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# ASYMPTOTIC BEHAVIOUR OF JACOBI POLYNOMIALS AND THEIR ZEROS

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ABSTRACT. We obtain the explicit form of the expansion of the Jacobi polynomial $P_n^{(\alpha,\beta)}(1-2x/\beta)$ in terms of the negative powers of $\beta$. It is known that the constant term in the expansion coincides with the Laguerre polynomial $L_n^{(\alpha)}(x)$. Therefore, the result in the present paper provides the higher terms of the asymptotic expansion as $\beta \to \infty$. The corresponding asymptotic relation between the zeros of Jacobi and Laguerre polynomials is also derived.

## 1. Introduction and statement of the main result

Denote by $P_n^{(\alpha,\beta)}(x)$, $C_n^{(\lambda)}(x)$, $L_n^{(\alpha)}(x)$, $H_n(x)$ the classical Jacobi, Gegenbauer (ultraspherical), Laguerre and Hermite orthogonal polynomials. We adopt the standard normalizations as in [6, 7]. For a fixed $n \in \mathbb{N}$, denote by $x_{nk}(\alpha,\beta)$, $u_{nk}(\lambda)$, $\ell_{n,k}(\alpha)$ and $h_{nk}$, $k = 1,\ldots,n$, the corresponding zeros of the above polynomials, all arranged in increasing order with respect to $k$.

The following limit relations are well known (see [6, p. 57]):

$$\lim_{\beta\to\infty} P_n^{(\alpha,\beta)}(1-\frac{2x}{\beta}) = L_n^{(\alpha)}(x) \tag{1}$$

and

$$\lim_{\lambda\to\infty} \lambda^{-n/2} C_n^{(\lambda)}(\frac{x}{\sqrt{\lambda}}) = \frac{H_n(x)}{n!}. \tag{2}$$

These imply the following asymptotic formulae for the corresponding zeros of the classical orthogonal polynomials:

$$\lim_{\beta\to\infty} \frac{\beta(1-x_{n,k}(\alpha,\beta))}{2} = \ell_{n,n+1-k}(\alpha), \quad k=1,\ldots,n, \tag{3}$$

and

$$\lim_{\lambda\to\infty} \sqrt{\lambda}\, u_{nk}(\lambda) = h_{nk}, \quad k=1,\ldots,n. \tag{4}$$

It is clear that (2) and (4) are rather straightforward consequences of (1) and (3). On the other hand, Elbert and Laforgia [4] obtained nice extensions of (2) and (4),

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which read as follows:

$$\lambda^{-n/2}C_n^{(\lambda)}(\frac{x}{\sqrt{\lambda}}) = \sum_{j=0}^{n-1} \lambda^{-j} Q_{nj}(x),$$

where $Q_{n0}(x) = H_n(x)/n!$ and $Q_{nj}(x)$, $j = 1, \ldots, n-1$, are defined in [4, (1.2)], and

$$\begin{aligned} \sqrt{\lambda}\, u_{n,k}^{(\lambda)} &= h_{n,k} - h_{n,k}\left(\frac{2n-1+2h_{n,k}^2}{8}\right)\lambda^{-1} \\ &+h_{n,k}\left(\frac{12n^2-12n+1}{128} + \frac{5n-2}{24}h_{n,k}^2 + \frac{5}{96}h_{n,k}^4\right)\lambda^{-2} + O(\lambda^{-3}). \end{aligned} \tag{5}$$

Our aim is to obtain analogues of these results concerning the limit relations (1) and (3). In order to formulate our main result, recall first that the Pochhammer symbol $(a)_n$ is defined by $(a)_0 := 1$ and $(a)_n = a(a+1)\cdots(a+n-1)$ when $n \geq 1$. The elementary symmetric functions $\sigma_j(a_1, \ldots, a_k)$ of $a_1, \ldots, a_k$ are given by

$$\sigma_j(a_1, \ldots, a_k) = \sum_{1 \leq i_1 < \cdots < i_j \leq k} a_{i_1} \cdots a_{i_j}$$

with the conventions $\sigma_0(a_1, \ldots, a_k) := 1$ and $\sigma_j(a_1, \ldots, a_k) := 0$ when $j > k$. In what follows we denote by $\sigma_j^{(k)}(n, \alpha, \mu)$ the elementary symmetric functions of the specific variables $n+\alpha+\mu+1, \ldots, n+\alpha+\mu+k$, that is,

$$\sigma_j^{(k)}(n, \alpha, \mu) := \sigma_j(n+\alpha+\mu+1, \ldots, n+\alpha+\mu+k).$$

Our main result reads as follows:

**Theorem 1.** *The Jacobi polynomial $P_n^{(\alpha,\beta+\mu)}(x)$ can be represented in the form*

$$P_n^{(\alpha,\beta+\mu)}(1 - \frac{2x}{\beta}) = \sum_{j=0}^{n} R_{nj}(x)\ \beta^{-j}, \tag{6}$$

*where*

$$R_{nj}(x) = (-1)^j \frac{(j+\alpha+1)_{n-j}}{(n-j)!}\ x^j \sum_{k=0}^{n-j} \frac{(j-n)_k}{(j+\alpha+1)_k} \sigma_j^{(j+k)}(n, \alpha, \mu) \frac{x^k}{(j+k)!}. \tag{7}$$

*In particular,*

$$R_{n,0}(x) = L_n^{(\alpha)}(x),$$

$$R_{n,1}(x) = (n+\alpha+\mu+1)x\frac{d}{dx}L_n^{(\alpha)}(x) + \frac{x^2}{2}\frac{d^2}{dx^2}L_n^{(\alpha)}(x), \tag{8}$$

$$\begin{aligned} R_{n,2}(x) &= \frac{(n+\alpha+\mu+1)(n+\alpha+\mu+2)}{2}x^2\frac{d^2}{dx^2}L_n^{(\alpha)}(x) \\ &+\frac{3n+3(\alpha+\mu)+5}{6}x^3\frac{d^3}{dx^3}L_n^{(\alpha)}(x) + \frac{x^4}{8}\frac{d^4}{dx^4}L_n^{(\alpha)}(x). \end{aligned} \tag{9}$$

Observe that while the expansion we need is obviously the one for $\mu = 0$, we obtain a little more general result which will allow us to derive the result of Elbert and Laforgia too.

**Theorem 2.** *Let $n \in \mathbb{N}$, $1 \le k \le n$, and $\alpha > -1$. The zeros of Jacobi polynomials possess the following asymptotic behaviour as $\beta \to \infty$:*

$$\frac{\beta(1 - x_{nk}(\alpha, \beta + \mu))}{2} = \ell_{n,n+1-k}(\alpha) + \frac{A}{\beta} + \frac{B}{\beta^2} + O\left(\frac{1}{\beta^3}\right), \tag{10}$$

*where*

$$A = -\frac{\ell_{n,n+1-k}(\alpha)}{2}\left(2(n+\mu) + \alpha + 1 + \ell_{n,n+1-k}(\alpha)\right)$$

*and*

$$\begin{aligned} B = {} & \frac{\ell_{n,n+1-k}(\alpha)}{24} \\ & \times \{5 + 7\alpha^2 + 24\mu(1+\mu) + 2\alpha(1+2\mu) + 24n(1+\alpha+n+2\mu) \\ & \quad + [13(1+\alpha+2n) + 24\mu]\ell_{n,n+1-k}(\alpha) + 4\ell^2_{n,n+1-k}(\alpha)\}. \end{aligned}$$

A similar procedure as in [4] yields the following:

**Corollary 1.** *The derivatives of the zeros of Jacobi polynomials with respect to the parameter $\beta$ obey the following limit relations:*

$$\lim_{\beta\to\infty} \beta^2 \frac{\partial}{\partial\beta} x_{n,k}(\alpha,\beta) = 2\,\ell_{n,n+1-k}(\alpha), \tag{11}$$

$$\lim_{\beta\to\infty} \beta^3 \frac{\partial^2}{\partial\beta^2} x_{n,k}(\alpha,\beta) = -4\,\ell_{n,n+1-k}(\alpha). \tag{12}$$

While the fact that all zeros of Jacobi polynomials are increasing functions of the parameter $\beta$ is well known, (11) reveals the exact behaviour of the derivative with respect to $\beta$ at infinity. The expression (12) is more interesting because it shows that $x_{n,k}(\alpha, \beta)$ are concave for large $\beta$. Though it is risky to conjecture that the zeros would be concave for the entire range of $\beta \in (-1, \infty)$ (see [1,5]), it would be of interest to study the problem about their convexity/concavity properties. We refer to [2] for some partial results and conjectures concerning convexity of the zeros of Gegenbauer and Laguerre polynomials with respect to the corresponding parameters and to [3] for monotonicity properties of $X_{n,k}(\alpha, \beta) = \beta(1 - x_{n,k}(\alpha, \beta))/2$, considered as functions of $\beta$. Since (11) and (12) are equivalent to

$$\lim_{\beta\to\infty} \frac{\partial}{\partial\beta}\left\{\beta\, X_{n,k}(\alpha,\beta)\right\} = \ell_{n,n+1-k}(\alpha),$$

$$\lim_{\beta\to\infty} \frac{\partial^2}{\partial\beta^2}\left\{\beta^2\, X_{n,k}(\alpha,\beta)\right\} = 2\,\ell_{n,n+1-k}(\alpha),$$

for some technical reasons described in [3], one could study the problem if the functions on the left-hand side of the latter relations are monotonic.

## 2. Proofs

Let us recall that the hypergeometric function is defined by

$${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; x) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{x^k}{k!}.$$

*Proof of Theorem* 1. The Jacobi polynomials $P_n^{(\alpha,\beta+\mu)}(x)$ have the following explicit representation in terms of the hypergeometric function [6, (1.8.1)]:

$$P_n^{(\alpha,\beta+\mu)}(x) = \frac{(\alpha+1)_n}{n!}\ {}_2F_1(-n, n+\alpha+\beta+\mu+1; \alpha+1; \frac{1-x}{2}).$$

Replacing $x$ by $1-2x/\beta$, we obtain

$$P_n^{(\alpha,\beta+\mu)}\left(1-\frac{2x}{\beta}\right) = \frac{(\alpha+1)_n}{n!}\sum_{k=0}^{n}\frac{(-n)_k}{(\alpha+1)_k}\frac{(\beta+n+\alpha+\mu+1)_k}{\beta^k}\frac{x^k}{k!}. \tag{13}$$

Let us consider the numerator in the second fraction in the sum as a polynomial of $\beta$ and expand it using Vieta's formulae to obtain

$$P_n^{(\alpha,\beta)}\left(1-\frac{2x}{\beta}\right) = \frac{(\alpha+1)_n}{n!}\sum_{k=0}^{n}\frac{(-n)_k}{(\alpha+1)_k}\sum_{j=0}^{k}\sigma_j^{(k)}(n,\alpha,\mu)\beta^{-j}\frac{x^k}{k!}. \tag{14}$$

Change of order of summation yields

$$P_n^{(\alpha,\beta)}\left(1-\frac{2x}{\beta}\right) = \frac{(\alpha+1)_n}{n!}\sum_{j=0}^{n}\beta^{-j}\sum_{k=j}^{n}\frac{(-n)_k}{(\alpha+1)_k}\sigma_j^{(k)}(n,\alpha,\mu)\frac{x^k}{k!}. \tag{15}$$

Therefore

$$\begin{aligned} R_{nj}(x) &= \frac{(\alpha+1)_n}{n!}\sum_{k=j}^{n}\frac{(-n)_k}{(\alpha+1)_k}\sigma_j^{(k)}(n,\alpha,\mu)\frac{x^k}{k!} \\ &= \frac{(-n)_j(\alpha+1)_n}{n!(\alpha+1)_j}\ x^j\sum_{k=0}^{n-j}\frac{(-n+j)_k}{(\alpha+j+1)_k}\sigma_j^{(j+k)}(n,\alpha,\mu)\frac{x^k}{(j+k)!} \\ &= \frac{(-1)^j}{(n-j)!}(\alpha+j+1)_{n-j}\ x^j\sum_{k=0}^{n-j}\frac{(-n+j)_k}{(\alpha+j+1)_k}\sigma_j^{(j+k)}(n,\alpha,\mu)\frac{x^k}{(j+k)!}. \end{aligned}$$

It remains to obtain the explicit representations of $R_{nj}(x)$ for $j=0,1,2$. Since the Laguerre polynomials possess the hypergeometric representation of the form (see [6, (1.11.1)])

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!}\ {}_1F_1(-n;\alpha+1;x),$$

it is easily seen that

$$R_{n0}(x) = \frac{(\alpha+1)_n}{n!}\sum_{k=0}^{n}\frac{(-n)_k}{(\alpha+1)_k}\frac{x^k}{k!} = \frac{(\alpha+1)_n}{n!}\ {}_1F_1(-n;\alpha+1;x) = L_n^{(\alpha)}(x).$$

Observe that

$$\sigma_1^{(k+1)}(n,\alpha,\mu) = (k+1)\left(n+\alpha+\mu+1+\frac{k}{2}\right).$$

Then

$$\begin{aligned} R_{n1}(x) &= -\frac{(\alpha+2)_{n-1}}{(n-1)!}\,x\sum_{k=0}^{n-1}\frac{(-n+1)_k}{(\alpha+2)_k}\left\{(n+\alpha+\mu+1)+\frac{k}{2}\right\}\frac{x^k}{k!} \\ &= -x\left\{(n+\alpha+\mu+1)L_{n-1}^{(\alpha+1)}(x)+\frac{x}{2}\frac{d}{dx}L_{n-1}^{(\alpha+1)}(x)\right\}. \end{aligned}$$

Now the well-known relation between the Laguerre polynomial and its derivatives

$$\frac{d^k}{dx^k}L_n^{(\alpha)}(x) = (-1)^k L_{n-k}^{(\alpha+k)}(x) \tag{16}$$

immediately yields (8). Similarly,

$$\frac{\sigma_2^{(k+2)}(n,\alpha,\mu)}{(k+1)(k+2)} = \frac{(n+\alpha+\mu+1)(n+\alpha+\mu+2)}{2} + \frac{k(3n+3(\alpha+\mu)+5)}{6} + \frac{k(k-1)}{8}$$

yields

$$\begin{aligned} R_{n2}(x) &= \frac{(\alpha+3)_{n-2}}{(n-2)!}\, x^2 \sum_{k=0}^{n-2} \frac{(-n+2)_k}{(\alpha+3)_k} \sigma_2^{(k+2)} \frac{x^k}{(k+2)!} \\ &= \frac{(n+\alpha+\mu+1)(n+\alpha+\mu+2)}{2}\, x^2\, L_{n-2}^{(\alpha+2)}(x) \\ &\quad + \frac{3n+3(\alpha+\mu)+5}{6}\, x^3\, \frac{d}{dx} L_{n-2}^{(\alpha+2)}(x) \\ &\quad + \frac{x^4}{8}\, \frac{d^2}{dx^2} L_{n-2}^{(\alpha+2)}(x). \end{aligned}$$

Then (16) implies (9). □

We shall need the following technical results in the proof of Theorem 2.

**Lemma 1.** *The Laguerre polynomials satisfy the linear differential equations*

$$x^j \frac{d^{j+1}(L_n^{(\alpha)}(x))}{dx^{j+1}} = a_{j,1}(x)\frac{d}{dx}L_n^{(\alpha)}(x) + a_{j,2}(x) L_n^{(\alpha)}(x), \ \ j = 1, 2, \ldots, n,$$

*where the polynomials $a_{j,1}(x)$ and $a_{j,2}(x)$ are generated by the recurrence relations*

$$\begin{aligned} a_{j,1}(x) &= (x-\alpha-j)\, a_{j-1,1}(x) + (j-1-n)\, x\, a_{j-2,1}(x), \\ a_{j,2}(x) &= (x-\alpha-j)\, a_{j-1,2}(x) + (j-1-n)\, x\, a_{j-2,2}(x), \end{aligned}$$

*for $j = 2, \ldots, n-1$, with initial conditions*

$$\begin{aligned} a_{0,1}(x) &= 1, \qquad a_{1,1}(x) = x-\alpha-1, \\ a_{0,2}(x) &= 0, \qquad a_{1,2}(x) = -n. \end{aligned}$$

*Proof.* The classical second order differential equation for the Laguerre polynomials is (see [7, p. 100])

$$x\frac{d^2}{dx^2}L_n^{(\alpha)}(x) = (x-\alpha-1)\frac{d}{dx}L_n^{(\alpha)}(x) - nL_n^{(\alpha)}(x).$$

Differentiating it we obtain

$$x\frac{d^{j+1}(L_n^{(\alpha)}(x))}{dx^{j+1}} = (x-\alpha-j)\frac{d^j(L_n^{(\alpha)}(x))}{dx^j} + (j-1-n)\frac{d^{j-1}(L_n^{(\alpha)}(x))}{dx^{j-1}}.$$

Multiplying this result by $x^{j-1}$, we derive

$$\begin{aligned}
x^j \frac{d^{j+1}(L_n^{(\alpha)}(x))}{dx^{j+1}} &= (x-\alpha-j)x^{j-1}\frac{d^j(L_n^{(\alpha)}(x))}{dx^j} + (j-1-n)x^{j-1}\frac{d^{j-1}(L_n^{(\alpha)}(x))}{dx^{j-1}} \\
&= (x-\alpha-j)[a_{j-1,1}(x)\frac{d}{dx}L_n^{(\alpha)}(x) + a_{j-1,2}(x)L_n^{(\alpha)}(x)] \\
&\quad + (j-1-n)x[a_{j-2,1}(x)\frac{d}{dx}L_n^{(\alpha)}(x) + a_{j-2,2}(x)L_n^{(\alpha)}(x)] \\
&= [(x-\alpha-j)a_{j-1,1}(x) + (j-1-n)xa_{j-2,1}(x)]\frac{d}{dx}L_n^{(\alpha)}(x) \\
&\quad + [(x-\alpha-j)a_{j-1,2}(x) + (j-1-n)xa_{j-2,2}(x)]L_n^{(\alpha)}(x) \\
&= a_{j,1}(x)\frac{d}{dx}L_n^{(\alpha)}(x) + a_{j,2}(x)L_n^{(\alpha)}(x).
\end{aligned}$$

Therefore, the polynomials $a_{j,1}(x)$ and $a_{j,2}(x)$ obey the announced recurrence relations. □

Straightforward calculations yield

$$\begin{aligned}
a_{2,1}(x) &= x^2 - [2(\alpha+1)+n]x + (\alpha+2)(\alpha+1), \\
a_{2,2}(x) &= -n(x-\alpha-2),
\end{aligned}$$

and

$$\begin{aligned}
a_{3,1}(x) &= x^3 - [3(\alpha+1)+2n]x^2 + [(\alpha+1)(2\alpha+3+n) + (\alpha+3)(\alpha+1+n)]x \\
&\quad -(\alpha+1)(\alpha+2)(\alpha+3), \\
a_{3,2}(x) &= -n[x^2 - (2\alpha+n+3)x + (\alpha+3)(\alpha+2)].
\end{aligned}$$

**Lemma 2.** *The polynomials $R_{n,1}(x)$, $R_{n,2}(x)$ and $R'_{n,1}(x)$ can be represented as*

$$\begin{aligned}
R_{n1}(x) &= f_1(x)\frac{d}{dx}L_n^{(\alpha)}(x) + f_2(x)L_n^{(\alpha)}(x), \\
R_{n2}(x) &= g_1(x)\frac{d}{dx}L_n^{(\alpha)}(x) + g_2(x)L_n^{(\alpha)}(x), \\
R'_{n1}(x) &= h_1(x)L_n^{(\alpha)\prime}(x) + h_2(x)L_n^{(\alpha)}(x),
\end{aligned}$$

*with*

$$\begin{aligned}
f_1(x) &= \frac{x}{2}(\alpha+2n+2\mu+x+1), \quad f_2(x) = -\frac{nx}{2}, \\
g_1(x) &= \frac{x}{24}\{3x^3 + (11+6n+3\alpha+12\mu)x^2 \\
&\quad +(12\mu^2 + 12(n+1)\mu - (1+2n+\alpha)(3\alpha-2))x \\
&\quad -(\alpha+1)(2+7\alpha+12\mu+3(4n^2+(\alpha+2\mu)^2+4n(1+\alpha+2\mu)))\}, \\
g_2(x) &= -\frac{nx}{24}\{-3x^2 - (11+9n+6\alpha+12\mu)x \\
&\quad -(2+7\alpha+12\mu+3(4n^2+(\alpha+2\mu)^2+4n(1+\alpha+2\mu)))\}, \\
h_1(x) &= \frac{1}{2}\{x^2 + (2+n+2\mu)x - \alpha(1+2n+\alpha+2\mu)\}, \\
h_2(x) &= -\frac{n}{2}\{2(1+n+\mu)+\alpha+x\}.
\end{aligned}$$

The explicit forms of $f_1(x), f_2(x), g_1(x)$ and $g_2(x)$ can be derived immediately from Lemma 1 and then, to obtain $h_1(x)$ and $h_2(x)$, it suffices to observe that

$$\begin{aligned} h_1(x) &= f_1'(x) + f_2(x) + \frac{f_1(x)a_{1,1}(x)}{x}, \\ h_2(x) &= \frac{f_1(x)}{x} a_{1,2}(x) + f_2'(x). \end{aligned}$$

*Proof of Theorem* 2. The problem of determining the asymptotic behaviour of the zeros of $\sum_{j=0}^{n} R_{nj}(x)\ \beta^{-j}$, when $\beta \to \infty$, is equivalent to obtaining an expression for the asymptotics of the zeros of

$$\sum_{j=0}^{n} R_{nj}(x)\ \delta^j, \tag{17}$$

when $\delta \to 0$. Since the zeros on the left-hand side of (6) are obviously real and distinct, so are the zeros $\zeta_k(\delta)$ of (17). Then, by the implicit function theorem, each $\zeta_k(\delta)$ is an analytic function of $\delta$ in a neighbourhood of the origin. Therefore, the zero $\zeta_k(\delta)$ has a Maclaurin expansion

$$\zeta_k(\delta) = \sum_{i=0}^{\infty} \zeta_{k,i}\delta^i.$$

The above expansion is equivalent to the fact that $\beta(1 - x_{nk}(\alpha, \beta + \mu))/2$ possesses a Laurent expansion of the form

$$\frac{\beta(1 - x_{nk}(\alpha, \beta + \mu))}{2} = \sum_{i=0}^{\infty} \zeta_{k,i}\beta^{-i}.$$

We determine the coefficients $\zeta_{k,0}$, $\zeta_{k,1}$ and $\zeta_{k,2}$.

Let us set $\bar{l} := \ell_{n,n+1-k}(\alpha)$. Substituting (10) in (6), we obtain

$$\begin{aligned} 0 &= R_{n0}\left(\bar{l} + \frac{A}{\beta} + \frac{B}{\beta^2} + O\left(\frac{1}{\beta^3}\right)\right) + \frac{1}{\beta}R_{n1}\left(\bar{l} + \frac{A}{\beta} + \frac{B}{\beta^2} + O\left(\frac{1}{\beta^3}\right)\right) \\ &\quad + \frac{1}{\beta^2}R_{n2}\left(\bar{l} + \frac{A}{\beta} + \frac{B}{\beta^2} + O\left(\frac{1}{\beta^3}\right)\right) + O\left(\frac{1}{\beta^3}\right) \\ &= \left(\frac{A}{\beta} + \frac{B}{\beta^2}\right) R_{n0}'(\bar{l}) + \frac{1}{2}\left(\frac{A}{\beta} + \frac{B}{\beta^2}\right)^2 R_{n0}''(\bar{l}) \\ &\quad + \frac{1}{\beta}\left[R_{n1}(\bar{l}) + \frac{A}{\beta}R_{n1}'(\bar{l})\right] + \frac{1}{\beta^2}R_{n2}(\bar{l}) + O\left(\frac{1}{\beta^3}\right) \\ &= \frac{1}{\beta}[A\,R_{n0}'(\bar{l}) + R_{n1}(\bar{l})] \\ &\quad + \frac{1}{\beta^2}\left[B\,R_{n0}'(\bar{l}) + \frac{1}{2}A^2\,R_{n0}''(\bar{l}) + A\,R_{n1}'(\bar{l}) + R_{n2}(\bar{l})\right] + O\left(\frac{1}{\beta^3}\right). \end{aligned}$$

A comparison of the coefficients of $1/\beta$ and $1/\beta^2$ on both sides gives

$$A = -\frac{R_{n1}\,(\bar{l})}{R_{n0}'\,(\bar{l})} \quad \text{and} \quad B = -\frac{1}{2}A^2\frac{R_{n0}''\,(\bar{l})}{R_{n0}'\,(\bar{l})} - A\frac{R_{n1}'\,(\bar{l})}{R_{n0}'\,(\bar{l})} - \frac{R_{n2}\,(\bar{l})}{R_{n0}'\,(\bar{l})}.$$

Using Lemma 2, we obtain

$$A = -f_1(\bar{l}) = -\frac{\bar{l}}{2}(\alpha + 2n + 2\mu + \bar{l} + 1)$$

and

$$\begin{aligned} B &= \frac{\bar{l}}{24}\{5+7\alpha^2+24\mu(1+\mu)+2\alpha(1+2\mu)+24n(1+\alpha+n+2\mu) \\ &\quad +[13(1+\alpha+2n)+24\mu]\bar{l}+4\bar{l}^2\}. \end{aligned}$$

This completes the proof of Theorem 2. □

Our final task is to show how Theorem 2 implies (5). According to [6, pp. 41 and 49], when $n=2m$, we have

$$C_{2m}^{(\lambda)}(x)=\frac{(-1)^m 2^{2m} m!(\lambda+1)_m}{(2m)!}P_m^{(-1/2,\lambda-1/2)}(1-2x^2),\ \lambda\neq 0,$$

and

$$H_{2m}(x)=(-1)^m 2^{2m} m! L_m^{(-1/2)}(x^2).$$

We shall need the following identities which involve symmetric functions and Stirling numbers.

**Lemma 3.** *Let $S_n^{(m)}$ be the Stirling numbers of the first kind. Then the identities*

$$S_{2m-j}^{(2m-j-t)}=\sum_{\nu=\max(0,t-m+1)}^{\min(t,m-j)}(-1)^\nu S_m^{(m+\nu-t)}\sigma_\nu^{(m-j)}(2m,-1/2,-1/2)$$

*hold for every $m,j,t\in\mathbb{N}$ with $0\le j\le 2m$ and $t=0,\dots,2m-j$.*

*Proof.* Using the generating function for the Stirling numbers of the first kind, we obtain

$$(x)_{2m-j}=x\dots(x-m+1)(x-m)\dots(x-m-(m-j)+1)=\sum_{t=0}^{2m-j}S_{2m-j}^{(2m-j-t)}x^{2m-j-t}$$

and

$$\begin{aligned} (x)_{2m-j} &= \left\{\sum_{k=1}^{m}S_m^{(k)}x^k\right\}\left\{\sum_{i=0}^{m-j}(-1)^i\sigma_i^{(m-j)}(2m,-1/2,-1/2)x^{m-j-i}\right\} \\ &= \sum_{t=0}^{2m-j}\left\{\sum_{\nu=\max(0,t-m+1)}^{\min(t,m-j)}(-1)^\nu S_m^{(m+\nu-t)}\sigma_\nu^{(m-j)}(2m,-1/2,-1/2)\right\}x^{2m-j-t}. \end{aligned}$$

A comparison implies the desired result.

Now we are ready to complete our alternative proof of (5) for $n = 2m$ as a consequence of Theorem 2 and the latter two facts. Using Theorem 2 with $\alpha = -1/2$, $\beta = \lambda$ and $\mu = -1/2$, and denoting $\kappa_m := (-4)^m m!/(2m)!$, we obtain

$$\begin{aligned}
\lambda^{-m} C_{2m}^{(\lambda)}\left(\frac{x}{\sqrt{\lambda}}\right) &= \kappa_m \lambda^{-m} (\lambda)_m P_m^{(-1/2,\lambda-1/2)}(1-2x^2/\lambda) \\
&= \kappa_m \left\{\sum_{i=1}^{m} (-1)^{i-m} S_m^{(i)} \lambda^{i-m}\right\} \left\{\sum_{j=0}^{m} R_{mj}(x^2)\lambda^{-j}\right\} \\
&= \kappa_m \left\{\sum_{j=0}^{m-1} (-1)^j S_m^{(m-j)} \lambda^{-j}\right\} \left\{\sum_{j=0}^{m} R_{mj}(x^2)\lambda^{-j}\right\} \\
&= \kappa_m \left\{\sum_{j=0}^{m-1} \delta_{m-j} \lambda^{-j}\right\} \left\{\sum_{j=0}^{m} R_{mj}(x^2)\lambda^{-j}\right\} \\
&= \kappa_m \left\{\sum_{t=0}^{2m-1} \left[\sum_{\nu=\max(0,t-m+1)}^{\min(m,t)} \delta_{m-(t-\nu)} R_{m\nu}(x^2)\right] \lambda^{-t}\right\}.
\end{aligned}$$

Thus we have derived a representation of the form

$$\lambda^{-m} C_{2m}^{(\lambda)}\left(\frac{x}{\sqrt{\lambda}}\right) = \sum_{t=0}^{2m-1} \tilde{Q}_{2m,t}(x)\, \lambda^{-t}$$

where

$$\tilde{Q}_{2m,t}(x) = \kappa_m \sum_{\nu=\widehat{m}}^{\overline{m}} \delta_{m+\nu-t} R_{m,\nu}(x^2).$$

Since

$$\begin{aligned}
R_{m\nu}(x^2) &= \frac{(1/2)_m}{m!} \sum_{k=\nu}^{m} \frac{(-m)_k}{(1/2)_k} \sigma_\nu^{(k)}(2m,-1/2,-1/2) \frac{x^{2k}}{k!} \\
&= \frac{(2m)!}{2^{2m} m!} \sum_{k=\nu}^{m} (-1)^k \sigma_\nu^{(k)}(2m,-1/2,-1/2) \frac{(2x)^{2k}}{(m-k)!(2k)!},
\end{aligned}$$

adopting the abbreviations $\widehat{m} = \max(0, t-m+1)$ and $\overline{m} = \min(m,t)$, we derive

$$\begin{aligned}
\tilde{Q}_{2m,t}(x) &= \sum_{\nu=\widehat{m}}^{\overline{m}} (-1)^m \delta_{m-(t-\nu)} \sum_{k=\nu}^{m} (-1)^k \sigma_\nu^{(k)}(2m,-1/2,-1/2) \frac{(2x)^{2k}}{(m-k)!(2k)!} \\
&= \sum_{\nu=\widehat{m}}^{\overline{m}} \sum_{k=\nu}^{m} (-1)^{m+k} \delta_{m-(t-\nu)} \sigma_\nu^{(k)}(2m,-1/2,-1/2) \frac{(2x)^{2k}}{(m-k)!(2k)!} \\
&= \sum_{\nu=\widehat{m}}^{\overline{m}} \sum_{j=0}^{m-\nu} (-1)^{t-j-\nu} S_m^{(m+\nu-t)} \sigma_\nu^{(m-j)}(2m,-1/2,-1/2) \frac{(2x)^{2m-2j}}{(2m-2j)!j!} \\
&= \sum_{j=0}^{m-\widehat{m}} \left( \sum_{\nu=\widehat{m}}^{\min(t,m-j)} (-1)^{t-j-\nu} S_m^{(m+\nu-t)} \sigma_\nu^{(m-j)}(2m,-1/2,-1/2) \right) \frac{(2x)^{2m-2j}}{(2m-2j)!j!} \\
&= \sum_{j=0}^{\min(m,2m-t-1)} \left( \sum_{\nu=\widehat{m}}^{\min(t,m-j)} (-1)^{t-j-\nu} S_m^{(m+\nu-t)} \sigma_\nu^{(m-j)}(2m,-1/2,-1/2) \right) \frac{(2x)^{2m-2j}}{(2m-2j)!j!}.
\end{aligned}$$

Now Lemma 3 yields

$$\tilde{Q}_{2m,t}(x) = \sum_{j=0}^{\min(m,2m-t-1)} (-1)^{t+j} S_m^{(2m-j-t)} \frac{(2x)^{2m-2j}}{(2m-2j)!j!},$$

and this is exactly the explicit form of the polynomials $Q_{2m,t}(x)$ obtained by Elbert and Laforgia in [4]. A similar procedure for $n = 2m+1$, using the identities

$$\begin{aligned}
C_{2m+1}^{(\lambda)}(x) &= \frac{\Gamma(\alpha+m+1)m!2^{2m+1}}{\Gamma(\alpha)(2m+1)!}\, x\, P_m^{(1/2,\lambda-1/2)}(2x^2-1), \\
H_{2m+1}(x) &= (-1)^m 2^{2m+1} m!\, x\, L_m^{(1/2)}(x^2),
\end{aligned}$$

shows that Theorem 2 yields the explicit form of the polynomials $Q_{2m+1,t}(x)$, obtained in [4] too.

The statement of the corollary is an immediate consequence of Theorem 2. Indeed, multiplying (10) by $\beta$ (by $\beta^2$, respectively) and differentiating once (twice), we obtain (11) and (12).

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