut}(P), \tag{3.1.1}$$

which is unique up to the $\mathrm{Inn}(P)$*-conjugation.*

Proof. By Frattini's argument, we have $G = HN_G(P_\gamma)$. Hence the inclusion $N_G(P_\gamma)$
$\subset G$ induces a group isomorphism

$$N_G(P_\gamma)/N_H(P_\gamma) \cong G/H. \tag{3.1.2}$$

Since $N_H(P_\gamma)/PC_H(P)$ is a p'-group (see [16]) and the index of H in G is coprime to p, the group $E_{G,\dot{G}}(P_\gamma)$ is a p'-group. Then this lemma follows from the Schur-Zassenhaus theorem. □

3.2. We fix a choice of the homomorphism (3.1.1). Then this gives an action of $E_{G,\dot{G}}(P_\gamma)$ on P. We lift this action to an action of $\hat{E}_{G,\dot{G}}(P_\gamma)$ on P through the canonical homomorphism $\hat{E}_{G,\dot{G}}(P_\gamma) \to E_{G,\dot{G}}(P_\gamma)$. Set

$$\hat{\mathcal{L}} = P \rtimes \hat{E}_{G,\dot{G}}(P_\gamma),\ \mathcal{L} = P \rtimes E_{G,\dot{G}}(P_\gamma),\ \hat{\mathcal{K}} = P \rtimes \hat{E}_H(P_\gamma) \text{ and } \mathcal{K} = P \rtimes E_H(P_\gamma).$$

The group $\hat{\mathcal{L}}$ becomes a k^*-group with k^*-quotient $\mathcal{L}$ with the inclusion $k^* \subset \hat{E}_{G,\dot{G}}(P_\gamma)$ (see (2.3.1)), and $\hat{\mathcal{K}}$ is a normal k^*-subgroup of it. The opposite group $\hat{\mathcal{L}}^\circ$ of $\hat{\mathcal{L}}$ is a k^*-group with k^*-quotient $\mathcal{L}$ with the injective group homomorphism $k^* \to \hat{\mathcal{L}}^\circ$, $\lambda \mapsto \lambda^{-1}$. The twisted group algebra $\mathcal{O}_*\hat{\mathcal{L}}^\circ$ is an $\mathcal{L}/\mathcal{K}$-graded algebra (see section 2.1). We have group isomorphisms

$$\mathcal{L}/\mathcal{K} \cong E_{G,\dot{G}}(P_\gamma)/E_H(P_\gamma) \cong N_G(P_\gamma)/N_H(P_\gamma) \cong G/H$$

and we identify $\mathcal{L}/\mathcal{K}$ and G/H. Then the twisted group algebra $\mathcal{O}_*\hat{\mathcal{L}}^\circ$ becomes a $\dot{G}$-graded algebra. The inclusion $P \subset \hat{\mathcal{L}}$ induces a P-interior algebra structure on $\mathcal{O}_*\hat{\mathcal{L}}^\circ$.

Proposition 3.3. *Keep the above notation. Assume that the index of H in G is coprime to p and that the block b is inertial. Then there is a P-interior full matrix algebra S over $\mathcal{O}$ such that*

$$\mathcal{A}_\gamma \cong S \otimes_{\mathcal{O}} \mathcal{O}_*\hat{\mathcal{L}}^\circ \tag{3.3.1}$$

as $\dot{G}$-graded P-interior algebras. In this case, S has a P-stable $\mathcal{O}$-basis containing the unity of S and it is unique up to P-algebra isomorphisms.

3.4. We begin the proof of this proposition. Since the block b is inertial, by [11] there is a P-interior full matrix algebra S over $\mathcal{O}$ such that there is a P-interior algebra isomorphism

$$\mathcal{B}_\gamma \cong S \otimes_{\mathcal{O}} \mathcal{O}_*\hat{\mathcal{K}}. \tag{3.4.1}$$

Moreover S has a P-stable $\mathcal{O}$-basis containing the unity of S and it is unique up to a P-algebra isomorphism. Set $\hat{\mathcal{A}} = S^\circ \otimes_{\mathcal{O}} \mathcal{A}$ and $\hat{\mathcal{B}} = S^\circ \otimes_{\mathcal{O}} \mathcal{B}$. Here S° is the opposite ring of S, and denoting by $\rho : P \to S^*$ the structural homomorphism of the P-interior algebra S, it becomes a P-interior algebra with the group homomorphism $P \to (S^\circ)^*$, $u \mapsto \rho(u^{-1})$. The algebras $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ are P-interior algebras with the group homomorphism $P \to (\hat{\mathcal{B}})^*$, $u \mapsto u1_S \otimes u1_{\mathcal{B}}$. Also, $\hat{\mathcal{A}}$ is a $\dot{G}$-graded algebra with the 1-component $\hat{\mathcal{B}}$. By [9, Theorem 5.3], P_γ determines a unique local pointed group $P_{\hat{\gamma}}$ on $\hat{\mathcal{B}}$ such that for some $\hat{i} \in \hat{\gamma}$,

$$\hat{i}(1 \otimes i) = (1 \otimes i)\hat{i} = \hat{i}. \tag{3.4.2}$$

Set $\hat{\mathcal{A}}_{\hat{\gamma}} = \hat{i}\hat{\mathcal{A}}\hat{i}$ and $\hat{\mathcal{B}}_{\hat{\gamma}} = \hat{i}\hat{\mathcal{B}}\hat{i}$. Both $\hat{\mathcal{A}}_{\hat{\gamma}}$ and $\hat{\mathcal{B}}_{\hat{\gamma}}$ are P-interior algebras with the same group homomorphism $P \to \hat{\mathcal{B}}_{\hat{\gamma}}^*$, $u \mapsto u\hat{i}$, and $\hat{\mathcal{A}}_{\hat{\gamma}}$ is a $\dot{G}$-graded algebra with the 1-component $\hat{\mathcal{B}}_{\hat{\gamma}}$.

3.5. For any $\dot{x} \in \dot{G}$, we denote by $\mathcal{N}^{\dot{x}}_{\hat{\mathcal{A}}^*_{\hat{\gamma}}}(P)$ the set of all invertible elements in the $\dot{x}$-component of $\hat{\mathcal{A}}_{\hat{\gamma}}$ normalizing $P\hat{i}$. Set $\mathcal{N}_{\hat{\mathcal{A}}^*_{\hat{\gamma}}}(P) = \bigcup_{\dot{x}\in\dot{G}} \mathcal{N}^{\dot{x}}_{\hat{\mathcal{A}}^*_{\hat{\gamma}}}(P)$. The set $\mathcal{N}_{\hat{\mathcal{A}}^*_{\hat{\gamma}}}(P)$ with the multiplication of $\hat{\mathcal{A}}_{\hat{\gamma}}$ becomes a group and $P(\hat{\mathcal{B}}^P_{\hat{\gamma}})^*$ and $P(\hat{i} + J(\hat{\mathcal{B}}^P_{\hat{\gamma}}))$ are normal subgroups of $\mathcal{N}_{\hat{\mathcal{A}}^*_{\hat{\gamma}}}(P)$. Set $\hat{\mathcal{F}}_{\hat{\mathcal{A}}}(P_{\hat{\gamma}}) = \mathcal{N}_{\hat{\mathcal{A}}^*_{\hat{\gamma}}}(P)/P(\hat{i}+J(\hat{\mathcal{B}}^P_{\hat{\gamma}}))$ and $\mathcal{F}_{\hat{\mathcal{A}}}(P_{\hat{\gamma}}) = \mathcal{N}_{\hat{\mathcal{A}}^*_{\hat{\gamma}}}(P)/P(\hat{\mathcal{B}}^P_{\hat{\gamma}})^*$. Similar to section 2.4, it is easy to show that $\hat{\mathcal{F}}_{\hat{\mathcal{A}}}(P_{\hat{\gamma}})$ is a k^*-group with k^*-quotient $\mathcal{F}_{\hat{\mathcal{A}}}(P_{\hat{\gamma}})$.

3.6. For any x in $N_G(P_\gamma)$, there is some invertible element a_x of $\mathcal{B}^P$ such that $xix^{-1} = a_x i a_x^{-1}$. Thus $a_x^{-1}x$ commutes with i and $(a_x^{-1}x)i$ belongs to $\mathcal{N}^{\dot{x}}_{\mathcal{A}^*_\gamma}(P)$. We set $d_x = (a_x^{-1}x)i$. Note that the $((a_x^{-1}x)i)$-conjugation induces a P-interior algebra isomorphism $\mathcal{B}_\gamma \cong \mathrm{Res}_{\varphi_x}(\mathcal{B}_\gamma)$, where φ_x denotes the group isomorphism $P \cong P$ mapping u onto xux^{-1} and $\mathrm{Res}_{\varphi_x}(\mathcal{B}_\gamma)$ denotes the restriction of the P-interior algebra $\mathcal{B}_\gamma$ through φ_x (see [6, Definition 3.1]). For convenience, we also say that the P-interior algebra $\mathcal{B}_\gamma$ is $N_G(P_\gamma)$-stable. The uniqueness of the P-algebra S implies that it is $N_G(P_\gamma)$-stable. Without loss of generality, we can adjust the structural homomorphism $\rho : P \to S^*$ of the P-interior algebra S such that $\rho(P)$ is contained in the kernel of the determinant map $S^* \to \mathcal{O}^*$ (see [11, 3.9]). The P-interior algebra S with such a homomorphism ρ is unique up to P-interior algebra isomorphism (see [16, Proposition 21.5]) and thus is $N_G(P_\gamma)$-stable. So there is an invertible element s_x in S such that $s_x u s_x^{-1} = \varphi_x(u)1_S$ for any $u \in P$. Clearly $\hat{i}^{(s_x\otimes d_x)^{-1}}$ is contained in some local point of P on $\hat{\mathcal{B}}$. Since

$$\hat{i}^{(s_x\otimes d_x)^{-1}}(1_S \otimes i) = (1_S \otimes i)\hat{i}^{(s_x\otimes d_x)^{-1}} = \hat{i}^{(s_x\otimes d_x)^{-1}},$$

by the uniqueness of $P_{\hat{\gamma}}$ (see section 3.4), $\hat{i}^{(s_x\otimes d_x)^{-1}}$ belongs to $\hat{\gamma}$. Therefore there is some invertible element e_x of $\hat{\mathcal{B}}^P$ such that $(s_x \otimes d_x)\hat{x}(s_x \otimes d_x)^{-1} = e_x\hat{i}e_x^{-1}$. Set $c_x = e_x^{-1}(s_x \otimes d_x)\hat{i}$. Then c_x is an invertible element of the $\dot{x}$-component of $\hat{\mathcal{A}}_{\hat{\gamma}}$ and $c_x u c_x^{-1} = \varphi_x(u)\hat{i}$ for any $u \in P$. In particular, $\hat{\mathcal{A}}_{\hat{\gamma}}$ is a crossed product of $\dot{G}$.

3.7. We claim that the correspondence $x \mapsto c_x$ induces a group isomorphism

$$E_{G,\dot{G}}(P_\gamma) \cong \mathcal{F}_{\hat{\mathcal{A}}}(P_{\hat{\gamma}}). \tag{3.7.1}$$

Let c'_x be another element constructed as in section 3.6 such that $c'_x u {c'_x}^{-1} = \varphi_x(u)\hat{i}$ for any $u \in P$. Clearly the product $c'_x c_x^{-1}$ is an invertible element of $\hat{\mathcal{B}}^P_{\hat{\gamma}}$, and the correspondence $\rho : N_G(P_\gamma) \to \mathcal{N}_{\hat{\mathcal{A}}^*_{\hat{\gamma}}}(P)/(\hat{\mathcal{B}}^P_{\hat{\gamma}})^*$ sending x onto the image of x in $\mathcal{N}_{\hat{\mathcal{A}}^*_{\hat{\gamma}}}(P)/(\hat{\mathcal{B}}^P_{\hat{\gamma}})^*$ is a well-defined group homomorphism. Let a be an element of $\mathcal{N}^{\dot{x}}_{\hat{\mathcal{A}}^*_{\hat{\gamma}}}(P)$. Since the map $P \to P\hat{i}$, $u \mapsto u\hat{i}$ is a group isomorphism, there is a group isomorphism $\psi : P \cong P$ such that $aua^{-1} = \psi(u)\hat{i}$. We have $(c_x^{-1}a)u(c_x^{-1}a)^{-1} = (\varphi_x^{-1} \circ \psi)(u)\hat{i}$ for any $u \in P$. Since $c_x^{-1}a$ lies in $\hat{\mathcal{B}}_{\hat{\gamma}}$, by [7, Theorem 3.1] there is some $y \in N_H(P_\gamma)$ such that $(c_x^{-1}a)u(c_x^{-1}a)^{-1} = \varphi_y(u)\hat{i}$ for any $u \in P$. Again, since the map $P \to P\hat{i}$, $u \mapsto u\hat{i}$ is a group isomorphism, we have $\psi = \varphi_x \circ \varphi_y = \varphi_{xy}$ and $c_{xy}uc_{xy}^{-1} = \psi(u)\hat{i} = aua^{-1}$ for any $u \in P$. Therefore the images of a and c_{xy} in $\mathcal{N}_{\hat{\mathcal{A}}^*_{\hat{\gamma}}}(P)/(\hat{\mathcal{B}}^P_{\hat{\gamma}})^*$ are the same and the homomorphism ρ is surjective. If x is in the kernel of ρ, then $c_x \in (\hat{\mathcal{B}}^P_{\hat{\gamma}})^*$. This implies that x lies in H and φ_x is the identity map on P; equivalently, x lies in $C_H(P)$. On the other hand, $C_H(P)$ is contained in the kernel of ρ. Therefore $C_H(P)$ is exactly the kernel of the homomorphism ρ and then ρ induces a group isomorphism $\hat{\rho}: N_G(P_\gamma)/C_H(P) \cong \mathcal{N}_{\hat{\mathcal{A}}^*_{\hat{\gamma}}}(P)/(\hat{\mathcal{B}}^P_{\hat{\gamma}})^*$. Since $\hat{\rho}$ maps $PC_H(P)/C_H(P)$ onto $P(\hat{\mathcal{B}}^P_{\hat{\gamma}})^*/(\hat{\mathcal{B}}^P_{\hat{\gamma}})^*$, $\hat{\rho}$ induces the desired group isomorphism.

Remark. The isomorphism (3.7.1) is a graded algebra version of the transfer of fusions developed by L. Puig (see [7, Theorem 3.1] and [9, Theorem 5.3]).

3.8. By the isomorphism (3.7.1), we get an obvious short exact sequence of group homomorphisms

$$1 \to P(\hat{\mathcal{B}}^P_{\hat{\gamma}})^*/(\hat{\mathcal{B}}^P_{\hat{\gamma}})^* \to \mathcal{N}_{\hat{\mathcal{A}}^*_{\hat{\gamma}}}(P)/(\hat{\mathcal{B}}^P_{\hat{\gamma}})^* \to E_{G,\dot{G}}(P_\gamma) \to 1.$$

Since $E_{G,\dot{G}}(P_\gamma)$ is a p'-group (see section 3.1), this sequence splits. In particular, $\mathcal{N}_{\hat{\mathcal{A}}^*_{\hat{\gamma}}}(P)$ contains a subgroup $\mathcal{F}$ containing $(\hat{\mathcal{B}}^P_{\hat{\gamma}})^*$ such that by restriction, the surjective homomorphism $\mathcal{N}_{\hat{\mathcal{A}}^*_{\hat{\gamma}}}(P)/(\hat{\mathcal{B}}^P_{\hat{\gamma}})^* \to E_{G,\dot{G}}(P_\gamma)$ induces a group isomorphism $\mathcal{F}/(\hat{\mathcal{B}}^P_{\hat{\gamma}})^* \cong E_{G,\dot{G}}(P_\gamma)$. Consider another short exact sequence of group homomorphisms

$$\{\hat{\imath}\} \to \hat{\imath} + J(\hat{\mathcal{B}}^P_{\hat{\gamma}}) \to \mathcal{F}/k^* \to E_{G,\dot{G}}(P_\gamma) \to 1. \tag{3.8.1}$$

Again, since $E_{G,\dot{G}}(P_\gamma)$ is a p'-group, the sequence (3.8.1) splits. In particular, $\mathcal{F}$ contains a subgroup $\hat{\mathfrak{L}}$ containing k^* such that by restriction, the homomorphism $\mathcal{F}/k^* \to E_{G,\dot{G}}(P_\gamma)$ induces a group isomorphism

$$\hat{\mathfrak{L}}/k^* \cong E_{G,\dot{G}}(P_\gamma). \tag{3.8.2}$$

Set $\mathfrak{L} = \hat{\mathfrak{L}}/k^*$. Then $\hat{\mathfrak{L}}$ is a k^*-group with k^*-quotient $\mathfrak{L}$.

3.9. Let $\hat{a} \in \hat{\mathfrak{L}}$ and let a be the image of $\hat{a}$ in $\mathfrak{L}$. We remark that if a corresponds to $\tilde{x}$ through the isomorphism (3.8.2), then $\hat{a}$ belongs to $\mathcal{N}^{\tilde{x}}_{\hat{\mathcal{A}}^*_{\hat{\gamma}}}(P)$ and $\hat{a}u\hat{a}^{-1} = \varphi_x(u)\hat{\imath}$ for any $u \in P$. Moreover, the correspondence $\mathfrak{L} \to \mathrm{Aut}(P)$, $a \mapsto \varphi_x$ is a group homomorphism. By composing the inverse of the isomorphism (3.8.2) and the homomorphism $\mathfrak{L} \to \mathrm{Aut}(P)$, we get a group homomorphism

$$E_{G,\dot{G}}(P_\gamma) \to \mathrm{Aut}(P), \tag{3.9.1}$$

which clearly is a lifting of the group homomorphism $E_{G,\dot{G}}(P_\gamma) \to \mathrm{Out}(P)$ (see section 3.1). Note that we have another such lifting (3.1.1). By Lemma 3.1, we can adjust the group homomorphism (3.1.1), so that the homomorphisms (3.9.1) and (3.1.1) coincide.

3.10. Denote by $\mathfrak{G}$ the semidirect product $P \rtimes \mathfrak{L}$ obtained through the homomorphism $\mathfrak{L} \to \mathrm{Aut}(P)$. Denote by $\hat{\mathfrak{G}}$ the subgroup of $\hat{\mathcal{A}}^*_{\hat{\gamma}}$ generated by $P\hat{\imath}$ and $\hat{\mathfrak{L}}$, which actually is the semidirect product $P\hat{\imath} \rtimes \hat{\mathfrak{L}}$. With the inclusion $k^* \subset \hat{\mathfrak{L}}$, the group $\hat{\mathfrak{G}}$ becomes a k^*-group with k^*-quotient isomorphic to $\mathfrak{G}$. Denote by $\mathfrak{K}$ the inverse image of $N_H(P_\gamma)/PC_H(P)$ through the isomorphism (3.8.2) and by $\hat{\mathfrak{K}}$ the inverse image of $\mathfrak{K}$ in $\hat{\mathfrak{L}}$. Set $\hat{\mathfrak{H}} = P\hat{\imath} \rtimes \hat{\mathfrak{K}}$ and $\mathfrak{H} = P \rtimes \mathfrak{K}$. Clearly $\mathfrak{H}$ is normal in $\mathfrak{G}$ and the twisted group algebra $\mathcal{O}_*\hat{\mathfrak{G}}$ is a $\mathfrak{G}/\mathfrak{H}$-graded aglebra (see section 2.1). We have group isomorphisms

$$\mathfrak{G}/\mathfrak{H} \cong \hat{\mathfrak{G}}/\hat{\mathfrak{H}} \cong \hat{\mathfrak{L}}/\hat{\mathfrak{K}} \cong N_G(P_\gamma)/N_H(P_\gamma) \cong G/H$$

and then identify $\dot{G}$ and $\mathfrak{G}/\mathfrak{H}$. Then the twisted group algebra $\mathcal{O}_*\hat{\mathfrak{G}}$ is a $\dot{G}$-graded algebra. In addition, $\mathcal{O}_*\hat{\mathfrak{G}}$ is a P-interior algebra with the inclusion $P \subset \hat{\mathfrak{G}}$. The inclusion $\hat{\mathfrak{G}} \subset \hat{A}_{\hat{\gamma}}$ induces a $\dot{G}$-graded P-interior algebra homomorphism

$$\mathcal{O}_*\hat{\mathfrak{G}} \to \hat{\mathcal{A}}_{\hat{\gamma}}. \tag{3.10.1}$$

We claim that this homomorphism actually is a $\dot{G}$-graded P-interior algebra isomorphism.

3.11. By (3.4.1) and (3.4.2), we get a P-interior algebra embedding

$$\hat{\mathcal{B}}_{\hat{\gamma}} \to S^\circ \otimes_{\mathcal{O}} S \otimes_{\mathcal{O}} \mathcal{O}_*(P \rtimes \hat{E}_H(P_\gamma)^\circ). \tag{3.11.1}$$

There is a P-interior algebra embedding $\mathcal{O} \to S^\circ \otimes_{\mathcal{O}} S$. By tensoring both sides of the embedding $\mathcal{O} \to S^\circ \otimes_{\mathcal{O}} S$, we get a P-interior algebra embedding

$$\mathcal{O}_*(P \rtimes \hat{E}_H(P_\gamma)^\circ) \to S^\circ \otimes_{\mathcal{O}} S \otimes_{\mathcal{O}} \mathcal{O}_*(P \rtimes \hat{E}_H(P_\gamma)^\circ). \tag{3.11.2}$$

Since $C_{P\rtimes E_H(P_\gamma)}(P) = Z(P)$, P has the unique local point on $\mathcal{O}_*(P \rtimes \hat{E}_H(P_\gamma)^\circ)$, consisting of the identity element. Then by [9, Theorem 5.3], P has a unique local point on $S^\circ \otimes_{\mathcal{O}} S \otimes_{\mathcal{O}} \mathcal{O}_*(P \rtimes \hat{E}_H(P_\gamma)^\circ)$. Note that $\{\hat{i}\}$ is the unique local point of P on $\hat{\mathcal{B}}_{\hat{\gamma}}$. Since embeddings preserve the localness of points (see [16]), the local points of P on $S^\circ \otimes_{\mathcal{O}} S \otimes_{\mathcal{O}} \mathcal{O}_*(P \rtimes \hat{E}_H(P_{\tilde{\gamma}})^\circ)$ respectively determined by the local points of P on $\hat{\mathcal{B}}_{\hat{\gamma}}$ and $\mathcal{O}_*(P \rtimes \hat{E}_H(P_\gamma)^\circ)$ through (3.11.1) and (3.11.2) have to coincide. Therefore by [8, 2.13.1], there is a P-interior algebra isomorphism

$$\hat{\mathcal{B}}_{\hat{\gamma}} \cong \mathcal{O}_*(P \rtimes \hat{E}_H(P_\gamma)^\circ). \tag{3.11.3}$$

3.12. Set $N_{\hat{\mathcal{B}}^*_{\hat{\gamma}}}(P) = \mathcal{N}^1_{\hat{\mathcal{B}}^*_{\hat{\gamma}}}(P)$. By restriction, the isomorphism (3.7.1) induces a group isomorphism $E_H(P_\gamma) \cong N_{\hat{\mathcal{B}}^*_{\hat{\gamma}}}(P)/P(\hat{\mathcal{B}}^P_{\hat{\gamma}})^*$. We get an obvious short exact sequence of group homomorphisms

$$1 \to P(\hat{\mathcal{B}}^P_{\hat{\gamma}})^*/(\hat{\mathcal{B}}^P_{\hat{\gamma}})^* \to N_{\hat{\mathcal{B}}^*_{\hat{\gamma}}}(P)/(\hat{\mathcal{B}}^P_{\hat{\gamma}})^* \to E_H(P_\gamma) \to 1.$$

Since $E_H(P_\gamma)$ is a p'-group, this sequence splits and any two sections of the homomorphism $N_{\hat{\mathcal{B}}^*_{\hat{\gamma}}}(P)/(\hat{\mathcal{B}}^P_{\hat{\gamma}})^* \to E_H(P_\gamma)$ are conjugate by some element of $P(\hat{\mathcal{B}}^P_{\hat{\gamma}})^*/(\hat{\mathcal{B}}^P_{\hat{\gamma}})^*$. Denote by $\hat{\mathfrak{K}}'$ the inverse image of $\hat{E}_H(P_\gamma)^\circ$ through the isomorphism $\hat{\mathcal{B}}_{\hat{\gamma}} \cong \mathcal{O}_*(P \rtimes \hat{E}_H(P_\gamma)^\circ)$. Clearly $\hat{\mathfrak{K}}$ and $\hat{\mathfrak{K}}'$ are contained in $N_{\hat{\mathcal{B}}^*_{\hat{\gamma}}}(P)$. By [7, Theorem 3.1] and [4, Lemma 1.17], it is easy to check that by restriction, the homomorphism $N_{\hat{\mathcal{B}}^*_{\hat{\gamma}}}(P)/(\hat{\mathcal{B}}^P_{\hat{\gamma}})^* \to E_H(P_\gamma)$ induces two group isomorphisms $\hat{\mathfrak{K}}(\hat{\mathcal{B}}^P_{\hat{\gamma}})^*/(\hat{\mathcal{B}}^P_{\hat{\gamma}})^* \cong E_H(P_\gamma)$ and $\hat{\mathfrak{K}}'(\hat{\mathcal{B}}^P_{\hat{\gamma}})^*/(\hat{\mathcal{B}}^P_{\hat{\gamma}})^* \cong E_H(P_\gamma)$, whose inverses are two sections of the homomorphism $N_{\hat{\mathcal{B}}^*_{\hat{\gamma}}}(P)/(\hat{\mathcal{B}}^P_{\hat{\gamma}})^* \to E_H(P_\gamma)$. Therefore $\hat{\mathfrak{K}}(\hat{\mathcal{B}}^P_{\hat{\gamma}})^*$ and $\hat{\mathfrak{K}}'(\hat{\mathcal{B}}^P_{\hat{\gamma}})^*$ are conjugate by some element of P. Without loss, we assume $\mathcal{K} = \hat{\mathfrak{K}}(\hat{\mathcal{B}}^P_{\hat{\gamma}})^* = \hat{\mathfrak{K}}'(\hat{\mathcal{B}}^P_{\hat{\gamma}})^*$. Since $\mathcal{K}/(\hat{\mathcal{B}}^P_{\hat{\gamma}})^* \cong E_H(P_\gamma)$, we have another short exact sequence

$$\{\hat{i}\} \to \hat{i} + J(\hat{\mathcal{B}}^P_{\hat{\gamma}}) \to \mathcal{K}/k^* \to E_H(P_\gamma) \to 1.$$

Again, this sequence splits and any two sections of the homomorphism $\mathcal{K}/k^* \to E_H(P_\gamma)$ are conjugate by some element of $\hat{i} + J(\hat{\mathcal{B}}^P_{\hat{\gamma}})$. By restriction, the homomorphism $\mathcal{K}/k^* \to E_H(P_\gamma)$ induces two group isomorphisms $\hat{\mathfrak{K}}/k^* \cong E_H(P_\gamma)$

and $\hat{\mathfrak{K}}'/k^* \cong E_H(P_\gamma)$, whose inverses are sections of the homomorphism $\mathcal{K}/k^* \to E_H(P_\gamma)$. Therefore $\hat{\mathfrak{K}}$ and $\hat{\mathfrak{K}}'$ are conjugate by some element of $\hat{i} + J(\hat{\mathcal{B}}_{\hat{\gamma}}^P)$. By the isomorphism (3.11.3), this implies that the P-interior algebra homomorphism $\mathcal{O}_*\hat{\mathfrak{H}} \to \hat{\mathcal{B}}_{\hat{\gamma}}$ induced by the restriction to $\mathcal{O}_*\hat{\mathfrak{H}}$ of the homomorphism (3.10.1) is surjective. Then, by comparing the $\mathcal{O}$-ranks of $\mathcal{O}_*\hat{\mathfrak{H}}$ and $\hat{\mathcal{B}}_{\hat{\gamma}}$, we see that the homomorphism $\mathcal{O}_*\hat{\mathfrak{H}} \to \hat{\mathcal{B}}_{\hat{\gamma}}$ is a P-interior algebra isomorphism. Finally, since $\dot{G}$-graded algebras $\mathcal{O}_*\hat{\mathfrak{G}}$ and $\hat{\mathcal{A}}_{\hat{\gamma}}$ are crossed products of $\dot{G}$, the homomorphism $\mathcal{O}_*\hat{\mathfrak{G}} \to \hat{\mathcal{A}}_{\hat{\gamma}}$ must be a $\dot{G}$-graded P-interior algebra isomorphism. The claim in section 3.10 is proved.

3.13. By (3.4.2), we have $\hat{\mathcal{A}}_{\hat{\gamma}} \subset S^\circ \otimes_{\mathcal{O}} \mathcal{A}_\gamma$ and the inclusion map $\hat{\mathcal{A}}_{\hat{\gamma}} \to S^\circ \otimes_{\mathcal{O}} \mathcal{A}_\gamma$ is a $\dot{G}$-graded P-interior algebra embedding. By tensoring S with both sides of the embedding $\hat{\mathcal{A}}_{\hat{\gamma}} \to S^\circ \otimes_{\mathcal{O}} \mathcal{A}_\gamma$, we get a $\dot{G}$-graded P-interior algebra embedding

$$S \otimes_{\mathcal{O}} \hat{\mathcal{A}}_{\hat{\gamma}} \to S \otimes_{\mathcal{O}} S^\circ \otimes_{\mathcal{O}} \mathcal{A}_\gamma. \tag{3.13.1}$$

By [9, Theorem 5.3], the obvious local point $\{\hat{i}\}$ of P on $\hat{\mathcal{B}}_{\hat{\gamma}}$ determines a local point $\hat{\gamma}'$ of P on $S \otimes_{\mathcal{O}} \hat{\mathcal{B}}_{\hat{\gamma}}$. Since embeddings preserve the localness of pointed groups (see [16]), there is a local point γ' of P on $S \otimes_{\mathcal{O}} S^\circ \otimes_{\mathcal{O}} \mathcal{B}_\gamma$ containing $\hat{\gamma}'$. On the other hand, by tensoring $\mathcal{A}_\gamma$ with both sides of the P-interior algebra embedding $\mathcal{O} \to S \otimes_{\mathcal{O}} S^\circ$ (see section 3.11), we get a $\dot{G}$-graded P-interior algebra embedding

$$\mathcal{A}_\gamma \to S \otimes_{\mathcal{O}} S^\circ \otimes_{\mathcal{O}} \mathcal{A}_\gamma. \tag{3.13.2}$$

Obviously $\{i\}$ is a local point of P on $\mathcal{B}_\gamma$. Again, since embeddings preserve the localness of pointed groups (see [16]), there is a local point γ'' of P on $S \otimes_{\mathcal{O}} S^\circ \otimes_{\mathcal{O}} \mathcal{B}_\gamma$ containing the image of i through the embedding (3.13.2). Since $\{i\}$ is also the unique local point of P on $\mathcal{B}_\gamma$, by [9, Theorem 5.3], P has a unique local point on $S \otimes_{\mathcal{O}} S^\circ \otimes_{\mathcal{O}} \mathcal{B}_\gamma$ and thus γ' and γ'' have to be equal. By [4, 2.11.2], the embedding (3.13.2) factors through the embedding (3.13.1) and we get a $\dot{G}$-graded P-interior algebra embedding $\mathcal{A}_\gamma \to S \otimes_{\mathcal{O}} \hat{\mathcal{A}}_{\hat{\gamma}}$. Combining the isomorphism (3.10.1), we get a $\dot{G}$-graded P-interior algebra embedding

$$\mathcal{A}_\gamma \to S \otimes_{\mathcal{O}} \mathcal{O}_*\hat{\mathfrak{G}}. \tag{3.13.3}$$

Since P has a unique local point on $S \otimes_{\mathcal{O}} \mathcal{O}_*\hat{\mathfrak{H}}$, which consists of the identity element of $S \otimes_{\mathcal{O}} \mathcal{O}_*\hat{\mathfrak{H}}$, the embedding (3.13.3) must be a $\dot{G}$-graded P-interior algebra isomorphism.

3.14. Denote by $N_{S^*}(P)$ the normalizer in S^* of the image of P in S^*. Set

$$F = N_{S^*}(P)/P(S^P)^* \text{ and } \hat{F} = N_{S^*}(P)/P(1_S + J(S^P)).$$

Let $\tilde{x}$ be an element of $E_{G,\dot{G}}(P_\gamma)$ and x a representative of $\tilde{x}$ in $N_G(P_\gamma)$. Since the P-interior algebra S is $N_G(P_\gamma)$-stable (see section 3.6), there is an invertible element $s_{\tilde{x}}$ in S such that $s_{\tilde{x}}(u1_S)s_{\tilde{x}}^{-1} = \varphi_x(u)1_S$ for any $u \in P$. It is easy to check that the correspondence $\theta : E_{G,\dot{G}}(P_\gamma) \to F$ mapping $\tilde{x}$ onto the image of $s_{\tilde{x}}$ in F is a group homomorphism. By [4, 2.12.4], the homomorphism θ can be lifted to a group homomorphism $\hat{\theta} : E_{G,\dot{G}}(P_\gamma) \to \hat{F}$.

3.15. Set $\hat{\theta}(\tilde{x}) = \tilde{s}_{\tilde{x}}$ and let $\hat{s}_{\tilde{x}}$ be a representative of $\tilde{s}_{\tilde{x}}$ in $N_{S^*}(P)$. Let $\hat{b}_{\tilde{x}}$ be an element of $\hat{\mathfrak{L}}$ such that the image of $\hat{b}_{\tilde{x}}$ in $\mathfrak{L}$ corresponds to $\tilde{x}$ through the isomorphism (3.8.2), and set $a_{\tilde{x}} = \hat{s}_{\tilde{x}} \otimes \hat{b}_{\tilde{x}}$. Then $\hat{b}_{\tilde{x}} u \hat{b}_{\tilde{x}}^{-1} = \varphi_x(u)\hat{\imath}$ for any $u \in P$ and $\hat{b}_{\tilde{x}}$ belongs to $\mathcal{N}_{\hat{\mathcal{A}}_{\hat{\gamma}}^*}(P)$ (see section 3.9), thus $a_{\tilde{x}} u a_{\tilde{x}}^{-1} = \varphi_x(u) i$ for any $u \in P$ and $a_{\tilde{x}}$ belongs to $\mathcal{N}_{\mathcal{A}_\gamma^*}(P)$. Then it is easy to check that the correspondence

$$\hat{\mathfrak{L}} \to \hat{\mathcal{F}}_{\mathcal{A},\,\dot{G}}(P_\gamma) \tag{3.15.1}$$

mapping $\hat{b}_{\tilde{x}}$ onto $\bar{a}_{\tilde{x}}$ is a well-defined injective k^*-group homomorphism, which induces an injective group homomorphism $\mathfrak{L} \to \mathcal{F}_{\mathcal{A},\,\dot{G}}(P_\gamma)$. Since the orders of $\mathfrak{L}$ and $\mathcal{F}_{\mathcal{A},\,\dot{G}}(P_\gamma)$ are the same (see (2.8.1) and (3.8.2)), the homomorphism $\mathfrak{L} \to \mathcal{F}_{\mathcal{A},\,\dot{G}}(P_\gamma)$ is a group isomorphism and thus the homomorphism (3.15.1) is a k^*-group isomorphism.

3.16. By composing the isomorphism (3.15.1) and the inverse of the isomorphism (2.5.1), we get a k^*-group isomorphism $\rho : \hat{\mathfrak{L}} \cong \hat{E}_{G,\,\dot{G}}(P_\gamma)^\circ$. We claim that the isomorphism ρ lifts the isomorphism (3.8.2). It suffices to prove that the inverse image of $\bar{a}_{\tilde{x}}$ through the isomorphism (2.5.1) is an inverse image of $\tilde{x}$. Let $\overline{(x,\, s_\gamma(a))}$ be an inverse image of $\tilde{x}$ in $\hat{E}_{G,\,\dot{G}}(P_\gamma)^\circ$. We adjust the choice of a in $\mathcal{B}^P$ such that xa^{-1} commutes with i (see section 2.6). Set $d_x = xa^{-1}i$. Then d_x belongs to $\mathcal{N}^{\dot{x}}_{\mathcal{A}_\gamma^*}(P)$ and $d_x^{-1}a_{\tilde{x}}$ belongs to $\mathcal{B}_\gamma^*$. Since $(d_x^{-1}a_{\tilde{x}})u(d_x^{-1}a_{\tilde{x}})^{-1} = d_x^{-1}\varphi_x(u)d_x = ui$ for any $u \in P$, $d_x^{-1}a_{\tilde{x}}$ belongs to $(\mathcal{B}_\gamma^P)^*$. Therefore there is $\lambda \in k^*$ such that $\bar{d}_x = \lambda \bar{a}_{\tilde{x}}$. Then the claim follows from the explicit correspondence of the isomorphism (2.5.1).

3.17. Recall that the homomorphism (3.1.1) coincides with the homomorphism (3.9.1) and that the homomorphism (3.9.1) is the composition of the homomorphism $\mathfrak{L} \to \mathrm{Aut}(P)$ and the inverse of the isomorphism (3.8.2). Therefore the isomorphism ρ induces a k^*-group isomorphism $\hat{\mathfrak{G}} = P\hat{\imath} \rtimes \hat{\mathfrak{L}} \cong P \rtimes \hat{E}_{G,\,\dot{G}}(P_\gamma)^\circ = \hat{\mathcal{L}}^\circ$. Now we have proved the isomorphism (3.3.1).

4. Proofs of the Theorem and the Corollary

In this section, we will prove the Theorem and the Corollary given in the introduction. So we need to use the notation there. We also continue to use the notation from section 2.1.

4.1. **Proof of the Theorem.** We denote by $(\mathcal{O}H)^P$ the centralizer of P in $\mathcal{O}H$ and by $\mathrm{Br}_P^{\mathcal{O}H}$ the Brauer homomorphism $(\mathcal{O}H)^P \to kC_H(P)$. Let γ' be a local point of P on $\mathcal{O}H'$ such that $\mathrm{Br}_P^{\mathcal{O}H}(\gamma) = \mathrm{Br}_P^{\mathcal{O}H'}(\gamma')$. Since b' is the Brauer correspondent of b in H', $\mathrm{Br}_P^{\mathcal{O}H}(b) = \mathrm{Br}_P^{\mathcal{O}H'}(b')$ and then it is easy to check that $P_{\gamma'}$ is a defect pointed group of the block b' of H'. Take $i' \in \gamma'$ and set $\mathcal{A}'_{\gamma'} = i'(\mathcal{O}G')i'$ and $\mathcal{B}'_{\gamma'} = i'(\mathcal{O}H')i'$. The subalgebra $\mathcal{A}'_{\gamma'}$ is a G'/H'-graded P-interior algebra (see section 2.1). Note that $N_G(P_\gamma) = N_{G'}(P_{\gamma'})$ and $E_{G,\,\dot{G}}(P_\gamma) = E_{G',\,G'/H'}(P_{\gamma'})$. We fix the group homomorphism (3.1.1), then get an action of $E_{G',\,G'/H'}(P_{\gamma'})$ on P, and lift this action to an action of $\hat{E}_{G',\,G'/H'}(P_{\gamma'})$ on P through the canonical homomorphism $\hat{E}_{G',\,G'/H'}(P_{\gamma'}) \to E_{G',\,G'/H'}(P_{\gamma'})$. By Proposition 3.3, there is a P-interior full matrix algebra S' such that there is a G'/H'-graded P-interior algebra isomorphism

$$\mathcal{A}'_{\gamma'} \cong S' \otimes_{\mathcal{O}} \mathcal{O}_*(P \rtimes \hat{E}_{G',\,G'/H'}(P_{\gamma'})^\circ). \tag{4.1.1}$$

In particular, $\mathcal{B}'_{\gamma'} \cong S' \otimes_{\mathcal{O}} \mathcal{O}_*(P \rtimes \hat{E}_{H'}(P_{\gamma'})^{\circ})$ as P-interior algebras. By [8, Proposition 14.6], S' has to be equal to $\mathcal{O}$. So without loss of generality, we can assume that the P-interior algebra $\mathcal{O}$ is just the trivial P-interior algebra $\mathcal{O}$.

Note that $\mathrm{Br}_P^{\mathcal{O}H}(\gamma)$ is a point of $kC_H(P)$ and that the simple factor $\mathcal{B}(P_\gamma)$ actually is isomorphic to the simple factor of $kC_H(P)$ determined by $\mathrm{Br}_P^{\mathcal{O}H}(\gamma)$. Since $\mathrm{Br}_P^{\mathcal{O}H}(\gamma) = \mathrm{Br}_P^{\mathcal{O}H'}(\gamma')$, $\mathcal{B}(P_\gamma)$ and $(\mathcal{O}H')(P_{\gamma'})$ are isomorphic as $N_G(P_\gamma)$-algebras. Then it is easy to check that the equality $E_{G,\dot{G}}(P_\gamma) = E_{G',G'/H'}(P_{\gamma'})$ can be lifted to a k^*-group isomorphism $\hat{E}_{G,\dot{G}}(P_\gamma) \cong \hat{E}_{G',G'/H'}(P_{\gamma'})$. Moreover, this k^*-group isomorphism extends to a P-interior algebra isomorphism

$$\mathcal{O}_*(P \rtimes \hat{E}_{G,\dot{G}}(P_\gamma)^{\circ}) \cong \mathcal{O}_*(P \rtimes \hat{E}_{G',G'/H'}(P_{\gamma'})^{\circ}). \tag{4.1.2}$$

The inclusion $G' \subset G$ induces a group isomorphism $\dot{G} \cong G'/H'$. Identify $\dot{G}$ and G'/H'. Then the isomorphism (4.1.1) becomes a $\dot{G}$-graded P-interior algebra isomorphism. Also, the isomorphism (4.1.2) is a $\dot{G}$-graded P-interior algebra isomorphism. Then by these two $\dot{G}$-graded P-interior algebra isomorphisms and the isomorphism (3.3.1) as well, we get a $\dot{G}$-graded P-interior algebra isomorphism

$$\mathcal{A}_\gamma \cong S \otimes_{\mathcal{O}} \mathcal{A}'_{\gamma'}. \tag{4.1.3}$$

We identify $\mathcal{A}_\gamma$ with $S \otimes_{\mathcal{O}} \mathcal{A}'_{\gamma'}$ through the isomorphism (4.1.3). Let V be an $\mathcal{O}P$-module such that $\mathrm{End}_{\mathcal{O}}(V) \cong S$ as P-interior algebras. Set

$$M = \mathcal{O}Gi \otimes_{\mathcal{A}_\gamma} (V \otimes_{\mathcal{O}} \mathcal{A}'_{\gamma'}) \otimes_{\mathcal{A}'_{\gamma'}} i'\mathcal{O}G' \text{ and } N = \mathcal{O}Hi \otimes_{\mathcal{B}_\gamma} (V \otimes_{\mathcal{O}} \mathcal{B}'_{\gamma'}) \otimes_{\mathcal{B}'_{\gamma'}} i'\mathcal{O}H'.$$

By [4, 2.14.1], the $\mathcal{O}(G\times G')$-module M induces a Morita equivalence between $\mathcal{O}Gb$ and $\mathcal{O}G'b'$, and by [6, Corollary 3.5], the $\mathcal{O}(H \times H')$-module N induces a Morita equivalence between $\mathcal{O}Hb$ and $\mathcal{O}H'b'$. Moreover, the inflation of the $\mathcal{O}P$-module V through the isomorphism $\Delta(P) \cong P$, $(x, x) \mapsto x$ is a source of the $\mathcal{O}(H \times H')$-module N.

The inclusions $\mathcal{O}Hi \subset \mathcal{O}Gi$, $\mathcal{B}_\gamma \subset \mathcal{A}_\gamma$, $i'\mathcal{O}H' \subset i'\mathcal{O}G'$ and $\mathcal{B}'_{\gamma'} \subset \mathcal{A}'_{\gamma'}$ induce an $\mathcal{O}(H \times H')$-module homomorphism

$$N \to \mathrm{Res}^{G\times G'}_{H\times H'}(M). \tag{4.1.4}$$

These inclusions have respective sections $\mathcal{O}Gi \to \mathcal{O}Hi$, $\mathcal{A}_\gamma \to \mathcal{B}_\gamma$, $\mathcal{A}'_{\gamma'} \to \mathcal{B}'_{\gamma'}$ and $i'\mathcal{O}G' \to i'\mathcal{O}H'$, which are homomorphisms of $(\mathcal{O}H, \mathcal{B}_\gamma)$-bimodules, $(\mathcal{B}_\gamma, \mathcal{B}_\gamma)$-bimodules, $(\mathcal{B}'_{\gamma'}, \mathcal{B}'{}_{\gamma'})$-bimodules and $(\mathcal{B}'_{\gamma'}, \mathcal{O}H')$-bimodules, respectively. Therefore the homomorphism (4.1.4) has a section, the $\mathcal{O}(H\times H')$-module homomorphism induced by the first four sections. Now we claim that the image of N is stabilized by the multiplication by $\Delta(N_G(P_\gamma))$.

Given $x \in N_G(P_\gamma)$, there are some invertible elements $a_x \in (\mathcal{O}H)^P$ and $b_x \in (\mathcal{O}H')^P$ such that $xix^{-1} = a_x i a_x^{-1}$ and $xi'x^{-1} = b_x i' b_x^{-1}$, therefore $a_x^{-1}x$ and $b_x^{-1}x$ centralize i and i', respectively, and then $a_x^{-1}xi$ and $b_x^{-1}xi'$ belong to $\mathcal{A}_\gamma$ and $\mathcal{A}'_{\gamma'}$, respectively. Hence, modifying the choice of a_x if necessary, we get $a_x^{-1}xi = s_x \otimes b_x^{-1}xi'$ for some s_x in S^*. Thus, for any $a \in \mathcal{O}Hi$, any $a' \in i'\mathcal{O}H'$, any $v \in V$ and any $d \in \mathcal{B}'{}_{\gamma'}$ in M, we have

$$\begin{aligned}(x,x) \ \cdot \ (a \otimes (v \otimes d) \otimes a') &= xa \otimes (v \otimes d) \otimes a'x^{-1} \\ &= xax^{-1}a_x(a_x^{-1}xi) \otimes (v \otimes d) \otimes (i'x^{-1}b_x)b_x^{-1}xa'x^{-1} \\ &= xax^{-1}a_x \otimes (s_x \cdot v \otimes (b_x^{-1}xi')d(i'x^{-1}b_x)) \otimes b_x^{-1}xa'x^{-1};\end{aligned}$$

since $xax^{-1}a_x$, $b_x^{-1}xa'x^{-1}$ and $(b_x^{-1}xi')d(i'x^{-1}b_x)$ belong to $\mathcal{O}Hi$, $i'\mathcal{O}H'$ and $\mathcal{B}'_{\gamma'}$, respectively, this proves our claim.

In particular, the homomorphism (4.1.4) actually becomes an $\mathcal{O}K$-module homomorphism. Moreover, the homomorphism (4.1.4) induces an $\mathcal{O}(G\times G')$-module isomorphism $\mathrm{Ind}_K^{G\times G'}(N)\cong M$.

4.2. **Proof of the Corollary.** Assume that the block b is inertial. We continue to use the $\mathcal{O}(G\times G')$-module M and the $\mathcal{O}(H\times H')$-module N from the proof of the Theorem. By the proof of the Theorem, the module M induces a Morita equivalence between $\mathcal{O}Gb$ and $\mathcal{O}G'b'$. Consequently, the block e determines a block e' of G', such that the $\mathcal{O}(G\times G')$-module eMe' induces a Morita equivalence between $\mathcal{O}Ge$ and $\mathcal{O}G'e'$. Again, by the proof of the Theorem, $M\cong \mathrm{Ind}_K^{G\times G'}(N)$ as $\mathcal{O}(G\times G')$-modules, and the $\mathcal{O}(H\times H')$-module N has vertex $\Delta(P)$. Since the index of H in G is coprime to p, the $\mathcal{O}(G\times G')$-module eMe' has vertex $\Delta(P)$. Then by [10, Theorem 6.9 and Corollary 7.4], there exist defect pointed groups P_δ of the block e, $P_{\delta'}$ of the block e' and an $\mathcal{O}P$-module W such that, setting $(\mathcal{O}G)_\delta=j(\mathcal{O}G)j$ for some $j\in\delta$ and $(\mathcal{O}G')_{\delta'}=j'(\mathcal{O}G')j'$ for some $j'\in\delta'$, we have a P-interior algebra embedding

$$(\mathcal{O}G)_\delta\to \mathrm{End}_{\mathcal{O}}(W)\otimes_{\mathcal{O}}(\mathcal{O}G')_{\delta'}. \tag{4.2.1}$$

Moreover, the $\mathcal{O}P$-module W is an endopermutation module.

Since the block e' has the normal defect group P, by [8, Proposition 14.6] there is a P-interior algebra isomorphism $(\mathcal{O}G')_{\delta'}\cong \mathcal{O}_*(P\rtimes \hat{E}_{G'}(P_{\delta'})^\circ)$. Since P has a unique local point on $\mathrm{End}_{\mathcal{O}}(W)\otimes_{\mathcal{O}}\mathcal{O}_*(P\rtimes\hat{E}_{G'}(P_{\delta'})^\circ)$, which consists of the identity element of $\mathrm{End}_{\mathcal{O}}(W)\otimes_{\mathcal{O}}\mathcal{O}_*(P\rtimes \hat{E}_{G'}(P_{\delta'})^\circ)$, and since embeddings preserve the localness of pointed groups (see [16]), the embedding (4.2.1) has to be a P-interior algebra isomorphism.

Set $T=\mathrm{End}_{\mathcal{O}}(W)$ and consider the P-interior algebra $\mathcal{C}=T^\circ\otimes_{\mathcal{O}}(\mathcal{O}G)_\delta$. By [9, Theorem 5.3], P has a unique local point ε on $\mathcal{C}$. Set $\mathcal{C}_\varepsilon=\ell\mathcal{C}\ell$ for some $\ell\in\varepsilon$. Then by a proof similar to section 3.11, we prove that there is a P-interior algebra isomorphism $\mathcal{C}_\varepsilon\cong\mathcal{O}_*(P\rtimes\hat{E}_{G'}(P_{\delta'})^\circ)$. On the other hand, by [7, Theorem 3.1] and [10, 7.6.3] $E_G(P_\delta)\cong E_{G'}(P_{\delta'})$ as groups. Therefore the $\mathcal{O}$-rank of $\mathcal{C}_\varepsilon$ is equal to the product of the orders of P and $E_G(P_\delta)$. Then by [11, Proposition 3.11], the block e of G is inertial.

Conversely, assume that the block e is inertial. Then the Corollary follows from [11].

References

[1] Michel Broué and Lluis Puig, *A Frobenius theorem for blocks*, Invent. Math. **56** (1980), no. 2, 117–128, DOI 10.1007/BF01392547. MR558864 (81d:20011)

[2] Everett C. Dade, *Group-graded rings and modules*, Math. Z. **174** (1980), no. 3, 241–262, DOI 10.1007/BF01161413. MR593823 (82c:16028)

[3] Shigeo Koshitani and Naoko Kunugi, *Broué's conjecture holds for principal 3-blocks with elementary abelian defect group of order 9*, J. Algebra **248** (2002), no. 2, 575–604, DOI 10.1006/jabr.2001.9048. MR1882112 (2002j:20021)

[4] Burkhard Külshammer and Lluis Puig, *Extensions of nilpotent blocks*, Invent. Math. **102** (1990), no. 1, 17–71, DOI 10.1007/BF01233419. MR1069239 (91i:20009)

[5] Andrei Marcus, *On equivalences between blocks of group algebras: reduction to the simple components*, J. Algebra **184** (1996), no. 2, 372–396, DOI 10.1006/jabr.1996.0265. MR1409219 (97m:20012)

[6] Lluis Puig, *Pointed groups and construction of characters*, Math. Z. **176** (1981), no. 2, 265–292, DOI 10.1007/BF01261873. MR607966 (82d:20015)

[7] Lluis Puig, *Local fusions in block source algebras*, J. Algebra **104** (1986), no. 2, 358–369, DOI 10.1016/0021-8693(86)90221-8. MR866781 (88c:20020)

[8] Lluis Puig, *Pointed groups and construction of modules*, J. Algebra **116** (1988), no. 1, 7–129, DOI 10.1016/0021-8693(88)90195-0. MR944149 (89e:20024)

[9] Lluis Puig, *Nilpotent blocks and their source algebras*, Invent. Math. **93** (1988), no. 1, 77–116, DOI 10.1007/BF01393688. MR943924 (89e:20023)

[10] Lluis Puig, *On the local structure of Morita and Rickard equivalences between Brauer blocks*, Progress in Mathematics, vol. 178, Birkhäuser Verlag, Basel, 1999. MR1707300 (2001d:20006)

[11] Lluis Puig, *Nilpotent extensions of blocks*, Math. Z. **269** (2011), no. 1-2, 115–136, DOI 10.1007/s00209-010-0718-1. MR2836062 (2012j:20030)

[12] Lluis Puig and Yuanyang Zhou, *Glauberman correspondents and extensions of nilpotent block algebras*, J. Lond. Math. Soc. (2) **85** (2012), no. 3, 809–837, DOI 10.1112/jlms/jdr069. MR2927809

[13] Jeremy Rickard, *Splendid equivalences: derived categories and permutation modules*, Proc. London Math. Soc. (3) **72** (1996), no. 2, 331–358, DOI 10.1112/plms/s3-72.2.331. MR1367082 (97b:20011)

[14] Raphaël Rouquier, *Block theory via stable and Rickard equivalences*, Modular representation theory of finite groups (Charlottesville, VA, 1998), de Gruyter, Berlin, 2001, pp. 101–146. MR1889341 (2003g:20018)

[15] Jean-Pierre Serre, *Local fields*, Graduate Texts in Mathematics, vol. 67, Springer-Verlag, New York-Berlin, 1979. Translated from the French by Marvin Jay Greenberg. MR554237 (82e:12016)

[16] Jacques Thévenaz, *G-algebras and modular representation theory*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1995. Oxford Science Publications. MR1365077 (96j:20017)

[17] Yuanyang Zhou, *A remark on Rickard complexes*, J. Algebra **399** (2014), 845–853, DOI 10.1016/j.jalgebra.2013.10.015. MR3144614

Department of Mathematics and Statistics, Central China Normal University, Wuhan, 430079, People's Republic of China

E-mail address: zhouyuanyang@mail.ccnu.edu.cn