

## COISOTROPIC SUBALGEBRAS OF COMPLEX SEMISIMPLE LIE BIALGEBRAS

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ABSTRACT. In his paper “A Construction for Coisotropic Subalgebras of Lie Bialgebras”, Marco Zambon gave a way to use a long root of a complex semisimple Lie bialgebra  $\mathfrak{g}$  to construct a coisotropic subalgebra of  $\mathfrak{g}$ . In this paper, we generalize Zambon’s construction. Our construction is based on the theory of Lagrangian subalgebras of the double  $\mathfrak{g} \oplus \mathfrak{g}$  of  $\mathfrak{g}$ , and our coisotropic subalgebras correspond to torus fixed points in the variety  $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$  of Lagrangian subalgebras of  $\mathfrak{g} \oplus \mathfrak{g}$ .

### 1. INTRODUCTION

In [14], Marco Zambon gave a way of using a long root  $\beta$  of a complex semisimple Lie bialgebra  $\mathfrak{g}$  to construct a coisotropic subalgebra of  $\mathfrak{g}$ . In this paper, we generalize Zambon’s construction. Let  $w_0$  be the long element of the Weyl group,  $W$ , of  $\mathfrak{g}$ . To each pair  $u, v \in W$  such that  $u \leq vw_0$  in the weak order, we construct a coisotropic subalgebra  $\mathfrak{c}_{u,v}$ . This generalizes Zambon’s construction as we show later. Our construction is based on the theory of Lagrangian subalgebras of the double  $\mathfrak{g} \oplus \mathfrak{g}$  of  $\mathfrak{g}$ , and our coisotropic subalgebras  $\mathfrak{c}_{u,v}$  correspond to torus fixed points in the variety  $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$  of Lagrangian subalgebras of  $\mathfrak{g} \oplus \mathfrak{g}$ . It would be interesting to understand further coisotropic subalgebras of  $\mathfrak{g}$  using the theory of Lagrangian subalgebras.

Let  $\mathfrak{g}$  be a complex semisimple Lie algebra. A Lie bialgebra structure on  $\mathfrak{g}$  is a map  $\delta : \mathfrak{g} \rightarrow \mathfrak{g} \wedge \mathfrak{g}$  such that  $\delta$  is a 1-cocycle for  $\mathfrak{g}$  and the map dual to  $\delta$  gives a Lie bracket on  $\mathfrak{g}^*$ . If  $\mathfrak{g}$  is a Lie bialgebra, then  $\mathfrak{g}$  and  $\mathfrak{g}^*$  have compatible Lie brackets. If  $\mathfrak{g}$  is a Lie bialgebra, then a subalgebra  $\mathfrak{m}$  of  $\mathfrak{g}$  is called a *coisotropic subalgebra* if the annihilator  $\mathfrak{m}^0$  of  $\mathfrak{m}$  in  $\mathfrak{g}^*$  is a Lie subalgebra of  $\mathfrak{g}$ . For this paper, we will focus on the standard Lie bialgebra structure on  $\mathfrak{g}$ , which corresponds to the so-called standard Poisson Lie group structure on  $G$ , the adjoint group of  $\mathfrak{g}$  (see [11, Chapter 11]).

We will study coisotropic subalgebras via their connection to Lagrangian subalgebras. Let  $\mathfrak{d}$  be a  $2n$ -dimensional complex Lie algebra with a symmetric, non-degenerate, ad-invariant bilinear form  $\langle \cdot, \cdot \rangle$ . A subalgebra  $\mathfrak{l}$  of  $\mathfrak{d}$  is called a *Lagrangian subalgebra* if  $\dim \mathfrak{l} = n$  and  $\mathfrak{l}$  is isotropic with respect to  $\langle \cdot, \cdot \rangle$ . A coisotropic subalgebra  $\mathfrak{m}$  of  $\mathfrak{g}$  gives rise to a Lagrangian subalgebra  $\mathfrak{m} \oplus \mathfrak{m}^0$  of  $\mathfrak{g} \oplus \mathfrak{g}^* \cong \mathfrak{g} \oplus \mathfrak{g}$ . The Lagrangian subalgebras of  $\mathfrak{g} \oplus \mathfrak{g}$  have been classified by E. Karolinsky (see [7]). Hence, one can study coisotropic subalgebras of  $\mathfrak{g}$  by studying certain Lagrangian

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subalgebras in  $\mathfrak{g} \oplus \mathfrak{g}$ . For  $\mathfrak{l}$  Lagrangian in  $\mathfrak{g} \oplus \mathfrak{g}$ , we say  $\mathfrak{l}$  is coisotropic if  $\mathfrak{l}$  can be written as  $\mathfrak{m} \oplus \mathfrak{m}^0$  for  $\mathfrak{m} \subset \mathfrak{g}$  coisotropic.

Let  $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}_-$  be the triangular decomposition. Let  $W$  be the Weyl group. For a subspace  $V$  of  $\mathfrak{h}$ , let  $V_\Delta = \{(x, x) | x \in V\}$  and  $V_{-\Delta} = \{(x, -x) | x \in V\}$ . Define

$$\mathfrak{l}_{V,u,v} := V_\Delta + (V^\perp)_{-\Delta} + (u \cdot \mathfrak{n}, 0) + (0, v \cdot \mathfrak{n}_-)$$

where  $V \subseteq \mathfrak{h}$ ,  $u, v \in W$ . For all such triples  $(V, u, v)$ ,  $\mathfrak{l}_{V,u,v}$  is Lagrangian in  $\mathfrak{g} \oplus \mathfrak{g}$ . In this paper, we determine when  $\mathfrak{l}_{V,u,v}$  is coisotropic. We prove the following two theorems which improve and clarify the results of Zambon.

**Theorem 1.1.** *For any  $w \in W$ , let  $\Phi_w$  be the set of positive roots made negative by  $w^{-1}$ . Then  $\mathfrak{l}_{V,u,v}$  is coisotropic iff  $\Phi_u \cap \Phi_v = \emptyset$ .*

For a root  $\beta$ , let  $s_\beta$  be the reflection through  $\beta$  and  $H_\beta \in \mathfrak{h}$  be the unique element such that  $\beta(H_\beta) = 2$  and  $H_\beta \in [\mathfrak{g}_\beta, \mathfrak{g}_{-\beta}]$ .

**Theorem 1.2.** *Zambon’s coisotropic subalgebras are of the form  $\mathfrak{l}_{\mathbb{C}H_\beta, s_\beta, e}$  or  $\mathfrak{l}_{\mathbb{C}H_\beta, e, s_\beta}$  where  $\beta \in \Phi^+$  is a long root. In particular, the coisotropic subalgebras described by Zambon are a specific example of the more general coisotropic subalgebras described in Theorem 1.1 with  $u = s_\beta$  and  $v = e$  or  $u = e$  and  $v = s_\beta$  respectively.*

In [14], Zambon computes his coisotropic subalgebras for the classical Lie algebras. Our construction computes Zambon’s coisotropic subalgebras for all complex semisimple Lie algebras using Chevalley bases.

Coisotropic subalgebras are interesting in part due to their relation to quantum homogeneous spaces. In particular, coisotropic subalgebras give rise to Poisson homogeneous spaces (see [2]). In [12], Ohayon gives the quantization of the Poisson homogeneous spaces related to Zambon’s coisotropic subalgebras in the classical cases. It would be interesting to quantize the coisotropic subalgebras described in Theorem 1.1.

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## 2. PRELIMINARIES

In this section, we recall the facts from the literature which will be necessary in later sections.

**2.1. Notation.** For this paper, let  $\mathfrak{g}$  be a complex semisimple Lie algebra. Let  $G$  be a connected Lie group with Lie algebra  $\mathfrak{g}$ . Fix a Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$  and a set  $\Phi^+$  of positive roots in  $\Phi$ , the set of all roots of  $(\mathfrak{g}, \mathfrak{h})$ . Let  $W$  denote the corresponding Weyl group. Write  $\Phi^-$  for  $-\Phi^+$ , the negative roots of  $\Phi$ . Let  $\Delta$  be the set of simple roots in  $\Phi^+$ . For  $\alpha \in \Phi$ , let  $\mathfrak{g}_\alpha$  be the root space corresponding to  $\alpha$ . Also, let  $H_\alpha \in \mathfrak{h}$  be the unique element such that  $\alpha(H_\alpha) = 2$  and  $H_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ . Let  $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}_-$  be the triangular decomposition of  $\mathfrak{g}$  where  $\mathfrak{n}$  and  $\mathfrak{n}_-$  are chosen so that the roots of  $\mathfrak{n}$  are positive roots.

For any subset  $\mathfrak{u} \subseteq \mathfrak{g}$ , we write  $\mathfrak{u}_\Delta := \{(x, x) | x \in \mathfrak{u}\}$  in  $\mathfrak{g} \oplus \mathfrak{g}$  and  $\mathfrak{u}_{-\Delta} := \{(x, -x) | x \in \mathfrak{u}\}$  in  $\mathfrak{g} \oplus \mathfrak{g}$ . Let  $\{E_\alpha | \alpha \in \Phi\} \cup \{H_i | 1 \leq i \leq \text{rank}(\mathfrak{g})\}$  be a Chevalley basis for  $\mathfrak{g}$  where  $E_\alpha \in \mathfrak{g}_\alpha$  for all  $\alpha \in \Phi$  (see [13, §VI.6]).

**2.2. Lie bialgebras.** It will be necessary to study Lie algebras,  $\mathfrak{r}$ , such that  $\mathfrak{r}^*$  has a Lie bracket compatible with the Lie bracket on  $\mathfrak{r}$ . For more details from this section, see for instance [8], [11], or [9].

**Definition 2.1.** A *Lie bialgebra*  $(\mathfrak{r}, \phi)$  is a Lie algebra  $\mathfrak{r}$  together with a map  $\phi : \mathfrak{r} \rightarrow \mathfrak{r} \wedge \mathfrak{r}$  such that

- (1) The dual map  $\phi^* : \mathfrak{r}^* \wedge \mathfrak{r}^* \rightarrow \mathfrak{r}^*$  is a Lie bracket on  $\mathfrak{r}^*$ .
- (2)  $\phi$  is a 1-cocycle on  $\mathfrak{r}$  where  $\mathfrak{r}$  acts on  $\mathfrak{r} \wedge \mathfrak{r}$  by the adjoint action.

Let  $\mathfrak{r}$  be a Lie bialgebra. If  $\mathfrak{m} \subset \mathfrak{r}$  is a Lie subalgebra of  $\mathfrak{r}$  such that the annihilator  $\mathfrak{m}^0$  of  $\mathfrak{m}$  in  $\mathfrak{r}^*$  is a Lie subalgebra of  $\mathfrak{r}^*$ , then  $\mathfrak{m}$  is called a *coisotropic subalgebra* of  $\mathfrak{r}$ .

**2.3. Standard Lie bialgebra structure.** Let  $\mathfrak{d}$  be a finite dimensional Lie algebra with a nondegenerate invariant bilinear form. Let  $\mathfrak{u}_1, \mathfrak{u}_2$  be Lie subalgebras of  $\mathfrak{d}$ . The triple  $(\mathfrak{d}, \mathfrak{u}_1, \mathfrak{u}_2)$  is called a *Manin triple* if  $\mathfrak{d} = \mathfrak{u}_1 \oplus \mathfrak{u}_2$  as vector spaces and  $\mathfrak{u}_1, \mathfrak{u}_2$  are maximal isotropic subspaces of  $\mathfrak{d}$ . There is a one-to-one correspondence between Manin triples and Lie bialgebras (see [8, Theorem 2.3.2]). In particular, if  $(\mathfrak{d}, \mathfrak{u}_1, \mathfrak{u}_2)$  is a Manin triple, then  $(\mathfrak{u}_1, \phi)$  is a Lie bialgebra where  $\phi$  is the dual map to the Lie bracket in  $\mathfrak{u}_2 \cong \mathfrak{u}_1^*$ .

We now provide an example of a Manin triple which will be useful throughout this paper. Consider the Lie algebra  $\mathfrak{g} \oplus \mathfrak{g}$  where  $\mathfrak{g}$  is our fixed semisimple Lie algebra. For  $x_i, y_i \in \mathfrak{g}$ , define

$$(2.1) \quad \langle (x_1, y_1), (x_2, y_2) \rangle = \langle x_1, x_2 \rangle - \langle y_1, y_2 \rangle$$

where  $\langle \langle, \rangle \rangle$  is a fixed nonzero scalar multiple of the Killing form of  $\mathfrak{g}$ . Then  $\langle, \rangle$  is a symmetric, nondegenerate, and ad-invariant bilinear form on  $\mathfrak{g} \oplus \mathfrak{g}$ .

**Lemma 2.2** ([8, Exercise 2.3.7]). *Let  $\mathfrak{g}_\Delta = \{(x, x) | x \in \mathfrak{g}\}$  and  $\mathfrak{u}_2 = \mathfrak{h}_{-\Delta} + (\mathfrak{n}, 0) + (0, \mathfrak{n}_-) = \{(x + y, -x + z) | x \in \mathfrak{h}, y \in \mathfrak{n}, z \in \mathfrak{n}_-\}$ . Then  $(\mathfrak{g} \oplus \mathfrak{g}, \mathfrak{g}_\Delta, \mathfrak{u}_2)$  is a Manin triple where the bilinear form on  $\mathfrak{g} \oplus \mathfrak{g}$  is given by equation (2.1).*

The Manin triple from Lemma 2.2 gives rise to a Lie bialgebra structure on  $\mathfrak{g}_\Delta \cong \mathfrak{g}$ . We call this the *standard Lie bialgebra structure* on  $\mathfrak{g}$ . Furthermore, since  $(\mathfrak{g} \oplus \mathfrak{g}, \mathfrak{g}_\Delta, \mathfrak{u}_2)$  is a Manin triple, we have  $\mathfrak{u}_2 \cong (\mathfrak{g}_\Delta)^* \cong \mathfrak{g}^*$ , i.e.,

$$\mathfrak{g}^* \cong \mathfrak{h}_{-\Delta} + (\mathfrak{n}, 0) + (0, \mathfrak{n}_-).$$

For the remainder of the paper, we will assume  $\mathfrak{g}$  has the standard Lie bialgebra structure. We identify  $\mathfrak{g}^*$  with  $\mathfrak{h}_{-\Delta} + (\mathfrak{n}, 0) + (0, \mathfrak{n}_-)$  and  $\mathfrak{g}$  with  $\mathfrak{g}_\Delta$ .

When  $\mathfrak{g}$  has the standard Lie bialgebra structure, the annihilator of  $\mathfrak{m}$  in  $\mathfrak{g}^*$  is

$$\mathfrak{m}^\perp := \{(x, y) \in \mathfrak{g}^* | \langle (x, y), (z, z) \rangle = 0 \text{ for all } z \in \mathfrak{m}\}.$$

Therefore,  $\mathfrak{m}$  is a coisotropic subalgebra of  $\mathfrak{g}$  iff  $\mathfrak{m}^\perp$  is a Lie subalgebra of  $\mathfrak{g}^*$ .

If  $\mathfrak{m}$  is a coisotropic subalgebra, then  $\mathfrak{m}_\Delta$  and  $\mathfrak{m}^\perp$  are subalgebras of  $\mathfrak{g}_\Delta$  and  $\mathfrak{g}^*$  respectively. One can show that  $\mathfrak{m}_\Delta \oplus \mathfrak{m}^\perp$  is a subalgebra of  $\mathfrak{g} \oplus \mathfrak{g} \cong \mathfrak{g}_\Delta \oplus \mathfrak{g}^*$ . Furthermore,  $\dim(\mathfrak{m}_\Delta \oplus \mathfrak{m}^\perp) = \dim(\mathfrak{g})$  and it is easy to check that  $\mathfrak{m}_\Delta \oplus \mathfrak{m}^\perp$  is isotropic in  $\mathfrak{g} \oplus \mathfrak{g}$ . This gives the following lemma.

**Lemma 2.3.** *If  $\mathfrak{m} \subset \mathfrak{g}$  is a coisotropic subalgebra, then  $\mathfrak{m}_\Delta \oplus \mathfrak{m}^\perp$  is a Lagrangian subalgebra of  $\mathfrak{g} \oplus \mathfrak{g}$ .*

Let  $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$  be the variety of Lagrangian subalgebras of  $\mathfrak{g} \oplus \mathfrak{g}$ . In [7], E. Karolinsky gave a classification of the elements in  $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$ . In order to understand coisotropic subalgebras of  $\mathfrak{g}$ , we consider the Lagrangian subalgebras of  $\mathfrak{g} \oplus \mathfrak{g}$  and decide when we can write them in the form  $\mathfrak{m}_\Delta \oplus \mathfrak{m}^\perp$  where  $\mathfrak{m}$  is a coisotropic subalgebra.

$G \times G$  acts on  $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$  by the adjoint action of  $G \times G$  on  $\mathfrak{g} \oplus \mathfrak{g}$ . The following proposition will motivate our process of searching for coisotropic subalgebras.

**Proposition 2.4** ([4, Corollary 2.26]). *For every Lagrangian subspace  $U$  of  $\mathfrak{h} \oplus \mathfrak{h}$ , the  $(G \times G)$ -orbit through  $U + (\mathfrak{n}, 0) + (0, \mathfrak{n}_-)$  is Poisson submanifold isomorphic to  $G/B \times G/B_-$  where the isomorphism is given by*

$$\begin{aligned} G/B \times G/B_- &\rightarrow (G \times G) \cdot (U + (\mathfrak{n}, 0) + (0, \mathfrak{n}_-)) \\ (g_1 B, g_2 B_-) &\mapsto (g_1, g_2) \cdot (U + (\mathfrak{n}, 0) + (0, \mathfrak{n}_-)). \end{aligned}$$

Furthermore, these are the only closed  $(G \times G)$ -orbits in  $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$ .

Let  $\mathfrak{l}(U) := U + (\mathfrak{n}, 0) + (0, \mathfrak{n}_-)$ . There are many closed  $G \times G$ -orbits,  $(G \times G) \cdot \mathfrak{l}(U)$ , inside  $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$ . In this paper, we focus on certain such orbits and determine when they give rise to coisotropic subalgebras of  $\mathfrak{g}$ .

Define

$$\mathcal{CL}(\mathfrak{g} \oplus \mathfrak{g}) = \{ \mathfrak{l} \in \mathcal{L}(\mathfrak{g} \oplus \mathfrak{g}) \mid \mathfrak{l} = \mathfrak{m}_\Delta \oplus \mathfrak{m}^\perp \text{ for a coisotropic subalgebra } \mathfrak{m} \subseteq \mathfrak{g} \}.$$

If  $\mathfrak{l} \in \mathcal{CL}(\mathfrak{g} \oplus \mathfrak{g})$ , we will write  $\mathfrak{c}(\mathfrak{l})$  to be the corresponding coisotropic subalgebra of  $\mathfrak{g}$  such that  $\mathfrak{l} = \mathfrak{c}(\mathfrak{l})_\Delta + \mathfrak{c}(\mathfrak{l})^\perp$ .

The following lemma gives us equivalent conditions for when  $\mathfrak{l} \in \mathcal{CL}(\mathfrak{g} \oplus \mathfrak{g})$  and will be useful for determining whether a Lagrangian subalgebra is coisotropic. The proof of the lemma is easy and is left to the reader.

**Lemma 2.5.** *Let  $\mathfrak{l}$  be a Lagrangian subalgebra of  $\mathfrak{g} \oplus \mathfrak{g}$ . The following are equivalent:*

- (1)  $\mathfrak{l} \in \mathcal{CL}(\mathfrak{g} \oplus \mathfrak{g})$ ,
- (2)  $\dim pr_{\mathfrak{g}_\Delta}(\mathfrak{l}) + \dim pr_{\mathfrak{g}^*}(\mathfrak{l}) = \dim(\mathfrak{g}) = \dim(\mathfrak{l})$ ,
- (3)  $\mathfrak{l} = (\mathfrak{l} \cap \mathfrak{g}_\Delta) \oplus (\mathfrak{l} \cap \mathfrak{g}^*)$ .

*Remark 2.6.* Let  $\mathfrak{l} \in \mathcal{CL}(\mathfrak{g} \oplus \mathfrak{g})$  and  $\mathfrak{m} \subset \mathfrak{g}$  be such that  $\mathfrak{m}_\Delta = \mathfrak{l} \cap \mathfrak{g}_\Delta$ . Then by the above lemma,  $\mathfrak{m}$  is a coisotropic subalgebra of  $\mathfrak{g}$ .

2.4. For this paper, it will be necessary to understand certain subsets of positive roots. For  $u \in W$ , define

$$\Phi_u := u(\Phi^-) \cap \Phi^+ = \{ \alpha \in \Phi^+ \mid u^{-1}(\alpha) \in \Phi^- \}.$$

This set consists of the positive roots made negative under  $u^{-1}$ . Let  $\Phi_u^c := \Phi^+ \setminus \Phi_u$ .

Recall that any Weyl group element can be written as a product of simple reflections. If  $u \in W$ , then the length of  $u$ , denoted  $\ell(u)$ , is the smallest number so that  $u$  is a product of  $\ell(u)$  simple reflections. Note that  $|\Phi_u| = \ell(u)$ ; see [6] §1.6-1.7.

**Lemma 2.7.** *For all  $u, v \in W$ , the following are equivalent:*

- (1)  $\Phi_u \cap \Phi_v = \emptyset$ ,
- (2)  $\Phi_u \subset \Phi_v^c$ ,
- (3)  $\Phi_v \subset \Phi_u^c$ ,
- (4)  $\ell(u) + \ell(v) = \ell(u^{-1}v)$ .

The first three equivalences of the lemma follow directly from the definition of  $\Phi_u$ . The equivalence of (1) and (4) follows from Exercise 1.13 and Proposition 4.4.6 of [1].

Note that we can define a partial order on  $W$ . In particular, for  $u, v \in W$ , say  $u \leq v$  if  $v = us_1 \dots s_k$  for some simple reflections  $s_i$  such that  $\ell(us_1s_2 \dots s_i) = \ell(u) + i$  for all  $0 \leq i \leq k$ . This defines the **weak order** on  $W$  (see [1]). Let  $w_0$  be the long element of the Weyl group. The following proposition gives an additional equivalent condition for  $\Phi_u \cap \Phi_v = \emptyset$  using the weak order.

**Proposition 2.8** ([1, Proposition 3.1.2]). *For  $u, v \in W$ ,  $u \leq vw_0$  in the weak order iff  $\Phi_u \cap \Phi_v = \emptyset$ .*

### 3. THE SUBVARIETY $\mathcal{CL}(\mathfrak{g} \oplus \mathfrak{g})$ OF $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$

In this section, we study the subset  $\mathcal{CL}(\mathfrak{g} \oplus \mathfrak{g})$  of  $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$ . We introduce the Poisson structure  $\Pi$  on  $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$  and use it to give a necessary condition for  $\mathfrak{l}$  to be in  $\mathcal{CL}(\mathfrak{g} \oplus \mathfrak{g})$ . This condition will then allow us to explicitly describe certain coisotropic subalgebras in  $\mathfrak{g}$ .

3.1. In [3], Evens and Lu show that if  $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{l}_1 \oplus \mathfrak{l}_2$  is a splitting of  $\mathfrak{g} \oplus \mathfrak{g}$  with  $\mathfrak{l}_1, \mathfrak{l}_2 \in \mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$ , then there exists a Poisson structure  $\Pi_{\mathfrak{l}_1, \mathfrak{l}_2}$  on  $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$ . We consider only the splitting  $\mathfrak{g} \oplus \mathfrak{g} \cong \mathfrak{g}_\Delta \oplus \mathfrak{g}^*$  and let  $\Pi$  be the corresponding Poisson structure. The action of  $G \times G$  on  $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$  gives a Lie algebra anti-homomorphism  $\kappa$  from  $\mathfrak{g} \oplus \mathfrak{g}$  to the space of vector fields on  $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$ . Let  $\{e_1, \dots, e_n\}$  be a basis of  $\mathfrak{g}$  and  $\{\eta_1, \dots, \eta_n\}$  be a basis for  $\mathfrak{g}^*$  with  $\langle e_i, \eta_j \rangle = \delta_{ij}$ , and define

$$(3.1) \quad R = \frac{1}{2} \sum_i \eta_i \wedge e_i.$$

Then  $\Pi = \wedge^2 \kappa(R)$ .

3.2. In this section, we prove that  $\mathfrak{l} \in \mathcal{CL}(\mathfrak{g} \oplus \mathfrak{g})$  implies  $\Pi(\mathfrak{l}) = 0$ . This will allow us to narrow down which Lagrangian subalgebras of  $\mathfrak{g} \oplus \mathfrak{g}$  can possibly be in  $\mathcal{CL}(\mathfrak{g} \oplus \mathfrak{g})$ . In particular, if  $\mathfrak{l} \in \mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$  and  $\Pi(\mathfrak{l}) \neq 0$ , then the results of this section show that  $\mathfrak{l} \notin \mathcal{CL}(\mathfrak{g} \oplus \mathfrak{g})$ .

**Proposition 3.1.** *If  $\mathfrak{l} \in \mathcal{CL}(\mathfrak{g} \oplus \mathfrak{g})$ , then  $\Pi(\mathfrak{l}) = 0$ .*

We begin with some facts that will lead to the proof of this proposition. For more details, see §2.2 of [3]. Let  $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^* \cong \mathfrak{g} \oplus \mathfrak{g}$  be the double of the Lie bialgebra  $\mathfrak{g}$  and  $D$  be the adjoint group so that  $\text{Lie}(D) = \mathfrak{g} \oplus \mathfrak{g}$ . For any  $\mathfrak{l} \in \mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$ , let  $\Pi(\mathfrak{l})^\sharp$  be the linear map

$$\Pi(\mathfrak{l})^\sharp : T_{\mathfrak{l}}^*(D \cdot \mathfrak{l}) \rightarrow T_{\mathfrak{l}}(D \cdot \mathfrak{l})$$

given by  $\Pi(\mathfrak{l})^\sharp(\lambda)(\mu) = \Pi(\mathfrak{l})(\lambda, \mu)$  for  $\lambda, \mu \in T_{\mathfrak{l}}^*(D \cdot \mathfrak{l})$ .

Note that by definition of  $\Pi$ , the  $D$ -orbits are Poisson submanifolds of  $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$  (see Example 4.3 of [4]).

Recall,  $N_{\mathfrak{d}}(\mathfrak{l}) = \{x \in \mathfrak{d} \mid [x, \mathfrak{l}] \subset \mathfrak{l}\}$  is the normalizer subalgebra of  $\mathfrak{l}$  in  $\mathfrak{d}$ . Let  $\kappa : \mathfrak{d} \rightarrow T_{\mathfrak{l}}(D \cdot \mathfrak{l})$  be the action map, which induces an isomorphism,  $\kappa_* : \mathfrak{d}/N_{\mathfrak{d}}(\mathfrak{l}) \rightarrow T_{\mathfrak{l}}(D \cdot \mathfrak{l})$ . Furthermore, the transpose map  $\kappa^* = (\kappa_*)^{tr} : T_{\mathfrak{l}}^*(D \cdot \mathfrak{l}) \rightarrow (\mathfrak{d}/N_{\mathfrak{d}}(\mathfrak{l}))^* \cong N_{\mathfrak{d}}(\mathfrak{l})^\perp$  is an isomorphism

For  $R \in \wedge^2 \mathfrak{d}$  as in equation (3.1), we have  $R^\sharp : \mathfrak{d}^* \rightarrow \mathfrak{d}$  given by  $R^\sharp(\lambda)(\mu) = R(\lambda, \mu)$  for  $\lambda, \mu \in \mathfrak{d}^*$ . We can consider the restriction of this map  $R^\sharp : N_{\mathfrak{d}}(\mathfrak{l})^\perp \rightarrow \mathfrak{d}$ .

Composition with the projection  $\mathfrak{d} \rightarrow \mathfrak{d}/N_{\mathfrak{d}}(\mathfrak{l})$ , gives a map  $N_{\mathfrak{d}}(\mathfrak{l})^{\perp} \rightarrow \mathfrak{d}/N_{\mathfrak{d}}(\mathfrak{l})$  which by abuse of notation we call  $R^{\sharp}$ . Putting these maps together gives the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{d}/N_{\mathfrak{d}}(\mathfrak{l}) & \xleftarrow{R^{\sharp}} & N_{\mathfrak{d}}(\mathfrak{l})^{\perp} \\ \kappa_* \downarrow & & \uparrow \kappa^* = \kappa_*^{tr} \\ T_{\mathfrak{l}}(D \cdot \mathfrak{l}) & \xleftarrow{\Pi(\mathfrak{l})^{\sharp}} & T_{\mathfrak{l}}^*(D \cdot \mathfrak{l}) \end{array}$$

**Lemma 3.2.** *Let  $x + \xi \in N_{\mathfrak{d}}(\mathfrak{l})^{\perp}$  with  $x \in \mathfrak{g}$  and  $\xi \in \mathfrak{g}^*$ . Then*

$$\Pi(\mathfrak{l})^{\sharp}((\kappa^*)^{-1}(x + \xi)) = \kappa_*(x + N_{\mathfrak{d}}(\mathfrak{l})).$$

*Proof.* Since  $\mathfrak{d}^* \cong \mathfrak{g}^* \oplus \mathfrak{g}$ ,

$$R^{\sharp}(\xi + x) = \frac{1}{2} \left[ \sum_j \langle x, \eta_j \rangle e_j - \sum_j \langle e_j, \xi \rangle \eta_j \right] = \frac{1}{2}(x - \xi) + N_{\mathfrak{d}}(\mathfrak{l}).$$

Now,  $\mathfrak{l}$  is a Lagrangian subalgebra of  $\mathfrak{g} \oplus \mathfrak{g}$ , so  $N_{\mathfrak{d}}(\mathfrak{l})^{\perp} \subseteq \mathfrak{l}^{\perp} = \mathfrak{l} \subseteq N_{\mathfrak{d}}(\mathfrak{l})$ . In particular,  $x + \xi \in N_{\mathfrak{d}}(\mathfrak{l})^{\perp} \subset N_{\mathfrak{d}}(\mathfrak{l})$ . Therefore,

$$\begin{aligned} R^{\sharp}(x + \xi) &= \frac{1}{2}(x - \xi) + \frac{1}{2}(x + \xi) + N_{\mathfrak{d}}(\mathfrak{l}) \\ &= x + N_{\mathfrak{d}}(\mathfrak{l}). \end{aligned}$$

Finally, by the above commutative diagram, we have  $\Pi(\mathfrak{l})^{\sharp} = \kappa_* R^{\sharp} \kappa^*$ . This completes the proof. □

Using the above notation, we now prove Proposition 3.1.

*Proof of Proposition 3.1.* Let  $\mathfrak{l} \in \mathcal{CL}(\mathfrak{g} \oplus \mathfrak{g})$  and let  $x + \xi \in N_{\mathfrak{d}}(\mathfrak{l})^{\perp}$  with  $x \in \mathfrak{g}$  and  $\xi \in \mathfrak{g}^*$ . Then by Lemma 3.2,  $\Pi(\mathfrak{l})^{\sharp}((\kappa^*)^{-1}(x + \xi)) = \kappa_*(x + N_{\mathfrak{d}}(\mathfrak{l}))$ .

$\mathfrak{l}$  is Lagrangian, so  $N_{\mathfrak{d}}(\mathfrak{l})^{\perp} \subseteq \mathfrak{l}$ , and thus  $x + \xi \in \mathfrak{l}$ . Since  $\mathfrak{l} \in \mathcal{CL}(\mathfrak{g} \oplus \mathfrak{g})$ , Lemma 2.5 gives  $\mathfrak{l} = (\mathfrak{l} \cap \mathfrak{g}) \oplus (\mathfrak{l} \cap \mathfrak{g}^*)$ , so  $x, \xi \in \mathfrak{l} \subset N_{\mathfrak{d}}(\mathfrak{l})$ . Therefore,

$$\Pi(\mathfrak{l})^{\sharp}((\kappa^*)^{-1}(x + \xi)) = \kappa_*(x + N_{\mathfrak{d}}(\mathfrak{l})) = \kappa_*(0) = 0.$$

It follows that  $\Pi(\mathfrak{l}) = 0$ . □

**3.3. Rank of  $\Pi(\mathfrak{l})$ .** The following lemma will enable us to identify some subalgebras  $\mathfrak{l}$  such that  $\Pi(\mathfrak{l}) = 0$ .

From Proposition 2.4, we can identify the  $G \times G$ -orbit through  $U + (\mathfrak{n}, 0) + (0, \mathfrak{n}_-)$  where  $U \in \mathcal{L}(\mathfrak{h} \oplus \mathfrak{h})$  with  $G/B \times G/B_-$ . Therefore, we view  $\Pi$  as a Poisson structure on  $G/B \times G/B_-$ .

**Lemma 3.3** ([4, Example 4.9]). *Let  $w, w_1, w_2 \in W$ . If  $\mathfrak{l} \in G_{\Delta} \cdot (eB, wB_-) \cap G^* \cdot (w_1B, w_2B_-)$ , then*

$$rk(\Pi(\mathfrak{l})) = \ell(w_1) + \ell(w_2) - \ell(w) - \dim(\mathfrak{h}^{-ww_2^{-1}w_1}).$$

The following proposition will be useful in the subsequent sections to determine some specific coisotropic subalgebras of  $\mathfrak{g}$ .

**Proposition 3.4.** For  $u, v \in W$ , let  $\mathfrak{l} \in G_\Delta \cdot (uB, vB_-) \cap G^* \cdot (uB, vB_-)$ . We have the following:

- (1)  $rk(\Pi(\mathfrak{l})) = \ell(u) + \ell(v) - \ell(u^{-1}v)$ .
- (2)  $rk(\Pi(\mathfrak{l})) = 0$  iff  $\Phi_u \cap \Phi_v = \emptyset$ .
- (3) If  $\mathfrak{l} \in \mathcal{CL}(\mathfrak{g} \oplus \mathfrak{g})$ , then  $\Phi_u \cap \Phi_v = \emptyset$ .

*Proof.* For (1), by assumption  $\mathfrak{l} \in G_\Delta \cdot (uB, vB_-) = G_\Delta \cdot (eB, u^{-1}vB_-)$  and  $\mathfrak{l} \in G^* \cdot (uB, vB_-)$ . Therefore, following the notation of Lemma 3.3, we have  $w = u^{-1}v$ ,  $w_1 = u$ , and  $w_2 = v$ . Applying Lemma 3.3 gives  $rk(\Pi(\mathfrak{l})) = \ell(u) + \ell(v) - \ell(u^{-1}v) - \dim(\mathfrak{h}^{-u^{-1}vv^{-1}u})$ . However,  $\mathfrak{h}^{-u^{-1}vv^{-1}u} = \mathfrak{h}^{-id} = 0$ , and the result follows.

(2) follows from Lemma 2.7. Finally, (3) follows from (2) and Proposition 3.1.  $\square$

*Remark 3.5.* The converse of Proposition 3.1 is false. For example, let  $\mathfrak{z}_{u,v} := (u, v) \cdot [\mathfrak{h}_\Delta + (\mathfrak{n}, 0) + (0, \mathfrak{n}_-)]$ . One can show that  $\mathfrak{z}_{u,v} \in \mathcal{CL}(\mathfrak{g} \oplus \mathfrak{g})$  iff  $\Phi_u \cap \Phi_v = \emptyset$  and  $(v^{-1}u)^2 = e$ . In particular, if  $u \in W$  such that  $u^2 \neq e$ , then  $\mathfrak{z}_{u,e} \notin \mathcal{CL}(\mathfrak{g} \oplus \mathfrak{g})$  by the above remarks, but by Proposition 3.4(2),  $\Pi(\mathfrak{z}_{u,e}) = 0$ . See [10] for more details.

**3.4. A family of coisotropic subalgebras.** We use Proposition 3.4 and further structure theory to determine when a class of Lagrangian subalgebras are coisotropic.

**Theorem 3.6.** Let  $V$  be a subspace of  $\mathfrak{h}$  and  $u, v \in W$ , and define

$$\mathfrak{l}_{V,u,v} := V_\Delta \oplus (V^\perp)_{-\Delta} + (u \cdot \mathfrak{n}, 0) + (0, v \cdot \mathfrak{n}_-).$$

Then  $\mathfrak{l}_{V,u,v} \in \mathcal{CL}(\mathfrak{g} \oplus \mathfrak{g})$  iff  $\Phi_u \cap \Phi_v = \emptyset$ . Furthermore, if  $\mathfrak{l}_{V,u,v} \in \mathcal{CL}(\mathfrak{g} \oplus \mathfrak{g})$ ,

$$\mathfrak{c}(\mathfrak{l}_{V,u,v}) = V + \bigoplus_{\alpha \in \Phi_u} \mathfrak{g}_{-\alpha} + \bigoplus_{\alpha \in \Phi_v} \mathfrak{g}_\alpha$$

is the corresponding coisotropic subalgebra of  $\mathfrak{g}$ .

*Proof.* Note that  $\mathfrak{l}_{V,u,v} = (u, v) \cdot [(u^{-1}, v^{-1})U + (\mathfrak{n}, 0) + (0, \mathfrak{n}_-)]$  where  $U = V_\Delta + (V^\perp)_{-\Delta}$ . In the identification  $(G \times G) \cdot \mathfrak{l}_{V,u,v} \cong G/B \times G/B_-$ ,  $\mathfrak{l}_{V,u,v} \in \mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$  corresponds to  $(uB, vB_-)$  in  $G/B \times G/B_-$ . Therefore,  $\mathfrak{l}_{V,u,v} \in G_\Delta \cdot (uB, vB_-) \cap G^* \cdot (uB, vB_-)$ . By Proposition 3.4, if  $\mathfrak{l}_{V,u,v} \in \mathcal{CL}(\mathfrak{g} \oplus \mathfrak{g})$ , then  $\Phi_u \cap \Phi_v = \emptyset$ .

Now, assume  $\Phi_u \cap \Phi_v = \emptyset$ . We use Lemma 2.5 to show  $\mathfrak{l}_{V,u,v} \in \mathcal{CL}(\mathfrak{g} \oplus \mathfrak{g})$ . For any  $u \in W$ ,  $u\Phi^+ = \Phi_u^c \amalg -\Phi_u$ . Hence,

$$(u \cdot \mathfrak{n}, 0) = \bigoplus_{\alpha \in \Phi_u^c} (\mathfrak{g}_\alpha, 0) + \bigoplus_{\alpha \in \Phi_u} (\mathfrak{g}_{-\alpha}, 0) = \bigoplus_{\alpha \in \Phi_u^c} (\mathfrak{g}_\alpha, 0) + \bigoplus_{\alpha \in \Phi_u} \mathbb{C} \cdot [(E_{-\alpha}, E_{-\alpha}) - (0, E_{-\alpha})]$$

and

$$(0, v \cdot \mathfrak{n}_-) = \bigoplus_{\alpha \in \Phi_v} (0, \mathfrak{g}_\alpha) + \bigoplus_{\alpha \in \Phi_v^c} (0, \mathfrak{g}_{-\alpha}) = \bigoplus_{\alpha \in \Phi_v} \mathbb{C} \cdot [(E_\alpha, E_\alpha) - (E_\alpha, 0)] + \bigoplus_{\alpha \in \Phi_v^c} (0, \mathfrak{g}_{-\alpha}).$$

Therefore,

$$\text{pr}_{\mathfrak{g}_\Delta}(\mathfrak{l}_{V,u,v}) = V_\Delta + \bigoplus_{\alpha \in \Phi_u} (\mathfrak{g}_{-\alpha})_\Delta + \bigoplus_{\alpha \in \Phi_v} (\mathfrak{g}_\alpha)_\Delta$$

and

$$\text{pr}_{\mathfrak{g}^*}(\mathfrak{l}_{V,u,v}) = (V^\perp)_{-\Delta} + \bigoplus_{\alpha \in \Phi_u^c} (\mathfrak{g}_\alpha, 0) + \bigoplus_{\alpha \in \Phi_u} (\mathfrak{g}_\alpha, 0) + \bigoplus_{\alpha \in \Phi_u} (0, \mathfrak{g}_{-\alpha}) + \bigoplus_{\alpha \in \Phi_v^c} (0, \mathfrak{g}_{-\alpha}).$$

Since  $\Phi_u \cap \Phi_v = \emptyset$ , Lemma 2.7 implies that  $\Phi_u \subset \Phi_u^c$  and  $\Phi_v \subset \Phi_v^c$ . Therefore,

$$\text{pr}_{\mathfrak{g}^*}(\mathfrak{l}_{V,u,v}) = (V^\perp)_{-\Delta} + \bigoplus_{\alpha \in \Phi_u^c} (\mathfrak{g}_\alpha, 0) + \bigoplus_{\alpha \in \Phi_v^c} (0, \mathfrak{g}_{-\alpha}).$$

Since  $V \subset \mathfrak{h}$ ,  $\dim(V_\Delta) + \dim(V^\perp)_{-\Delta} = \dim(V) + \dim(V^\perp) = \dim(\mathfrak{h})$ . Thus,

$$\begin{aligned} \dim(\text{pr}_{\mathfrak{g}_\Delta}(\mathfrak{l}_{V,u,v})) + \dim(\text{pr}_{\mathfrak{g}^*}(\mathfrak{l}_{V,u,v})) &= \dim(\mathfrak{h}) + |\Phi_u| + |\Phi_v| + |\Phi_u^c| + |\Phi_v^c| \\ &= \dim(\mathfrak{h}) + 2|\Phi^+| = \dim(\mathfrak{g}). \end{aligned}$$

Therefore, by Lemma 2.5,  $\mathfrak{l}_{V,u,v} \in \mathcal{CL}(\mathfrak{g} \oplus \mathfrak{g})$  which completes the first part of the theorem.

Now, assume  $\mathfrak{l}_{V,u,v} \in \mathcal{CL}(\mathfrak{g} \oplus \mathfrak{g})$ , so  $\Phi_u \cap \Phi_v = \emptyset$  by Proposition 3.4. Let  $\mathfrak{m} := V + \bigoplus_{\alpha \in \Phi_u} \mathfrak{g}_{-\alpha} + \bigoplus_{\alpha \in \Phi_v} \mathfrak{g}_\alpha$ . It is easy to check that

$$\mathfrak{m}_\Delta = V_\Delta + \bigoplus_{\alpha \in \Phi_u} (\mathfrak{g}_{-\alpha})_\Delta + \bigoplus_{\alpha \in \Phi_v} (\mathfrak{g}_\alpha)_\Delta$$

and  $\mathfrak{m}^\perp$  in  $\mathfrak{g}^*$  is given by

$$\mathfrak{m}^\perp = (V^\perp)_{-\Delta} + \bigoplus_{\alpha \in \Phi_u^c} (\mathfrak{g}_\alpha, 0) + \bigoplus_{\alpha \in \Phi_v^c} (0, \mathfrak{g}_{-\alpha}).$$

Therefore,  $\mathfrak{m}_\Delta + \mathfrak{m}^\perp = \mathfrak{l}_{V,u,v} \in \mathcal{CL}(\mathfrak{g} \oplus \mathfrak{g})$  and indeed  $\mathfrak{m} = \mathfrak{c}(\mathfrak{l}_{V,u,v})$ , the coisotropic subalgebra of  $\mathfrak{g}$  corresponding to  $\mathfrak{l}_{V,u,v}$ .  $\square$

*Remark 3.7.* By the proof of Theorem 3.6, if  $\mathfrak{l}_{V,u,v} \in \mathcal{CL}(\mathfrak{g} \oplus \mathfrak{g})$ , then  $\mathfrak{l}_{V,u,v} = \mathfrak{m}_\Delta \oplus \mathfrak{m}^\perp$  where

$$(3.2) \quad \mathfrak{m}_\Delta = V_\Delta + \bigoplus_{\alpha \in \Phi_u} (\mathfrak{g}_{-\alpha})_\Delta + \bigoplus_{\alpha \in \Phi_v} (\mathfrak{g}_\alpha)_\Delta$$

and

$$(3.3) \quad \mathfrak{m}^\perp = (V^\perp)_{-\Delta} + \bigoplus_{\alpha \in \Phi_u^c} (\mathfrak{g}_\alpha, 0) + \bigoplus_{\alpha \in \Phi_v^c} (0, \mathfrak{g}_{-\alpha}).$$

Let  $H_\Delta = \{(x, x) | x \in H\}$ . Note that  $(G/B \times G/B_-)^{H_\Delta} = \{(uB, vB_-) | u, v \in W\}$  and  $(uB, vB_-)$  corresponds to  $(u, v) \cdot (U + (\mathfrak{n}, 0) + (0, \mathfrak{n}_-)) \in \mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$  for any Lagrangian subspace  $U$  of  $\mathfrak{h} \oplus \mathfrak{h}$ . Therefore, whether  $(uB, vB_-) \in \mathcal{CL}(\mathfrak{g} \oplus \mathfrak{g})$  depends not only on  $u, v$  but on  $U$  also.

For  $V, u, v$  as in Theorem 3.6, let

$$\mathfrak{s}_{V,u,v} := V_\Delta + (V^\perp)_{-\Delta} + (u \cdot \mathfrak{n}, 0) + (0, v \cdot \mathfrak{n}).$$

Note that  $\mathfrak{s}_{V,u,v} = \mathfrak{l}_{V,u,vw_0}$  where  $w_0$  is the long element of the Weyl group. Combining Proposition 2.8 and Theorem 3.6, we have  $\mathfrak{s}_{V,u,v} \in \mathcal{CL}(\mathfrak{g} \oplus \mathfrak{g})$  iff  $u \leq v$  in the weak Bruhat order. This makes determining all pairs  $u, v$  such that  $\mathfrak{s}_{V,u,v} \in \mathcal{CL}(\mathfrak{g} \oplus \mathfrak{g})$  into a tabulation for each  $v$  of the  $u$  such that  $u \leq v$ . This is standard to do.

#### 4. THE COISOTROPIC SUBALGEBRAS OF ZAMBON

In this section, we show that Zambon's coisotropic subalgebras are a special case of the coisotropic subalgebras described in Theorem 3.6 and compute his coisotropic subalgebras in more generality. We begin by restating the main result of [14].



**Theorem 4.1** ([14, Proposition 4.3]). *Let  $\mathfrak{g}$  be a complex semisimple Lie algebra with Killing form  $K(\cdot, \cdot)$ . Let  $\beta \in \Phi^+$  be such that for all  $\alpha \in \Phi$ ,  $(\alpha + \mathbb{Z}\beta) \cap \Phi$  does not contain a string of three consecutive elements. Then*

$$\mathfrak{u}_\beta := [E_\beta, \pi]^\sharp \mathfrak{g}^* \text{ and } \mathfrak{u}_{-\beta} := [E_{-\beta}, \pi]^\sharp \mathfrak{g}^*$$

are coisotropic subalgebras of  $\mathfrak{g}$  where

$$\pi = \sum_{\alpha \in \Phi^+} \lambda_\alpha E_\alpha \wedge E_{-\alpha}$$

with  $\lambda_\alpha = \frac{1}{K(E_\alpha, E_{-\alpha})}$ . We refer to such  $\mathfrak{u}_\beta$  and  $\mathfrak{u}_{-\beta}$  as **Zambon coisotropic subalgebras**.

In [14], the Zambon coisotropic subalgebras are determined for the classical Lie algebras. As a consequence of Theorem 4.11, we are able to determine the Zambon coisotropic subalgebras for every simple Lie algebra.

**4.1. Long roots.** We need to understand which roots  $\beta \in \Phi$  satisfy the condition that  $(\alpha + \mathbb{Z}\beta) \cap \Phi$  does not contain three consecutive elements for all  $\alpha \in \Phi$ .

**Definition 4.2.** A root  $\beta \in \Phi$  is called a **long root** if there is no  $\alpha \in \Phi$  in the same irreducible component of  $\Phi$  as  $\beta$  such that  $(\beta, \beta) < (\alpha, \alpha)$ . Otherwise,  $\beta$  is called a **short root**.

We show that the long roots are exactly those that satisfy the above condition.

**Lemma 4.3.** *If  $\Phi$  is an irreducible root system in  $E$  and  $\beta$  is a short root, then there is some long root  $\alpha \in \Phi$  such that  $(\alpha, \beta) \neq 0$ .*

*Proof.* Let  $\gamma$  be a long root. Then  $\sigma(\gamma)$  is a long root for all  $\sigma \in W$  since the Weyl group preserves the length of roots. Furthermore,  $\{\sigma(\gamma) | \sigma \in W\}$  spans  $E$  (see [5, §10.4 Lemma B]). Therefore, there is some  $\sigma \in W$  such that  $\alpha := \sigma(\gamma)$  is not orthogonal to  $\beta$ . □

**Proposition 4.4.** *Let  $\beta \in \Phi$ . For all  $\alpha \in \Phi$ ,  $(\alpha + \mathbb{Z}\beta) \cap \Phi$  does not contain a string of three consecutive elements iff  $\beta$  is a long root.*

*Proof.* ( $\Leftarrow$ ) If  $\beta$  is a long root, then  $\frac{2(\beta, \alpha)}{(\beta, \beta)} = 0, \pm 1$  for all  $\alpha \in \Phi$ . Therefore, the length of the  $\alpha$ -root string through  $\beta$  is at most 2.

( $\Rightarrow$ ) Let  $\beta \in \Phi$  be such that  $(\alpha + \mathbb{Z}\beta) \cap \Phi$  does not contain a string of three consecutive elements for all  $\alpha$  and suppose for a contradiction that  $\beta$  is not a long root. Hence, there exists  $\alpha$  in the same irreducible component as  $\beta$  such that

$$(4.1) \quad (\beta, \beta) < (\alpha, \alpha).$$

By Lemma 4.3, we can assume  $\alpha$  and  $\beta$  are not orthogonal. Therefore, replacing  $\alpha$  with  $-\alpha$  if necessary,  $s_\beta(\alpha) = \alpha + k\beta$  where  $k > 0$ . But equation (4.1) implies that  $k \geq 2$ . Therefore,  $\alpha + 2\beta$  or  $\alpha + 3\beta$  is a root and  $\{\alpha, \alpha + \beta, \alpha + 2\beta\}$  is a root string of three elements, a contradiction. □

Zambon’s coisotropic subalgebras are all of the form  $[E_\beta, \pi]^\sharp \mathfrak{g}^*$  where  $\beta$  is a long root. In the next several sections, we explicitly compute  $[E_\beta, \pi]^\sharp \mathfrak{g}^*$ .

**4.2. Computing  $[E_\beta, \pi]$ .** In order to understand Theorem 4.1, we now compute  $[E_\beta, \pi]$  where  $\beta$  is a long root. We begin with several necessary facts.

**Lemma 4.5.** *Let  $\lambda_\alpha := \frac{1}{K(E_\alpha, E_{-\alpha})}$ . If  $\beta \in \Phi$  is a long root and  $\alpha \in \Phi^+$ , then  $\lambda_\alpha = \lambda_{s_\beta(\alpha)}$ . In particular,*

- (1) *If  $\alpha + \beta$  is a root, then  $\lambda_\alpha = \lambda_{\alpha+\beta}$ .*
- (2) *If  $\alpha - \beta$  is a root, then  $\lambda_\alpha = \lambda_{\alpha-\beta}$ .*

Since the Killing form is a symmetric bilinear form, we have  $\lambda_\gamma = \lambda_{-\gamma}$  for all  $\gamma \in \Phi$ .

*Proof.* We prove (1) and (2) follows in the same fashion. Since  $\{E_\alpha | \alpha \in \Phi\} \cup \{H_i\}$  is a Chevalley basis for  $\mathfrak{g}$ ,  $[E_\alpha, E_{-\alpha}] = H_\alpha$  for all  $\alpha \in \Phi^+$ . Since  $\beta$  is a long root and  $\alpha + \beta$  is a root,  $s_\beta(\alpha) = \alpha + \beta$ . Therefore,  $(\alpha, \alpha) = (s_\beta(\alpha), s_\beta(\alpha)) = (\alpha + \beta, \alpha + \beta)$ .

For  $\gamma \in \Phi$ , let  $t_\gamma \in \mathfrak{h}$  be the unique element satisfying  $\gamma(h) = K(t_\gamma, h)$  for all  $h \in \mathfrak{h}$ . We have  $t_\alpha = \frac{H_\alpha}{K(E_\alpha, E_{-\alpha})}$  and  $t_\alpha = \frac{(\alpha, \alpha)}{2} H_\alpha$  (see [5, Proposition 8.3]).

Therefore,  $\frac{(\alpha, \alpha)}{2} = \frac{1}{K(E_\alpha, E_{-\alpha})} = \lambda_\alpha$ . Hence

$$\lambda_\alpha = \frac{(\alpha, \alpha)}{2} = \frac{(\alpha + \beta, \alpha + \beta)}{2} = \frac{1}{K(E_{\alpha+\beta}, E_{-\alpha-\beta})} = \lambda_{\alpha+\beta}.$$

□

**Lemma 4.6.** *Let  $\beta$  be a long root and  $\alpha \in \Phi$  such that  $\alpha + \beta$  is a root. If  $c_{\alpha, \beta} \in \mathbb{Z}$  is such that  $[E_\alpha, E_\beta] = c_{\alpha, \beta} E_{\alpha+\beta}$ , then*

$$c_{\beta, \alpha} = -c_{\beta, -\alpha-\beta}.$$

Furthermore, if  $\alpha - \beta$  is a root, then

$$c_{\beta, -\alpha} = -c_{\beta, \alpha-\beta}.$$

*Proof.* Assume  $\beta$  is a long root and  $\alpha + \beta$  is a root; then the  $\beta$ -string through  $\alpha$  is  $\alpha, \alpha + \beta$ . Furthermore,  $s_\beta(\alpha) = \alpha + \beta$ . By Lemma 25.2 of [5],

$$[E_{-\beta}, [E_\beta, E_\alpha]] = E_\beta.$$

On the other hand, by definition of  $c_{\beta, \alpha}$ , we have  $[E_{-\beta}, [E_\beta, E_\alpha]] = [E_{-\beta}, c_{\beta, \alpha} E_{\alpha+\beta}] = c_{\beta, \alpha} c_{-\beta, \alpha+\beta} E_\beta$ . Furthermore, by definition of a Chevalley basis,  $c_{-\beta, \alpha+\beta} = -c_{\beta, -\alpha-\beta}$ . Therefore,

$$(4.2) \quad 1 = -c_{\beta, \alpha} c_{\beta, -\alpha-\beta}.$$

Since the  $c_{\beta, \alpha}$  are integers, we have  $c_{\beta, \alpha} = -c_{\beta, -\alpha-\beta}$ . The second statement follows in a similar fashion. □

For the remainder of this section, let  $\beta \in \Phi^+$  be a fixed long root. We are now in a position to compute  $[E_\beta, \pi]$ . First, let

$$Q_+ := \{\alpha \in \Phi^+ | s_\beta(\alpha) \in \Phi^+; s_\beta(\alpha) \neq \alpha\},$$

$$Q_0 := \{\alpha \in \Phi^+ | s_\beta(\alpha) = \alpha\},$$

and

$$Q_- := \{\alpha \in \Phi^+ | s_\beta(\alpha) \in \Phi^-\}.$$

Then  $\Phi^+ = Q_+ \cup Q_0 \cup Q_-$  is a disjoint union. Furthermore,  $Q_- = \Phi_{s_\beta}$ . Let  $\pi_+ := \sum_{\alpha \in Q_+} \lambda_\alpha E_\alpha \wedge E_{-\alpha}$  and define  $\pi_0$  and  $\pi_-$  similarly. Then,

$$(4.3) \quad \pi = \sum_{\alpha \in \Phi^+} \lambda_\alpha E_\alpha \wedge E_{-\alpha} = \pi_+ + \pi_0 + \pi_-.$$

Therefore,

$$(4.4) \quad [E_\beta, \pi] = [E_\beta, \pi_+] + [E_\beta, \pi_0] + [E_\beta, \pi_-].$$

If  $\alpha \in Q_0$ , then  $s_\beta(\alpha) = \alpha$  and  $\alpha \pm \beta \notin \Phi$ . It follows easily that  $[E_\beta, \pi_0] = 0$ . It remains to compute  $[E_\beta, \pi_+]$  and  $[E_\beta, \pi_-]$ .

**Lemma 4.7.**  $[E_\beta, \pi_+] = 0$ .

*Proof.* Note that  $\alpha \in Q_+$  implies  $s_\beta(\alpha) \in Q_+$ . In particular,  $s_\beta$  acts on the elements of  $Q_+$  and the orbits are of the form  $\{\alpha, s_\beta(\alpha)\}$ .

Let  $\alpha \in Q_+$ . Without loss of generality, we can assume  $s_\beta(\alpha) = \alpha + \beta$ . Indeed, if  $s_\beta(\alpha) \neq \alpha + \beta$ , then  $s_\beta(\alpha) = \alpha - \beta$ . In this case, replace  $\alpha$  with  $\alpha - \beta$ .

By Lemmas 4.5 and 4.6,

$$\begin{aligned} [E_\beta, \lambda_\alpha E_\alpha \wedge E_{-\alpha} + \lambda_{s_\beta(\alpha)} E_{s_\beta(\alpha)} \wedge E_{-s_\beta(\alpha)}] &= \lambda_\alpha [E_\beta, E_\alpha \wedge E_{-\alpha} + E_{\alpha+\beta} \wedge E_{-\alpha-\beta}] \\ &= \lambda_\alpha (c_{\beta, \alpha} + c_{\beta, -\beta-\alpha}) E_{\alpha+\beta} \wedge E_{-\alpha} \\ &= 0. \end{aligned}$$

Hence,  $[E_\beta, \pi_+] = 0$ . □

Combining equation (4.4) with Lemma 4.7 gives

$$[E_\beta, \pi] = [E_\beta, \pi_-].$$

Thus it only remains to compute  $[E_\beta, \pi_-]$ .

Recall,  $Q_- = \Phi_{s_\beta}$ , so  $\pi_- = \sum_{\alpha \in \Phi_{s_\beta}} \lambda_\alpha E_\alpha \wedge E_{-\alpha}$ . Note  $\beta \in \Phi_{s_\beta}$  and

$$(4.5) \quad [E_\beta, \lambda_\beta E_\beta \wedge E_{-\beta}] = \lambda_\beta E_\beta \wedge H_\beta.$$

For  $\alpha \in \Phi_{s_\beta} \setminus \{\beta\}$ ,  $s_\beta(\alpha) = \alpha - \beta < 0$  and  $\alpha + \beta$  is not a root. Note that  $-s_\beta$  acts on  $\Phi_{s_\beta} \setminus \{\beta\}$ . The orbits of  $-s_\beta$  on  $\Phi_{s_\beta} \setminus \{\beta\}$  are of the form  $\{\alpha, \beta - \alpha\}$  where  $\alpha \in \Phi_{s_\beta} \setminus \{\beta\}$ . In particular, the elements of  $\Phi_{s_\beta} \setminus \{\beta\}$  come in pairs.

Therefore, if  $\alpha \in \Phi_{s_\beta} \setminus \{\beta\}$ ,

$$(4.6) \quad \begin{aligned} [E_\beta, \lambda_\alpha E_\alpha \wedge E_{-\alpha} + \lambda_{\beta-\alpha} E_{\beta-\alpha} \wedge E_{\alpha-\beta}] \\ &= \lambda_\alpha c_{\beta, -\alpha} E_\alpha \wedge E_{\beta-\alpha} + \lambda_{\beta-\alpha} c_{\beta, \alpha-\beta} E_{\beta-\alpha} \wedge E_\alpha \\ &= (\lambda_\alpha c_{\beta, -\alpha} - \lambda_{\beta-\alpha} c_{\beta, \alpha-\beta}) E_\alpha \wedge E_{\beta-\alpha} \end{aligned}$$

where  $\beta - \alpha > 0$ .

Since the Killing form is a symmetric bilinear form, Lemma 4.5 gives  $\lambda_\alpha = \lambda_{\beta-\alpha}$ . Furthermore, by Lemma 4.6, we have  $c_{\beta, -\alpha} = -c_{\beta, \alpha-\beta}$ . In particular,  $\lambda_\alpha c_{\beta, -\alpha} - \lambda_{\beta-\alpha} c_{\beta, \alpha-\beta} = 2\lambda_\alpha c_{\beta, -\alpha}$ .

Combining equations (4.5) and (4.6) gives

$$[E_\beta, \pi_-] = E_\beta \wedge H_\beta + \sum_{\substack{\alpha \in \Phi_{s_\beta} \\ \alpha \neq \beta}} \lambda_\alpha c_{\beta, -\alpha} E_\alpha \wedge E_{\beta-\alpha}.$$

Therefore, we have the following proposition.

**Proposition 4.8.** *If  $\beta \in \Phi^+$  is a long root, then*

$$(4.7) \quad [E_\beta, \pi] = E_\beta \wedge H_\beta + \sum_{\substack{\alpha \in \Phi_{s_\beta} \\ \alpha \neq \beta}} \lambda_\alpha c_{\beta, -\alpha} E_\alpha \wedge E_{\beta-\alpha}.$$

**4.3. Computations of Zambon coisotropic subalgebras.** We are now in the position to understand  $\mathfrak{u}_\beta = [E_\beta, \pi]^\sharp \mathfrak{g}^*$  where  $\beta$  is a long root. We begin by computing  $[E_\beta, \pi]^\sharp \mathfrak{g}^*$  using the results of the previous section. We then prove  $(\mathfrak{u}_\beta)_\Delta \oplus \mathfrak{u}_\beta^\perp$  is in  $\mathcal{CL}(\mathfrak{g} \oplus \mathfrak{g})$  for all long roots  $\beta$  which recovers Theorem 4.1.

**Proposition 4.9.** *If  $\beta \in \Phi^+$  is a long root, then*

$$(4.8) \quad \mathfrak{u}_\beta := [E_\beta, \pi]^\sharp \mathfrak{g}^* = \mathbb{C}H_\beta + \bigoplus_{\alpha \in \Phi_{s_\beta}} \mathfrak{g}_\alpha$$

and

$$(4.9) \quad \mathfrak{u}_{-\beta} := [E_{-\beta}, \pi]^\sharp \mathfrak{g}^* = \mathbb{C}H_\beta + \bigoplus_{\alpha \in \Phi_{s_\beta}} \mathfrak{g}_{-\alpha}.$$

*Proof.* Let  $\{E_\alpha^* | \alpha \in \Phi\} \cup \{H_\beta^*\} \cup \{f_i | 2 \leq i \leq \text{rank}(\mathfrak{g})\}$  be a basis of  $\mathfrak{g}^*$  where the  $f_i$  are linear functions on  $\mathfrak{h}$  such that  $f_i(H_\beta) = 0$  for all  $i$ , and  $H_\beta^*$  vanishes on  $\mathfrak{n}$  and on  $\mathfrak{n}_-$  and evaluates as 1 against  $H_\beta$ . From equation (4.7),  $[E_\beta, \pi]^\sharp(f_i) = 0$  for all  $i$  and  $[E_\beta, \pi](H_\beta^*) = -E_\beta$ . Since  $[E_\beta, \pi]$  involves only positive roots,  $[E_\beta, \pi]^\sharp(E_{-\gamma}^*) = 0$  for all  $\gamma \in \Phi^+$ . Furthermore,  $[E_\beta, \pi]^\sharp(E_\beta^*) = H_\beta$ . Finally,  $\alpha \in \Phi_{s_\beta} \setminus \{\beta\}$  implies  $[E_\beta, \pi]^\sharp(E_\alpha^*) = E_{\beta-\alpha}$  and  $[E_\beta, \pi]^\sharp(E_{\beta-\alpha}^*) = -E_\alpha$ . Therefore,

$$\begin{aligned} [E_\beta, \pi]^\sharp \mathfrak{g}^* &= \mathbb{C}H_\beta + \mathbb{C}E_\beta + \sum_{\alpha \in \Phi_{s_\beta} \setminus \{\beta\}} \mathbb{C}E_\alpha \\ &= \mathbb{C}H_\beta + \sum_{\alpha \in \Phi_{s_\beta}} \mathbb{C}E_\alpha. \end{aligned}$$

Equation (4.9) follows in a similar fashion. □

We now show that  $\mathfrak{u}_\beta$  and  $\mathfrak{u}_{-\beta}$  are coisotropic subalgebras. The following proposition also proves that Zambon’s coisotropic subalgebras are a special case of those described in Theorem 3.6.

**Proposition 4.10.** *Let  $\beta \in \Phi^+$  be a long root. Then*

$$(\mathfrak{u}_\beta)_\Delta + \mathfrak{u}_\beta^\perp = (\mathbb{C}H_\beta)_\Delta + (\mathbb{C}H_\beta)^\perp_{-\Delta} + (\mathfrak{n}, 0) + (0, s_\beta \cdot \mathfrak{n}_-) = \mathfrak{l}_{\mathbb{C}H_\beta, e, s_\beta}$$

and

$$(\mathfrak{u}_{-\beta})_\Delta + \mathfrak{u}_{-\beta}^\perp = (\mathbb{C}H_\beta)_\Delta + (\mathbb{C}H_\beta)^\perp_{-\Delta} + (v \cdot \mathfrak{n}, 0) + (0, \mathfrak{n}_-) = \mathfrak{l}_{\mathbb{C}H_\beta, s_\beta, e}.$$

Furthermore, both of these subspaces are in  $\mathcal{CL}(\mathfrak{g} \oplus \mathfrak{g})$ . In particular, since  $\Phi_w \cap \Phi_e \subset \Phi_e = \emptyset$  for all  $w \in W$ , Zambon’s coisotropic subalgebras are a special case of the form of Theorem 3.6.

*Proof.* By Proposition 4.9,  $\mathfrak{u}_\beta = \mathbb{C}H_\beta + \bigoplus_{\alpha \in \Phi_{s_\beta}} \mathfrak{g}_\alpha$ . Now, compute  $\mathfrak{u}_\beta^\perp$  in  $\mathfrak{g}^*$ . First, note that

$$\begin{aligned} \mathfrak{u}_\beta^\perp &= \left\{ \left( H + \sum_{\gamma \in \Phi^+} c_\gamma E_\gamma, -H + \sum_{\eta \in \Phi^+} b_\eta E_{-\eta} \right) \in \mathfrak{g}^* \mid \right. \\ &\quad \left. \langle (x, x), \left( H + \sum_{\gamma \in \Phi^+} c_\gamma E_\gamma, -H + \sum_{\eta \in \Phi^+} b_\eta E_{-\eta} \right) \rangle = 0 \text{ for all } x \in \mathfrak{u}_\beta \right\}. \end{aligned}$$

It is easy to check that  $\mathfrak{u}_\beta^\perp$  contains

$$(4.10) \quad (V^\perp)_{-\Delta} + (\mathfrak{n}, 0) + \bigoplus_{\alpha \in \Phi_{s_\beta}^c} (0, \mathfrak{g}_{-\alpha})$$

where  $V = \mathbb{C}H_\beta$ . Furthermore,  $\dim(\mathfrak{u}_\beta^\perp) = \dim(\mathfrak{g}^*) - \dim(\mathfrak{u}_\beta)$  and it follows that  $\mathfrak{u}_\beta^\perp$  is exactly equal to the set in equation (4.10). Hence,

$$\begin{aligned} (\mathfrak{u}_\beta)_\Delta \oplus \mathfrak{u}_\beta^\perp &= V_\Delta + (V^\perp)_{-\Delta} + \bigoplus_{\alpha \in \Phi_{s_\beta}} (\mathfrak{g}_\alpha)_\Delta + (\mathfrak{n}, 0) + \bigoplus_{\alpha \in \Phi_{s_\beta}^c} (0, \mathfrak{g}_{-\alpha}) \\ &= V_\Delta + (V^\perp)_{-\Delta} + (e \cdot \mathfrak{n}, 0) + (0, s_\beta \cdot \mathfrak{n}_-) \\ &= \mathfrak{l}_{V, e, s_\beta} \end{aligned}$$

where the next to last equality follows from the proof of Theorem 3.6. In particular, for any  $v \in W$ ,  $\Phi_e \cap \Phi_v = \emptyset$ , so by Theorem 3.6,  $(\mathfrak{u}_\beta)_\Delta \oplus \mathfrak{u}_\beta^\perp \in \mathcal{CL}(\mathfrak{g} \oplus \mathfrak{g})$ . Furthermore,  $\mathfrak{u}_\beta$  is a coisotropic subalgebra of  $\mathfrak{g}$  by Remark 2.6. The statement for  $\mathfrak{u}_{-\beta}$  can be seen similarly.  $\square$

Combining Propositions 4.9 and 4.10, we have

**Theorem 4.11.** *If  $\beta \in \Phi^+$  is a long root, then*

$$(4.11) \quad \mathfrak{u}_\beta := [E_\beta, \pi]^\sharp \mathfrak{g}^* = \mathbb{C}H_\beta + \bigoplus_{\alpha \in \Phi_{s_\beta}} \mathfrak{g}_\alpha$$

and

$$(4.12) \quad \mathfrak{u}_{-\beta} := [E_{-\beta}, \pi]^\sharp \mathfrak{g}^* = \mathbb{C}H_\beta + \bigoplus_{\alpha \in \Phi_{s_\beta}} \mathfrak{g}_{-\alpha}$$

are a special case of the coisotropic subalgebras of Theorem 3.6.

In particular, we have recovered Zambon’s theorem, Theorem 4.1, about coisotropic subalgebras and have shown that his coisotropic subalgebras are a special case of Theorem 3.6. In [14], Zambon only explicitly describes his coisotropic subalgebras in the case of classical Lie algebras. Note that Theorem 4.11 applied to all semisimple Lie algebras, not just the classical Lie algebras.

*Remark 4.12.* By the proof of Proposition 4.7, the elements of  $\Phi_{s_\beta} \setminus \{\beta\}$  come in pairs. Therefore,  $|\Phi_{s_\beta}|$  is odd, and  $\dim(\mathfrak{u}_\beta) = 1 + |\Phi_{s_\beta}|$  is even. Thus, Zambon coisotropic subalgebras are all even dimensional.

*Remark 4.13.* As noted in [14], there are odd dimensional coisotropic subalgebras. This follows easily from Theorem 3.6.

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