NON-ISOMORPHIC COMPLEMENTED SUBSPACES OF THE REFLEXIVE ORLICZ FUNCTION SPACES \( L^\Phi[0,1] \)

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Abstract. In this note we show that the number of isomorphism classes of complemented subspaces of a reflexive Orlicz function space \( L^\Phi[0,1] \) is uncountable, as soon as \( L^\Phi[0,1] \) is not isomorphic to \( L^2[0,1] \). Also, we prove that the set of all separable Banach spaces that are isomorphic to such an \( L^\Phi[0,1] \) is analytic non-Borel. Moreover, by using the Boyd interpolation theorem we extend some results on \( L^p[0,1] \) spaces to the rearrangement invariant function spaces under natural conditions on their Boyd indices.

1. Introduction

Let \( C = \bigcup_{n=1}^{\infty} \mathbb{N}^n \). Consider the Cantor group \( G = \{-1,1\}^C \) equipped with the Haar measure. The dual group is the discrete group formed by Walsh functions \( w_F = \prod_{c \in F} r_c \) where \( F \) is a finite subset of \( C \) and \( r_c \) is the Rademacher function, that is, \( r_c(x) = x(c), x \in G \). These Walsh functions generate \( L^p(G) \) for \( 1 \leq p < \infty \), and the reflexive Orlicz function spaces \( L^\Phi(G) \), where \( \Phi \) is an Orlicz function.

A measurable function \( f \) on \( G \) only depends on the coordinates \( F \subset C \), provided \( f(x) = f(y) \) whenever \( x, y \in G \) with \( x(c) = y(c) \) for all \( c \in F \). A measurable subset \( S \) of \( G \) depends only on the coordinates \( F \subset C \) provided \( \chi_S \) does. Moreover, for \( F \subset C \) the sub-\( \sigma \)-algebra \( \mathcal{G}(F) \) contains all measurable subsets of \( G \) that depend only on the \( F \)-coordinates. A branch in \( C \) will be a subset of \( C \) consisting of mutually comparable elements. For more the reader is referred to [BRS81], [Bou81] and [DK14].

In [BRS81], the authors considered the subspace \( X^p_C \) which is the closed linear span in \( L^p(G) \) over all finite branches \( \Gamma \) in \( C \) of all those functions in \( L^p(G) \) which depend only on the coordinates of \( \Gamma \). In addition, they proved that \( X^p_C \) is complemented in \( L^p(G) \) and isomorphic to \( L^p(G) \), for \( 1 < p < \infty \). Moreover, for a tree \( T \) on \( \mathbb{N} \), the space \( X^p_T \) is the closed linear span in \( L^p(G) \) over all finite branches \( \Gamma \) in \( T \) of all those functions in \( L^p(G) \) which depend only on the coordinates of \( \Gamma \). Hence, \( X^p_T \) is a one-complemented subspace of \( X^p_C \) by the conditional expectation operator with respect to the sub-\( \sigma \)-algebra \( \mathcal{G}(T) \) which contains all measurable subsets of \( G \) that depend only on the \( T \)-coordinates. Bourgain in [Bou81] showed that the tree \( T \) is well-founded if and only if the space \( X^p_T \) does not contain a copy of \( L^p[0,1] \), for \( 1 < p < \infty \) and \( p \neq 2 \). Consequently, it was shown that if \( B \) is a universal separable

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Banach space for the elements of the class \( \{ X_T^p; T \text{ is a well-founded tree} \} \), then \( B \) contains a copy of \( L^p[0,1] \). It follows that there are uncountably many mutually non-isomorphic members in this class.

In this note we will show that these results extend to the case of the reflexive Orlicz function spaces \( L^\Phi[0,1] \), where \( \Phi \) is an Orlicz function. Moreover, some of the results extend to rearrangement invariant function spaces under some conditions on the Boyd indices.

2. Notation

A rearrangement invariant function space \( X \) (r.i. function space) on the interval \( I = [0,1] \) or \( [0,\infty] \) is a Banach space of equivalence classes of measurable functions on \( I \) such that:

(i) \( X(I) \) is a Banach lattice with respect to the pointwise order.

(ii) For every automorphism \( \tau \) of \( I \) (i.e., an invertible transformation \( \tau \) from \( I \) onto itself so that, for any measurable subset \( E \) of \( I \), \( \mu(\tau^{-1}E) = \mu(E) \)) and every \( f \in X(I) \), we have \( f(\tau) \in X(I) \) and \( \| f(\tau) \| = \| f \| \).

(iii) For \( I = [0,1] \) we have \( L^\infty([0,1]) \subset X[0,1] \subset L^1([0,1]) \) with norm one embeddings. Moreover, for \( I = [0,\infty] \) we have \( L^1[0,\infty] \cap L^\infty[0,\infty] \subset X[0,\infty] \subset L^1[0,\infty] + L^\infty[0,\infty] \) with norm one embeddings.

(iv) \( L^\infty([0,1]) \) is dense in \( X[0,1] \). For \( [0,\infty) \), the simple functions with bounded support are dense in \( X[0,\infty) \).

For more about the definition of the r.i. function space \( X(\Omega,\Sigma,\nu) \) on the separable measure space \( (\Omega,\Sigma,\nu) \) see [LT79] pp. 114-117.

We recall the definition of the indices introduced by D. Boyd in [Boy69]. If \( X \) is an r.i. function space on \([0,1]\), define the dilation mapping \( D_s \), \( 0 < s < \infty \), by the formula

\[
(D_s f)(t) = f(st), \quad t \in [0,1], \quad f \in X.
\]

In order to make this definition meaningful, the function \( f \) is extended to \([0,\infty)\) by \( f(u) = 0 \) for \( u > 1 \). Now define the indices

\[
\alpha_X = \inf_{0 < s < 1} \left( \frac{-\log \| D_s \|}{\log s} \right); \quad \beta_X = \sup_{1 < s < \infty} \left( \frac{-\log \| D_s \|}{\log s} \right).
\]

The numbers \( \alpha_X, \beta_X \) belong to the closed interval \([0,1]\) and are called the Boyd indices of \( X \). The Boyd indices of the r.i. function space in [LT79] Definition 2.b.1 are taken to be the reciprocals of the ones we use here (i.e., \( p_X = \frac{1}{\alpha_X} \), and \( q_X = \frac{1}{\beta_X} \)). We recall that a Banach lattice is super-reflexive if and only if it is \( p \)-convex and \( q \)-concave for some \( p > 1 \) and \( q < \infty \). Therefore, the Boyd indices of a super-reflexive r.i. function space \( X \) satisfy \( 0 < \beta_X \leq \alpha_X < 1 \), (see [JMST79] pp. 207-208).

We need a weaker version of Boyd’s interpolation theorem, [JMST79] p. 208:

Let \( X[0,1] \) be a rearrangement invariant function space, let \( p,q \) be such that \( 0 < \frac{1}{q} < \beta_X \leq \alpha_X < \frac{1}{p} < 1 \), and \((\Omega,\mathcal{F},\mathbb{P})\) be a probability space. A linear transformation \( L \), which is bounded from \( L^p(\Omega,\mathcal{F},\mathbb{P}) \) to itself and from \( L^q(\Omega,\mathcal{F},\mathbb{P}) \) to itself defines a bounded operator from \( X(\Omega,\mathcal{F},\mathbb{P}) \) into itself.

The most commonly used r.i. function spaces on \([0,1]\) besides the \( L^p \) spaces, \( 1 \leq p \leq \infty \), are the Orlicz function spaces. We recall their definition.
Let \((\Omega, \mathcal{F}, \mu)\) be a separable measure space and \(\Phi\) be an Orlicz function. The Orlicz function space \(L^\Phi(\Omega, \mathcal{F}, \mu)\) is the space of all (equivalence classes of) \(\mathcal{F}\)-measurable functions \(f\) so that
\[
\int_\Omega \Phi\left(\frac{|f|}{\rho}\right) < \infty
\]
for some \(\rho > 0\). The norm is defined by
\[
\|f\|_\Phi = \inf\{\rho > 0; \int_\Omega \Phi\left(\frac{|f|}{\rho}\right) \leq 1\}.
\]
In addition, we require the normalization \(\Phi(1) = 1\), in order that \(\|\chi_{(0,1)}\|_\Phi = 1\) in \(L^\Phi[0,1]\) and \(L^\Phi(0,\infty)\).

It is known that the Orlicz function space \(L^\Phi[0,1]\) is reflexive if and only if there exist \(\lambda_0 > 0\) and \(t_0 > 0\) such that for every \(\lambda > \lambda_0\) there exist positive constants \(c_\lambda, C_\lambda\) such that
\[
c_\lambda \Phi(t) \leq \Phi(\lambda t) \leq C_\lambda \Phi(t), \quad \forall t \geq t_0.
\]
Moreover, the Orlicz function space \(L^\Phi[0,1]\) is reflexive if and only if \(\Phi\) is super-reflexive (see [KP98, Corollary 2]). The Boyd indices for the Orlicz function space \(L^\Phi[0,1]\) are non-trivial (i.e., \(0 < \beta_\Phi < \alpha_\Phi < 1\)) if and only if it is reflexive. Furthermore, we recall that any Orlicz function space \(L^\Phi[0,1]\) is isomorphic to \(L^2[0,1]\) if and only if there exists \(t_0\) such that \(\Phi(t)\) is equivalent to \(t^2\) for all \(t \geq t_0\).

Let \(\{\Phi_n\}_{n=1}^\infty\) be a sequence of Orlicz functions. The space \(\ell_{\Phi_n}\) is the Banach space of all sequences \(x = (x_n)\) with
\[
\sum_{n=1}^\infty \Phi_n\left(\frac{|x_n|}{\rho}\right) < \infty
\]
for some \(\rho > 0\), equipped with the norm
\[
\|x\|_{\Phi_n} = \inf\{\rho > 0; \sum_{n=1}^\infty \Phi_n\left(\frac{|x_n|}{\rho}\right) \leq 1\}.
\]
The space \(\ell_{\Phi_n}\) is called a modular sequence space. An important subspace of \(\ell_{\Phi_n}\) is \(h_{\Phi_n}\), which consists of those sequences \(x = (x_1, x_2, \ldots)\) such that
\[
\sum_{n=1}^\infty \Phi_n\left(\frac{|x_n|}{\rho}\right) < \infty
\]
for every \(\rho > 0\).

Let \(L^\Phi[0,1]\) be a reflexive Orlicz space. The space \(L^\Phi[0,\infty)\) is isomorphic to \(L^\Phi[0,1]\) where \(\Phi\) is equivalent to \(t^2\) at 0 and to \(\Phi\) at \(\infty\), for example: \(\overline{\Phi}(t) = t^2\chi_{[0,1]}(t) + (2\Phi(t) - 1)\chi_{[1,\infty]}(t)\). Let \(\{f_n\}_{n=1}^\infty\) be a sequence of independent mean zero random variables in \(L^\Phi[0,1]\). We denote by \(\{\overline{f}_n\}_{n=1}^\infty\) the sequence of \(L^\Phi[0,\infty)\) defined by
\[
\overline{f}_n(t) = f_n(t-n+1)\chi_{[n-1,n)}(t) \quad \text{where} \quad t > 0.
\]
Thus, the sequence \(\{\overline{f}_n\}_{n=1}^\infty\) is a disjointification of the sequence \(\{f_n\}_{n=1}^\infty\). By [JS89, Theorem 1] we can find a constant \(C \geq 1\), which depends only on \(\Phi\), such that if \(\{f_n\}_{n=1}^\infty\) is a sequence of independent mean zero random variables, then:
\[
C^{-1}\|\sum_{n=1}^m \overline{f}_n\|_{L^\Phi[0,\infty)} \leq \|\sum_{n=1}^m f_n\|_{L^\Phi[0,1]} \leq C\|\sum_{n=1}^m \overline{f}_n\|_{L^\Phi[0,\infty)}.
\]
For more about this inequality and its applications see [JS89], [Rui91] and [ASW11].

**Proposition 2.1.** Let \(L^\Phi[0,1]\) be a reflexive Orlicz space and \(\{f_n\}_{n=1}^\infty\) be a sequence of independent mean zero random variables in \(L^\Phi[0,1]\). Then \(\{f_n\}_{n=1}^\infty\) is equivalent to the sequence of unit vectors \(\{e_n\}_{n=1}^\infty\) of the modular sequence space \(\ell_{\Phi_n}\) for some \(\{\varphi_n\}_{n=1}^\infty\).
Proof. Let \((\mathcal{f}_n)_{n=1}^\infty\) be a disjointification sequence of \((f_n)_{n=1}^\infty\) such that \(\mathcal{f}_n(t) = f_n(t-n+1)\chi_{|m-n|,n}(t)\) where \(t > 0\). Moreover, define the Orlicz functions \(\varphi_n(s) = \int_{n-1}^{n} \Phi(s|f_n(t-n+1)|)dt, n \in \mathbb{N}\). Now, assume \((e_n)_{n=1}^\infty\) is the sequence of the unit vectors in \(\ell(\varphi_n)\). Furthermore, let \((a_n)_{n=1}^m\) be real numbers. Then for any \(\lambda > 0\), we have
\[
\sum_{n=1}^{m} \varphi_n\left(\frac{|a_n|}{\lambda}\right) = \sum_{n=1}^{m} \int_{n-1}^{n} \Phi\left(\frac{|a_n|}{\lambda} |f_n(t-n+1)|\right)dt
\]
\[
= \int_{0}^{\infty} \Phi\left(\frac{|\sum_{n=1}^{m} a_n \mathcal{f}_n(t)|}{\lambda}\right)dt.
\]
Therefore, \(\|\sum_{n=1}^{m} a_n e_n\|_{(\varphi_n)} = \|\sum_{n=1}^{m} a_n \mathcal{f}_n\|_{L\Phi[0,\infty)}\). Thus, inequality (2.1) implies that the sequence \((f_n)_{n=1}^\infty\) is equivalent to \((e_n)_{n=1}^\infty\) in \(\ell(\varphi_n)\). \(\Box\)

Now, we need the following results about subspaces of \(X[0,1]\) that are isomorphic to \(X[0,1]\).

**Theorem 2.2 ([JMST79, Theorem 9.1]).** Let \(X[0,1]\) be an r.i. function space such that \(X[0,1]\) is \(q\)-concave for some \(q < \infty\), the index \(\alpha_X < 1\) and the Haar system in \(X[0,1]\) is not equivalent to a sequence of disjoint functions in \(X[0,1]\). Then any subspace of \(X[0,1]\) which is isomorphic to \(X[0,1]\) contains a further subspace which is complemented in \(X[0,1]\) and isomorphic to \(X[0,1]\). In particular, the theorem holds for \(X[0,1] = L^p[0,1], 1 < p < \infty\), and more generally, for every reflexive Orlicz function space \(L^\Phi[0,1]\).

The proof of the following corollary in the case of \(L^p[0,1], 1 < p < \infty\), is a straightforward consequence of Theorem 2.2 and Pelczyński’s decomposition method; it works also for the reflexive Orlicz function spaces \(L^\Phi[0,1]\), since the Haar basis cannot be equivalent to the unit vector basis of a modular sequence space, unless \(L^\Phi[0,1]\) is the Hilbert space (see Theorem 1.6 below).

**Corollary 2.3 ([JMST79, Corollary 9.2]).** Let \(X[0,1]\) be an r.i. function space satisfying the assumptions of Theorem 2.2. If \(Y\) is a complemented subspace of \(X[0,1]\) which contains an isomorphic copy of \(X[0,1]\), then \(X[0,1]\) is isomorphic to \(Y\).

Let \(N\) be a set of natural numbers. Given \(n \in N\), the power \(N^n\) is the set of all sequences (also called nodes) \(s = (s(0), \cdots, s(n-1))\) of length \(n\) of elements from \(N\). If \(m < n\), we let \(s|m = (s(0), \cdots, s(m)) \in N^m\). In this situation, we say that \(t = s|m\) is an initial segment of \(s\), writing \(t \leq s\). Two nodes are compatible if one is an initial segment of the other.

Let \(C = \bigcup_{n=0}^{\infty} N^n\). A tree \(T\) on \(N\) is a subset of \(C\) closed under initial segments. The relation \(\leq\) defined above induces a partial ordering on \(T\). The tree \(\overline{T}\) on \(N\) is well-founded provided there is no sequence \((x_n)_{n=1}^{\infty}\) satisfying \((x_1, \cdots, x_n) \in T\) for each \(n\). For a well-founded tree, we inductively define a transfinite sequence of trees \((T^\alpha)\) on a set \(N\) as follows:

\[T^0 = T,\]
\[T^{\alpha+1} = \{ (x_1, \cdots, x_n); (x_1, \cdots, x_n, x) \in T^\alpha \text{ for some } x \in N\},\]
\[T^\alpha = \bigcap_{\beta < \alpha} T^\beta \text{ for a limit ordinal } \alpha.\]
Since $T$ is well-founded, $(T^\alpha)$ is a strictly decreasing sequence, and thus $T^\alpha = \emptyset$ for some ordinal $\alpha$. We define the ordinal index $\omega[T] = \min\{\alpha; T^\alpha = \emptyset\}$. In addition, if $T$ is not well-founded, we let $\omega[T] = \omega_1$, where $\omega_1$ is the first uncountable ordinal. A branch in $C$ is a subset of $C$ consisting of mutually comparable elements.

3. Complemented embedding of separable rearrangement invariant function spaces into spaces with unconditional Schauder decompositions

We aim to extend \cite{BRS81} Theorem 1.1 to the r.i. function space $X[0,1]$ with the Boyd indices $0 < \beta_X \leq \alpha_X < 1$. Our proof heavily relies on the proof of \cite{BRS81}. We will use interpolation arguments to extend it.

**Theorem 3.1.** Let $X[0,1]$ be an r.i. function space whose Boyd indices satisfy $0 < \beta_X \leq \alpha_X < 1$. Suppose $X[0,1]$ is isomorphic to a complemented subspace of a Banach space $Y$ with an unconditional Schauder decomposition $(Y_j)$. Then one of the following holds:

1. There is an $i$ so that $X[0,1]$ is isomorphic to a complemented subspace of $Y_i$.
2. A block basic sequence of the $Y_i$'s is equivalent to the Haar basis of $X[0,1]$ and has closed linear span complemented in $Y$.

We first need some facts about unconditional bases and decompositions that were mentioned in \cite{BRS81}. Given a Banach space $B$ with unconditional basis $(b_i)$ and $(x_i)$ a sequence of non-zero elements in $B$, say that $(x_i)$ is disjoint if there exist disjoint subsets $M_1, M_2, \cdots$ of $\mathbb{N}$ with $x_i \in [b_j]_{j \in M_i}$ for all $i$. Say that $(x_i)$ is essentially disjoint if there exists a disjoint sequence $(y_i)$ such that $\Sigma\|x_i - y_i\|/\|x_i\| < \infty$. If $(x_i)$ is essentially disjoint, then $(x_i)$ is essentially a block basis of a permutation of $(b_i)$. Also, $(x_i)$ is an unconditional basic sequence.

Throughout this paper, if $\{b_i\}_{i \in I}$ is an indexed family of elements of a Banach space $B$, $[b_i]_{i \in I}$ denotes the closed linear span of $\{b_i\}_{i \in I}$ in $B$. We recall the definition of the Haar system $(h_n)$ which is normalized in $L^\infty$: let $h_1 \equiv 1$ and for $n = 2^k + j$ with $k \geq 0$ and $1 \leq j \leq 2^k$,

$$h_n = \chi_{[\frac{j-1}{2^k}, \frac{j-1}{2^k}]} - \chi_{[\frac{j+1}{2^k}, \frac{j+1}{2^k}]}.$$ 

Moreover, the Haar system is an unconditional basis of the r.i. function space $X[0,1]$ if and only if the Boyd indices of $X[0,1]$ satisfy $0 < \beta_X \leq \alpha_X < 1$; see \cite{LT79} Theorem 2.c.6. We use $[f = a]$ for $\{t; f(t) = a\}$; $\mu$ is the Lebesgue measure. For a measurable function $f$, supp $f = [f \neq 0]$.

The following three results are proved in \cite{BRS81}.

**Lemma 3.2.** Let $(b_n)$ be an unconditional basis for the Banach space $B$ with biorthogonal functionals $(b_n^*)$, $T : B \to B$ an operator, $\epsilon > 0$, and $(b_{n_i})$ a subsequence of $(b_n)$ so that $(Tb_{n_i})$ is essentially disjoint and $|b_{n_i}^* T b_{n_i}| \geq \epsilon$ for all $i$. Then $(Tb_{n_i})$ is equivalent to $(b_{n_i})$ and $[Tb_{n_i}]$ is complemented in $B$.

**Lemma 3.3.** Let $Z$ and $Y$ be Banach spaces with unconditional Schauder decompositions $(Z_i)$ and $(Y_i)$ respectively; and let $(P_i)$ (resp. $Q_i$) be the natural projection from $Z$ (resp. $Y$) onto $Z_i$ (resp. $Y_i$). Then if $T : Z \to Y$ is a bounded linear operator, so is $\sum Q_i T P_i$. 
Scholium 3.4. Let $Y$ be a Banach space with unconditional Schauder decomposition with corresponding projections $(Q_i)$, and let $Z$ be a complemented subspace of $Y$ with unconditional basis $(z_i)$ with biorthogonal functionals $(z^*_i)$. Suppose there exist $\varepsilon > 0$, a projection $U : Y \to Z$ and disjoint subsets $M_1, M_2, \ldots$ of $\mathbb{N}$ with the following properties:

(a) $(UQ_i z_i)_{i \in M, i \in \mathbb{N}}$ is an essentially disjoint sequence.
(b) $|z^*_i (UQ_i z_i)| \geq \varepsilon$ for all $l \in M, i \in \mathbb{N}$.

Then $(Q_i z_i)_{i \in M, i \in \mathbb{N}}$ is equivalent to $(z_i)_{i \in M, i \in \mathbb{N}}$ and $[Q_i z_i]_{i \in M, i \in \mathbb{N}}$ is complemented in $Y$.

The next lemma is proved in [LT79, Theorem (2.d.10)] and it is the extension of the fundamental result of Gamlen and Gaudet [GG73] to separable r.i function spaces $X[0,1]$.

Lemma 3.5. Let $I \subset \mathbb{N}$ such that if $E = \{ t \in [0,1] ; t \in \text{supp } h_i \text{ for infinitely many } i \in I \}$, then $E$ is of positive Lebesgue measure. Then $[h_i]_{i \in I}$ is isomorphic to $X[0,1]$.

Recall that $X(\ell^2)$ is the completion of the space of all sequences $(f_1, f_2, \cdots)$ of elements of $X$ which are eventually zero, with respect to the norm 

$$\|(f_1, f_2, \cdots)\|_{X(\ell^2)} = \|(\sum |f_i|^2)^{\frac{1}{2}}\|_X.$$ 

Let $X[0,1]$ be a separable r.i function space with $0 < \beta_X \leq \alpha_X < 1$. Let $\{h_i\}_{i}$ be the Haar basis of $X[0,1]$, fix $i$ and let $(h_{ij})$ be the element of $X(\ell^2)$ whose $j$-th coordinate equals $h_i$, all other coordinates 0. Then $(h_{ij})_{i,j}$ is an unconditional basis for $X(\ell^2)$ ([LT79, Proposition 2.d.8]). Next, we recall [JMST79] Lemma 9.7 and for more see [LT79].

Scholium 3.6. There is a constant $K$ so that for any function $j : \mathbb{N} \mapsto \mathbb{N}$, $(h_{ij(i)})_{i \geq 1}$ in $X(l_2)$ is $K-$ equivalent to $(h_i)$ in $X$.

The following is a consequence of the proof of [LT79, Theorem (2.d.11)] that $X[0,1]$ is primary. Let $(h_{ij(i)})_{i,j}$ denote the biorthogonal functionals to $(h_{ij})_{i,j}$.

Scholium 3.7. Let $T : X(l_2) \mapsto X(l_2)$ be a given operator. Suppose there is a $c > 0$ so that when $I = \{ i : |h^*_i Th_{ij}| \geq c \text{ for infinitely many } j \}$, then $E$ has positive Lebesgue measure, where 

$$E = \{ t \in [0,1] ; t \in \text{supp } h_i \text{ for infinitely many } i \in I \}$$ 

Then there is a subspace $Y$ of $X(l_2)$ with $Y$ isomorphic to $X$, $T|Y$ an isomorphism, and $TY$ complemented in $X(l_2)$.

Proof. Fix $i \in I$. By the definition of $I$, there is a sequence $j_1 < j_2 < \cdots$ with $\|Th_{ij_k}\| \geq c > 0$ for all $k$. Since $\{h_{ij_k}\}_{k=1}^\infty$ is equivalent to the unit vectors in $l_2$, then it is weakly null and so $\{Th_{ij_k}\}_{k \geq 1}$ is weakly null. Thus, there exists $j : I \mapsto \mathbb{N}$ such that $\{Th_{ij(i)}\}_{i \in I}$ is equivalent to a block basis $(z_i)_{i \geq 1}$ and we can choose it such that $\sum_{i \in I} \frac{\|z_i - Th_{ij(i)}\|}{\|h_{ij(i)}\|} < \infty$ by [BP58, Theorem (3)]. Thus $\{Th_{ij(i)}\}_{i \in I}$ is essentially disjoint with respect to $\{h_{ij}\}_{i,j = 1}^\infty$ and $|h^*_i Th_{ij(i)}| \geq c$. Then by Lemma 3.2, $\{Th_{ij(i)}\}_{i \in I}$ is complemented in $X(l_2)$, and $(Th_{ij(i)})_{i \in I}$ is equivalent to $(h_{ij(i)})_{i \in I}$, which is equivalent to $(h_i)_{i \in I}$ by Scholium 3.6. In turn, $[h_i]_{i \in I}$ is isomorphic to $X[0,1]$, by Lemma 3.5. \hfill \Box
Lemma 3.10. Let $T : X[0,1] \mapsto X[0,1]$ be a given operator. Then for $S = T$ or $I - T$, there exists a subspace $Y$ of $X[0,1]$ with $Y$ isomorphic to $X[0,1]$, $S|Y$ an isomorphism, and $S(Y)$ complemented in $X[0,1]$.

Proof. Since $X[0,1]$ is isomorphic to $X(l_2)$ by [LT79, Proposition 2.4.5], we can prove the statement with respect to $X(l_2)$. For each $i$, $j$, at least one of the numbers $|h^*_i Th_{ij}|$ and $|h^*_i (I - T)h_{ij}|$ is $\geq \frac{1}{2}$. Let $I_1 = \{ i : |h^*_i Th_{ij}| \geq \frac{1}{2} \text{ for infinitely many } j \}$, $I_2 = \{ i : |h^*_i (I - T)h_{ij}| \geq \frac{1}{2} \text{ for infinitely many } j \}$. Then $N = I_1 \cup I_2$. Hence, for $k = 1$ or $2$, $E_k = \{ t \in [0,1] ; t \in \supp h_i \text{ for infinitely many } i \in I_k \}$ has positive Lebesgue measure and by Scholium 3.7 we get the result. \hfill \Box

Theorem 3.9. Let $Z$ and $Y$ be given Banach spaces. If $X[0,1]$ is isomorphic to a complemented subspace of $Z \oplus Y$, then $X[0,1]$ is isomorphic to a complemented subspace of $Z$ or to a complemented subspace of $Y$.

Proof. Let $P$ (resp. $Q$) denote the natural projection from $Z \oplus Y$ onto $Z$ (resp. $Y$). Hence $P + Q = I$. Let $K$ be a complemented subspace of $Z \oplus Y$ isomorphic to $X[0,1]$ and let $U : Z \oplus Y \mapsto K$ be a projection. Since $UP|K + UQ|K = I|K$, Corollary 3.8 shows that there is a subspace $W$ of $K$ with $W$ isomorphic to $X[0,1]$, $T W$ an isomorphism, and $T W$ complemented in $K$, where $T = UP|K$ or $T = UQ|K$. Suppose for instance the former: Let $S$ be a projection from $K$ onto $T W$ and $R = (T W)^{-1}$. Then $I|W = RSUP|W$; hence, since the identity on $W$ may be factored through $Z$, $W$ is isomorphic to a complemented subspace of $Z$. \hfill \Box

The next lemma is [GG73, Lemma 4] and it is pointed out in the proof of [LT79 Theorem (2.4.10)] for separable r.i function space.

Lemma 3.10. Let $(x_i)$ be a measurable function on $[0,1]$ with $x_1 \{ 0,1 \}$-valued and $x_i \{ 1,0,-1 \}$-valued for $i > 1$. Suppose there exist positive constants $a$ and $b$ so that, for all positive $l$, with $k$ the unique integer, $1 \leq k \leq l$, and $\alpha$ the unique choice of $+1$ or $-1$ so that $\supp h_{i+1} = [h_k = \alpha]$. Then

(a) $[x_k = \alpha] = \supp x_{i+1}$.

(b) $\frac{a}{2} \int |h_k| \leq \mu([x_{i+1} = \beta]) \leq \frac{b}{2} \int |h_k|$ for $\beta = \pm 1$.

Then $(x_n)$ is equivalent to $(h_n)$ in $X[0,1]$, and $(x_n)$ is the range of a one-norm projection defined on $X[0,1]$.

Proof. Let $A_n = \supp x_n$; then the subspace $[x_n]$ is complemented in $X[0,1]$ since it is the range of the projection $P(f) = \chi_A \mathbb{E}_F(f)$ of norm one, where $A = \bigcup_{n=1}^\infty A_n$ and $\mathbb{E}_F$ denotes the conditional expectation operator with respect to the sub-$\sigma$-algebra generated by the measurable sets $A_n$.

Since $(x_n)$ and $(h_n)$ are equivalent in $L^p[0,1]$ for all $1 < p < \infty$, by [GG73 Lemma 4], then the operator $R_1 : L^p[0,1] \mapsto L^p[0,1]$ defined by $R_1(\sum_{n=1}^\infty a_n h_n) = \sum_{n=1}^\infty a_n x_n$ is bounded for all $1 < p < \infty$. Therefore, the Boyd interpolation theorem implies that $R_1$ will be bounded on every r.i. function space $X[0,1]$ such that $0 < \beta_X \leq \alpha_X < 1$. Now, we will define a bounded operator $R_2$ on $L^p[0,1]$ as follows: if $P(f) = \sum_{n=1}^\infty a_n x_n$, then $R_2(f) = \sum_{n=1}^\infty a_n h_n$. Again, the Boyd interpolation theorem implies that $R_2$ will be bounded on every r.i. function space $X[0,1]$ such that $0 < \beta_X \leq \alpha_X < 1$, and this clearly yields the equivalence of the sequences $(x_n)$ and $(h_n)$ in $X[0,1]$. \hfill \Box

Scholium 3.11. Let $(z_i)$ be a sequence of measurable functions on $[0,1]$ such that $z_1$ is $\{ 0,1 \}$-valued non-zero in $L^1$ and $z_i$ is $\{ 0,-1,1 \}$-valued with $\int z_i = 0$ for all
i > 1. Suppose that for all positive l, letting k be the unique integer, 1 \leq k \leq l, and \alpha the unique choice of 1 or -1 so that supp h_{l+1} = \{ t; h_k(t) = \alpha \}, then

\[
supp z_{l+1} \subset \{ t; z_k(t) = \alpha \}
\]

and \( \mu(\{ t; z_k(t) = \alpha \} \sim supp z_{l+1}) \leq \epsilon_l \int |z_1|, \) where \( \epsilon_l = \frac{1}{2^{2l}} \) and “\( \sim \)” denotes the set difference. Then \( (z_n) \) is equivalent to \((h_n)\), and \([z_n]\) is complemented in \(X\).

**Proof.** The proof is depending on [BRSS1] Scholium (1.11) and the Boyd interpolation theorem with the same procedure as in the proof of Lemma 3.10. \(\square\)

Now we will use the same procedure of [BRSS1] Theorem 1.1 in order to prove Theorem 3.1.

We assume that \(X(\ell^2)\) is a complemented subspace of \(Y\); let \(U : Y \to X(\ell^2)\) be a projection. Let \((Y_i)_i\) be an unconditional decomposition of \(Y\). Suppose that (1) fails, that is, there is no \(i\) with \([0,1]\) isomorphic to a complemented subspace of \(Y_i\). We shall then construct a blocking of the decomposition \((Y_i)\) with corresponding projections \((Q_i)\), finite disjoint subsets \(M_1, M_2, \ldots \) of \(\mathbb{N}\), and a map \(j : \bigcup_{i=1}^n M_i \to \mathbb{N}\) so that:

(i) \((Q_k h_{ij(i)})_{i \in M_k, k \in \mathbb{N}}\) is equivalent to \((h_i)_{i \in M_k, k \in \mathbb{N}}\) with \([Q_k h_{ij(i)}]_{i \in M_k, k \in \mathbb{N}}\) complemented in \(Y\).

(ii) \((z_k)\) is equivalent to the Haar basis and \([z_k]\) is complemented in \([0,1]\), where \(z_k = \sum_{i \in M_k} h_i\) for all \(k\).

We simply let \(b_k = \sum_{i \in M_k} Q_k h_{ij(i)}\) for all \(k\); then \((b_k)\) is the desired block basic sequence equivalent to the Haar basis with \([b_k]\) complemented.

Let \(P_i\) be the natural projection from \(Y\) onto \(Y_i\). More generally, for \(F\) a subset of \(\mathbb{N}\) we let \(P_F = \sum_{i \in F} P_i\). Also, we let \(R_n = I - \sum_{i=1}^n P_i\) \((= P_{(n,\infty)})\). We first draw a consequence from our assumption that no \(Y_i\) contains a complemented isomorphic copy of \([0,1]\).

**Lemma 3.12.** For each \(n\), let

\[
(3.1) \quad \mathcal{I} = \{ i \in \mathbb{N}; h_i^* UR_n h_{ij} > \frac{1}{2} \text{ for infinitely many integers } j \}.
\]

Let \(E_I = \{ t \in [0,1]; t \text{ belongs to the support of } h_i \text{ for infinitely many integers } i \in I \}. \) Then \(\mu(E_I) = 1\) \((\text{where } \mu \text{ is the Lebesgue measure})\).

Indeed, let \(L = \{ i \in \mathbb{N}; h_i^* UP_{[1,n]} h_{ij} \geq \frac{1}{2} \text{ for infinitely many integers } j \}; \) then \(I \cup L = \mathbb{N}\).

Hence \(E_I \cup E_L = [0,1]\). So if \(\mu(E_I) < 1, \mu(E_L) > 0\). But then \(T = UP_{[1,n]}\) satisfies the hypotheses of Scholium 3.7. Hence there is a subspace \(Z\) of \(X(\ell^2)\), with \(Z\) isomorphic to \([0,1]\) and \(TZ\) complemented in \(X(\ell^2)\). It follows easily that \(P_{[1,n]}|Z\) is an isomorphism with \(P_{[1,n]}|Z\) complemented; that is, \([0,1]\) embeds as a complemented subspace of \(Y_1 \oplus \cdots \oplus Y_n\). Hence by Theorem 3.9 \([0,1]\) embeds as a complemented subspace of \(Y_i\) for some \(i\), a contradiction.

**Lemma 3.13.** Let \(I \subset \mathbb{N}\), \(E_I\) be as in Lemma 3.12 with \(\mu(E_I) = 1\), and \(S \subset [0,1]\) with \(S\) a finite union of disjoint left-closed dyadic intervals. Then there exists a \(J \subset I\) so that \(\text{supp } h_i \cap \text{supp } h_l = \emptyset\), for all \(i \neq l\), \(i, l \in J\), with \(S \supset \bigcup_{j \in J} \text{supp } h_j\) and \(S \sim \bigcup_{j \in J} \text{supp } h_j \) of measure zero.

**Proof.** It suffices to prove the result for \(S\) is equal to the left-closed dyadic interval. Now any two Haar functions either have disjoint supports, or the support of one is
contained in that of the other. Moreover, for all but finitely many \( i \in I \), \( \text{supp } h_i \subset S \) or \( \text{supp } h_i \cap S = \emptyset \). Hence \( S \) differs from \( \bigcup \{ \text{supp } h_j : \text{supp } h_i \subset S, j \in I \} \) by a measure-zero set. Now simply let \( J = \{ j \in I : \text{supp } h_j \subset S \} \) and there is no \( l \in I \) with \( \text{supp } h_j \subset \text{supp } h_l \subset S \). \( \square \)

We now choose \( M_1, M_2, \ldots \) disjoint finite subsets of \( \mathbb{N} \), a map \( j : \bigcup_{i=1}^{\infty} M_i \mapsto \mathbb{N} \), and \( 1 = m_0 < m_1 < m_2 < \cdots \) with the following properties:

A. For each \( k \), the \( h_i \)'s for \( i \in M_k \) are disjointly supported. Set \( z_k = \sum_{i \in M_k} h_i \).

B. Let \( Q_k = P_{[m_k-1,m_k]} \) for all \( k \). Then \( (UQ_k h_{ij(i)})_{i \in M_k, k \in \mathbb{N}} \) is essentially disjoint and \( h_{ij(i)}^* UQ_k h_{ij(i)} > \frac{1}{2} \) for all \( i \in M_k, k \in \mathbb{N} \).

Having accomplished this, we set \( b_k = \sum_{i \in M_k} Q_k h_{ij(i)} \) for all \( k \). Then by B, \( (b_k) \) is a block basic sequence of the \( Y_i \)'s. By Scholium 3.4

\[ (Q_k h_{ij(i)})_{i \in M_k, k \in \mathbb{N}} \approx (h_{ij(i)})_{i \in M_k, k \in \mathbb{N}} \approx (h_i)_{i \in M_k, k \in \mathbb{N}} \]

where “\( \approx \)” denotes the equivalence of basic sequences; the last equivalence follows from Scholium 3.3 i.e., the unconditionality of the Haar basis. Hence by the definitions of \( (b_k) \) and \( (z_k), (b_k) \) is equivalent to \( (z_k) \) which is equivalent to \( (h_k) \), the Haar basis, by Scholium 3.11. Also, since \( [z_k] \) is complemented in \( X[0,1] \) by Scholium 3.11 \( [b_k] \) is complemented in \( (Q_k h_{ij(i)})_{i \in M_k, k \in \mathbb{N}} \) by 3.2. Again by Scholium 3.4 \( [Q_k h_{ij(i)}]_{i \in M_k, k \in \mathbb{N}} \) is complemented in \( Y \), hence also \( [b_k] \) is complemented in \( Y \).

It remains now to choose the \( M_i \)'s, \( m_i \)'s and the map \( j \). To ensure B, we shall also choose a sequence \( (f_i)_{i \in M_k, k \in \mathbb{N}} \) of disjointly supported elements of \( X(\ell^2) \) (disjointly supported with respect to the basis \( (h_{ij}) \)) so that

\[ \sum_{i \in M_k} \| UQ_k h_{ij(i)} - f_i \| \| UQ_k h_{ij(i)} \| < \frac{1}{2^k}, \text{ for all } k. \]

To start, we let \( M_1 = \{ 1 \} \) and \( j(1) = 1 \). Thus \( z_1 = 1 \); we also set \( f_1 = h_{11} \). Then \( h_{11} = U h_{11} = \lim_{n \to \infty} U P_{[1,n]} h_{11} \). So it is obvious that we can choose \( m_1 > 1 \) such that \( \| U P_{[1,m_1]} h_{11} - h_{11} \| < \frac{1}{2} \); hence \( h_{11}^* U P_{[1,m_1]} h_{11} > \frac{1}{2} \). Thus, the first step is essentially trivial.

Now suppose \( l \geq 1, M_1, \ldots, M_l, m_1 < \cdots < m_l, j : \bigcup_{i=1}^{l} M_i \mapsto \mathbb{N} \) and \( (f_i)_{i \in M_k, k \leq l} \) have been chosen. We set \( z_i = \sum_{j \in M_i} h_j \) for all \( i, 1 \leq i \leq l \).

Let \( 1 \leq k \leq l \) be the unique integer and \( \alpha \) the unique choice of \( \pm 1 \) so that \( \text{supp } h_{l+1} = [h_k = \alpha] \). Let \( S = [z_k = \alpha] \). Set \( n = m_l \) and let \( I \) be as in Lemma 3.12. Since \( S \) is a finite union of disjoint left-closed dyadic intervals, by Lemma 3.13 we may choose a finite set \( M_{l+1} \subset I \), disjoint from \( \bigcup_{i=1}^{l} M_i \), so that the \( h_i \)'s for \( i \in M_{l+1} \) are disjointly supported with \( \text{supp } h_i \subset S \) for \( i \in M_{l+1} \)

\[ \mu(S \sim \bigcup_{i \in M_{l+1}} \text{supp } h_i) \leq \epsilon_l \]

(where \( \epsilon_j = \frac{1}{2^j} \) for all \( j \)). At this point, we have that \( z_{l+1} = \sum_{i \in M_{l+1}} h_i \) satisfies the conditions of Scholium 3.11.

By the definition of \( I \), for each \( i \in M_{l+1} \) there is an infinite set \( J_i \) with

\[ h_{ij}^* U R_n h_{ij} > \frac{1}{2}, \text{ for all } j \in J_i. \]

Now \( (UR_n h_{ij})_{j=1}^{\infty} \) is a weakly null sequence; hence it follows that we may choose \( j : M_{l+1} \mapsto \mathbb{N} \) and disjointly finitely supported elements \( (f_i)_{i \in M_{l+1}} \), with supports
(relative to the $h_{ij}$'s) disjoint from those of \{fi : i ∈ ∪i=1M∈\}, so that
\[
\sum_{i ∈ M_{l+1}} \frac{\|UR_{n}h_{ij(i)} - f_{i}\|}{\|UR_{n}h_{ij(i)}\|} < \frac{1}{2^{l+1}}.
\]

At last, since $R_{n}g = \lim_{k→∞}P_{[m_{l},k]}g$ for any $g ∈ X(ℓ^{2})$, we may choose an $m_{l+1} > m_{l}$ so that setting $Q_{l+1} = P_{[m_{l},m_{l+1}]}$, (3.3) holds for $k = l + 1$ and also
\[
h_{ij}UQ_{k}h_{ij(i)} > \frac{1}{2}, \text{ for all } i ∈ M_{k}.
\]

This completes the construction of the $M_{i}$'s, $m_{i}$'s and map $j$. Since (3.2) holds, A and B hold. Thus (2) of Theorem 3.1 holds; thus the proof is complete.

4. MAIN RESULTS

In this section we will extend [Bou81 Theorem (4.30)] to the reflexive Orlicz function spaces $L^{Φ}[0,1]$. We will use in particular the Boyd interpolation theorem and characterizations of Hilbert spaces among r.i. spaces as subspaces of modular spaces (or Kalton’s result [Kal93]; see Theorem 4.6).

Again we let $C = ∪_{n=1}^{∞} N^{n}$. The space $X(G)$ is an r.i. function space defined on the separable measure space consisting of the Cantor group $G = \{-1, 1\}^{C}$ equipped with the Haar measure. The Walsh functions $w_{p}$ where $F$ is a finite subset of $C$ generate the $L^{p}(G)$ spaces for all $1 ≤ p < ∞$. Then they also generate the r.i. function space $X(G)$.

We consider an r.i. function space $X[0,1]$ such that the Boyd indices satisfy $0 < β_{X} ≤ α_{X} < 1$. The subspace $X_{C}$ is the closed linear span in the r.i. function space $X(G)$ over all finite branches $Γ$ in $C$ of the functions which depend only on the $Γ$-coordinates. Thus $X_{C}$ is a subspace of $X(G)$ generated by Walsh functions \{w_{p} = \prod_{c ∈ Γ} r_{c}; \ Γ$ is a finite branch of $C\}.

Proposition 4.1. Let $X[0,1]$ be an r.i. function space such that the Boyd indices satisfy $0 < β_{X} ≤ α_{X} < 1$. Then $X_{C}$ is a complemented subspace in $X(G)$.

Proof. The authors in [Bou81] and [BR81] express the orthogonal projection $P$ on $X_{C}^{p}$ which is bounded in $L^{p}$-norm for all $1 < p < ∞$, by taking $β_{θ} = \text{trivial algebra}$ and $β_{c} = \mathcal{S}(d ∈ C; d ≤ c)$ for each $c ∈ C$. For $c ∈ C$ and $|c| = 1$, let $c' = ∅$ and for $c ∈ C$ such that $|c| > 1$, let $c'$ be the predecessor of $c$ in $C$.

The orthogonal projection $P$ is given by
\[
P(f) = E[f|β_{θ}] + \sum_{c ∈ C}(E[f|β_{c}] - E[f|β_{c'}])
\]
for every $f ∈ L^{p}(G), 1 < p < ∞$.

Let the Boyd indices of $X[0,1]$ satisfy $0 < β_{X} ≤ α_{X} < 1$. Then the Boyd interpolation theorem implies that the map $P$ is a bounded projection on $X_{C}$ for all r.i. function spaces $X(G)$. Therefore, $X_{C}$ is a complemented subspace of $X(G)$.

Since the elements of any finite subset of the infinite branch $Γ_{∞}$ are mutually comparable, then it is a branch. Thus, the subspace $X_{Γ_{∞}}$ is isometrically isomorphic to $X(\{-1, 1\}^{Γ_{∞}})$ and a one-complemented subspace of $X(G)$ by the conditional expectation operator.
A tree $T$ on $\mathbb{N}$ is seen as a subset of $\mathcal{C}$. We define the subspace $X_T$ of $X(G)$ as a closed linear span in $X(G)$ over all finite branches $\Gamma$ in $T$ of all those functions in $X(G)$ which depend only on the coordinates of $\Gamma$.

By using the conditional expectation with respect to the sub-$\sigma$-algebra generated by a tree $T$ of $\mathcal{C}$, one can find that $X_T$ is a one-complemented subspace of $X_C$ and so it is a complemented subspace of $X(G)$. Therefore, the next result is true.

**Theorem 4.2.** Let $X[0,1]$ be an r.i. function space such that the Boyd indices satisfy $0 < \beta_X \leq \alpha_X < 1$, and $T$ be a tree on $\mathbb{N}$. Then $X_T$ is a complemented subspace of $X(G)$.

The next proposition is a direct consequence of Corollary 2.3.

**Proposition 4.3.** Let $X[0,1]$ be an r.i. function space such that $X[0,1]$ is $q$-concave for some $q < \infty$, the index $\alpha_X < 1$ and the Haar system in $X[0,1]$ is not equivalent to a sequence of disjoint function in $X[0,1]$. Then $X_C$ is isomorphic to $X(G)$.

**Theorem 4.4.** Let $L^\Phi[0,1]$ be a reflexive Orlicz function space which is not isomorphic to $L^2[0,1]$. Then $L^\Phi[0,1]$ does not embed in $X_T^\Phi$ if and only if $T$ is a well-founded tree.

*Proof.* If $T$ contains an infinite branch, then obviously $L^\Phi[0,1]$ embeds in $X_T^\Phi$.

We want to show that if $T$ is well-founded, then $L^\Phi[0,1]$ does not embed in $X_T^\Phi$. We will use Theorem 2.2 and Theorem 3.1.

We proceed by induction on $\circ[T]$. Assume the conclusion fails. Let $T$ be a well-founded tree such that $\circ[T] = \alpha$ and $L^\Phi[0,1]$ embeds in $X_T^\Phi$, where $\alpha = \min\{\circ[T]; L^\Phi[0,1]$ embeds in $X_T^\Phi\}$. We write $T = \bigcup_n (n, T_n)$, with $\circ[T] = \sup_n(\circ[T_n] + 1)$.

The space $X_T^\Phi$ is generated by the sequence of probabilistically mutually independent spaces $B_n = X_{(n,T_n)}^\Phi$. In particular, $\bigoplus_n B_n$ is an unconditional decomposition of $X_T^\Phi$ by the inequality in [BG70 Corollary(5.4)], that is: let $x_1, x_2, \cdots$ be an independent sequence of random variables, each with expectation zero; then for every $n \geq 1$

\begin{equation}
\frac{1}{2} \int_{\Omega} \Phi \left( \sum_{k=1}^{n} x_k^2 \right)^{1/2} \leq \int_{\Omega} \Phi \left( \sum_{k=1}^{n} x_k \right) \leq C \int_{\Omega} \Phi \left( \sum_{k=1}^{n} x_k^2 \right)^{1/2},
\end{equation}

By Theorem 2.2, the space $L^\Phi[0,1]$ embeds complementably in $X_T^\Phi$. Application of Theorem 3.1 implies that $A$ or $B$ below is true:

**A.** There is some $n$ such that $L^\Phi[0,1]$ is isomorphic to a complemented subspace of $B_n$.

**B.** There is a block basic sequence $(b_r)$ of the $B_n$’s which is equivalent to the Haar system of $L^\Phi[0,1]$.

**Assume (A):** It is easily seen that $B_n$ is isomorphic to $X_{T_n}^\Phi \oplus X_{T_n}^\Phi$. So by another application of Theorem 3.1, $L^\Phi[0,1]$ should embed complementally in $X_{T_n}^\Phi$. This however is impossible by induction hypothesis since $\circ[T_n] < \circ[T]$.

**Assume (B):** A block basic sequence of the $B_n$’s is a sequence of probabilistically independent functions which is equivalent to the Haar system of $L^\Phi[0,1]$. By Proposition 2.1, we have that $L^\Phi[0,1]$ is isomorphic to a modular sequence space $\ell(\varphi_n)$. This is impossible by [LT79 Theorem 2.14] and its remark on page 165. This contradiction concludes the proof. □
Remark. An alternative argument in “Assume (B)” works also with property (M) of Banach spaces (see e.g. [Kal93] and [KW95]). Here we recall the definition of the property (M) and some required results.

A Banach space $X$ has property (M) if whenever $u, v \in X$ with $\|u\| = \|v\|$ and $(x_n)$ is weakly null sequence in $X$, then

$$\limsup_{n \to \infty} \|u + x_n\| = \limsup_{n \to \infty} \|v + x_n\|.$$ 

Proposition 4.5. [Kal93, Proposition 4.1] A modular sequence space $X = \ell(\Phi_n)$ can be equivalently normed to have property (M).

Theorem 4.6. [Kal93, Theorem 4.3] Let $X$ be a separable order-continuous non-atomic Banach lattice. If $X$ has an equivalent norm with property (M), then $X$ is lattice-isomorphic to $L^2$.

Let $P$ be a Polish space, and $\mathcal{O}$ be a basis of open subsets of $P$. We denote by $\mathcal{F}(P)$ the set of all closed subsets of $P$ equipped with the Effros-Borel structure (i.e. the canonical Borel structure generated by the family $\{F \in \mathcal{F}(P) : F \cap U \neq \emptyset\}$ for $U \in \mathcal{O}$ (see [Kec95]).

Let $\mathcal{T}$ be the set of all trees on $\mathbb{N}$ which is a closed subset of the Cantor space $\Delta = 2^{\omega < \omega}$. In addition, we denote $\mathcal{SE}$ the set of all closed subspaces of $C(\Delta)$ equipped with the standard Effros-Borel structure. For more about the application of descriptive set theory in the geometry of Banach spaces see, e.g., [Bos02], or [AGR03].

Lemma 4.7. Suppose $\psi : \mathcal{T} \mapsto \mathcal{SE}$ is a Borel map, such that if $T$ is a well-founded tree and $S$ is a tree and $\psi(T) \cong \psi(S)$, then the tree $S$ is well-founded. Then there are uncountably many mutually non-isomorphic members in the class $\{\psi(T) ; T \text{ is a well-founded tree}\}$.

Proof. Assume by contradiction that the number of the non-isomorphic members in the class

$$\{\psi(T) ; T \text{ is a well-founded tree}\}$$ 

is countable, then there exists a countable sequence of well-founded trees $(T_i)_{i=1}^{\infty}$ such that for any well-founded tree $T$ there exists $i$ such that $\psi(T)$ is isomorphic to $\psi(T_i)$.

Consider $B_i = \{X \in \mathcal{SE} ; X \cong \psi(T_i)\}$; then $B_i$ is an analytic subset of $\mathcal{SE}$ for all $i \geq 1$ because of the analyticity of the isomorphism relation. Moreover, since $\psi$ is a Borel map, then $A_i = \{T \in \mathcal{T} ; \psi(T) \cong \psi(T_i)\}$ is an analytic subset of $\mathcal{T}$ for all $i \geq 1$.

From our hypothesis we get that $\{T ; T \text{ is a well-founded tree}\} = \bigcup_{i \geq 1} A_i$ is analytic which is a contradiction. \qed

Lemma 4.8. The map $\psi : \mathcal{T} \mapsto \mathcal{SE}$ defined by $\psi(T) = X_T^{\Phi}$ is Borel.

Proof. Let $U$ be an open set of $C(\Delta)$ and $(\Gamma_i)_{i=1}^{\infty}$ be a sequence of all the finite branches of $C$. It is sufficient to prove that $\mathcal{B} = \{T \in \mathcal{T} ; \psi(T) \cap U \neq \emptyset\}$ is Borel. It is clear that $\psi(T) \cap U \neq \emptyset$ if and only if there exists $\Delta = (\lambda_i)_{i=1}^{n} \in \mathbb{Q}^{<\mathbb{N}}$ such that $\sum_{i=0}^{n} \lambda_i w_{\Gamma_i} \in U$ and $\lambda_i = 0$ when $\Gamma_i \not\in T$. 

Let $\Lambda = \{ \lambda \in \mathbb{Q}^{<\mathbb{N}}; \sum \lambda_i w_{T_i} \in U \}$ and for $\Lambda \in \mathbb{Q}^{<\mathbb{N}}$ set $\text{supp}(\Lambda) = \{ i \in \mathbb{N}; \lambda_i \neq 0 \}$. Then

$$B = \bigcup_{\Lambda \in \Lambda} \bigcap_{i \in \text{supp}(\Lambda)} \{ T \in \mathcal{T}; \Gamma_i \subset T \}.$$  

(4.3)

Since $\{ T \in \mathcal{T}; \Gamma_i \subset T \} = \bigcap_{c \in \mathcal{C}} \{ T \in \mathcal{T}; c \in T \}$ is Borel (because for $c \in \mathcal{C}$ the set $\{ T \in \mathcal{T}; c \in T \}$ is clopen subset in $\mathcal{T}$), then $B$ is Borel.

We recall that in [JMST79, p. 235], it is shown that the Orlicz function $\Phi(t) = t^2 \exp(f_0(\log(t)))$, where $f_0(u) = \sum_{k=1}^{\infty}(1 - \cos \frac{\pi u}{2k})$ is such that the associated Orlicz function spaces $L^\Phi[0, 1]$ and $L^\Phi(0, \infty)$ are isomorphic. Moreover, this space is $(2-\epsilon)$-convex and $(2+\epsilon)$-concave for all $\epsilon > 0$. Hence, the Boyd indices satisfy $\alpha_\Phi = \beta_\Phi = \frac{1}{2}$. In addition, the space $L^\Phi[0, 1]$ is not isomorphic to $L^2[0, 1]$. Also, Orlicz function spaces are constructed in [HP86] and [HR89] which do not contain any complemented copy of $\ell^p$ for $p \geq 1$. Thus, the next corollary is not a straightforward consequence of [BRSS1].

**Corollary 4.9.** Let $L^\Phi[0, 1]$ be a reflexive Orlicz function space which is not isomorphic to $L^2[0, 1]$. Then there exists an uncountable family of mutually non-isomorphic complemented subspaces of $L^\Phi[0, 1]$.

**Proof.** Let $\psi$ be the Borel map defined by $\psi(T) = X^\Phi_T$. Now, let $T$ be a well-founded tree and $T_0$ be a tree such that $X^\Phi_{T_0}$ is isomorphic to a subspace of $X^\Phi_T$. Then Theorem 1.3 implies that $T_0$ is well-founded. By Theorem 4.2, the spaces $X^\Phi_T$ are complemented in $L^\Phi(G)$. Hence, there exists an uncountable family of mutually non-isomorphic complemented subspaces of $L^\Phi[0, 1]$ by Lemma 4.7.

It was mentioned before that the set of all well-founded trees is co-analytic non-Borel and so the set of all trees which are not well-founded (ill-founded) is analytic non-Borel. Following [Bos02], if $X$ is a separable Banach space, then $\langle X \rangle$ denotes the equivalence class $\{ Y \in \mathcal{SE}; Y \sim X \}$. We have the following result.

**Corollary 4.10.** Let $L^\Phi[0, 1]$ be a reflexive Orlicz function space which is not isomorphic to $L^2[0, 1]$. Then $\langle L^\Phi[0, 1] \rangle$ is analytic non-Borel.

**Proof.** Since the isomorphism relation $\{ (X, Y); X \sim Y \}$ is analytic in $\mathcal{SE}^2$ by [Bos02, Theorem 2.3], then the class $\langle L^\Phi[0, 1] \rangle$ is analytic. Moreover, since $\psi$ is Borel and $\psi^{-1}(\langle L^\Phi[0, 1] \rangle) = \{ T; T \text{ is ill-founded} \}$ by Theorem 4.4, then the class $\langle L^\Phi[0, 1] \rangle$ is non-Borel.

In [Bos02], it has been shown that $\langle \ell^2 \rangle$ is Borel. It is unknown whether this condition characterizes the Hilbert space, and thus we recall [Bos02, Problem 2.9]: Let $X$ be a separable Banach space whose isomorphism class $\langle X \rangle$ is Borel. Is $X$ isomorphic to $\ell^2$? A special case of this problem seems to be of particular importance, namely: Is the isomorphism class $\langle c_0 \rangle$ of $c_0$ Borel? For more about this question and analytic sets of Banach spaces see [God10].

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References


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