MULTIPlicity FORMULA FOR RESTRICTION OF REPRESENTATIONS OF GL₂₁(F) TO SL₂₁(F)

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Abstract. In this note we prove a certain multiplicity formula regarding the restriction of an irreducible admissible genuine representation of a 2-fold cover GL₂₁(F) of GL₂₁(F) to the 2-fold cover SL₂₁(F) of SL₂₁(F), and find in particular that this multiplicity may not be one, a result that was recently observed for certain principal series representations in the work of Szpruch (2013). The proofs follow the standard path via Waldspurger’s analysis of theta correspondence between SL₂₁(F) and PGL₂₁(F).

1. Introduction

This paper will be concerned with the representation theory of a certain 2-fold cover GL₂₁(F) of GL₂₁(F) to be called the metaplectic covering of GL₂₁(F), where F is a non-Archimedean local field. We recall that there is a unique (up to isomorphism) non-trivial 2-fold cover of SL₂₁(F) called the metaplectic cover and denoted by SL₂₁(F) in this paper, but there are many inequivalent 2-fold coverings of GL₂₁(F) which extend this 2-fold covering of SL₂₁(F). We fix a covering of GL₂₁(F) as follows.

Observe that GL₂₁(F) is the semi-direct product of SL₂₁(F) and F×, where F× sits inside GL₂₁(F) as e → \( \left( \begin{array}{cc} e & 0 \\ 0 & e \end{array} \right) \). This action of F× on SL₂₁(F) lifts uniquely to an action of F× on SL₂₁(F). Denote GL₂₁(F) = SL₂₁(F) × F× and call this the metaplectic cover of GL₂₁(F). Thus the metaplectic cover of GL₂₁(F) that we consider in this paper is that cover of GL₂₁(F) which extends the metaplectic cover of SL₂₁(F) and is further split on the subgroup \( \{ \left( \begin{array}{cc} e & 0 \\ 0 & e \end{array} \right) : e \in F× \} \). We have the following short exact sequence of locally compact topological groups:

\[ 1 \rightarrow \{ \pm 1 \} \rightarrow \widetilde{GL}_2(F) \rightarrow GL_2(F) \rightarrow 1. \]

For any subset X of GL₂₁(F), we write \( \tilde{X} \) for its inverse image in GL₂₁(F). A key property of the cover GL₂₁(F) is that for

\[ A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \]

and

\[ B = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}. \]
We have

\[ \tilde{A} \tilde{B} \tilde{A}^{-1} \tilde{B}^{-1} = (a, c)(b, d)(ab, cd), \]

where \( \tilde{A} \) and \( \tilde{B} \) are arbitrary lifts of the matrices \( A \) and \( B \) in \( \GL_2(F) \) to \( \widehat{\GL}_2(F) \), and the commutator belongs to the kernel of the map from \( \GL_2(F) \) to \( \GL_2(F) \) which is identified to \( \pm 1 \), and \((-,-)\) denotes the Hilbert symbol which is a non-degenerate bilinear form \( F^\times /F^\times 2 \times F^\times /F^\times 2 \to \mu_2 \). In particular, for later use, we note that for

\[ A = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \]

and

\[ B = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \]

we have

\[ \tilde{A} \tilde{B} \tilde{A}^{-1} \tilde{B}^{-1} = (a, z). \]

Let \( Z \) be the center of \( \GL_2(F) \) which we identify with \( F^\times \). From the commutation relation above, it follows that \( \tilde{Z} \) is an abelian subgroup of \( \GL_2(F) \) but is not the center of \( \widehat{\GL}_2(F) \). The center of \( \widehat{\GL}_2(F) \) is \( \tilde{Z}^2 \). The centralizer of \( \widehat{\SL}_2(F) \) inside \( \widehat{\GL}_2(F) \) is \( \tilde{Z} \). Let \( \widehat{\GL}_2(F)_+ = \tilde{Z} \cdot \widehat{\SL}_2(F) \). Let \( \mu \) be a genuine character of \( \tilde{Z} \) and \( \tau \) an irreducible admissible genuine representation of \( \widehat{\SL}_2(F) \). We say that \( \mu \) and \( \tau \) are compatible if \( \mu|_{\{\pm 1\}} = \omega_\tau \) where \( \omega_\tau \) is the central character of \( \tau \), i.e., the character by which \( \{\pm 1\} \) operates on the vector space underlying \( \tau \). If \( \mu \) and \( \tau \) are compatible, we define a representation \( \mu \tau \) of \( \widehat{\GL}_2(F)_+ \) whose restriction to \( \widehat{\SL}_2(F) \) is \( \tau \) and on which \( \tilde{Z} \) operates by \( \mu \). One may choose the representatives of the quotient \( \widehat{\GL}_2(F)/\widehat{\GL}_2(F)_+ \cong F^\times /F^\times 2 \) to be

\[ g(a) := \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \]

for \( a \in F^\times \) representing a coset of \( F^\times 2 \). We write \( (\mu \tau)^a \) for the representation of \( \widehat{\GL}_2(F)_+ \) which is the conjugate representation of \( \mu \tau \) by the element \( g(a) \). From the commutation relation above, it follows that \( \mu^a(\tilde{z}) = \mu(\tilde{z})(a, z) \) with \( z = p(\tilde{z}) \), and therefore since the quadratic Hilbert symbol is non-degenerate, for \( a \in F^\times - F^\times 2 \), \( \mu \neq \mu^a \).

It follows that if \( a \in F^\times - F^\times 2 \), then \( \mu \tau \not\cong (\mu \tau)^a \), as the central characters of \( \mu \tau \) and \((\mu \tau)^a\) are different. By Clifford theory, \( \hat{\mu} := \text{ind}_{\widehat{\SL}_2(F)}^{\widehat{\GL}_2(F)} (\mu \tau) \) is an irreducible admissible genuine representation of \( \widehat{\GL}_2(F) \). Moreover, every irreducible admissible genuine representation of \( \widehat{\GL}_2(F) \) arises in this fashion.

Let \( \hat{\mu} \) be an irreducible admissible genuine representation of \( \widehat{\GL}_2(F) \). Write \( \hat{\mu} = \text{ind}_{\widehat{\SL}_2(F)}^{\widehat{\GL}_2(F)} (\mu \tau) \) for suitable \( \mu \) and \( \tau \). Then by Mackey theory it is easy to see that

\[ (1) \quad \hat{\mu}|_{\widehat{\GL}_2(F)_+} = \bigoplus_{a \in F^\times /F^\times 2} (\mu \tau)^a. \]

Since \( (\mu \tau)^a \not\cong (\mu \tau)^b \) if \( ab^{-1} \notin F^\times 2 \), the restriction of \( \hat{\mu} \) to \( \widehat{\GL}_2(F)_+ \) is multiplicity free.
Note that the restriction of \((\mu \tau)^a\) to \(\widetilde{SL}_2(F)\) is \(\tau^a\). Therefore, the restriction of \(\tilde{\pi}\) to \(SL_2(F)\) is given by
\[
\tilde{\pi}|_{\widetilde{SL}_2(F)} = \bigoplus_{a \in F^*/F^2} \tau^a.
\]
This identification of \(\tilde{\pi}\) restricted to \(\widetilde{SL}_2(F)\) allows us to study the multiplicity of the restriction of a representation of \(\widetilde{GL}_2(F)\) to \(\widetilde{SL}_2(F)\), and in particular shows that it may be greater than one as we show below after discussing some preliminaries about theta correspondence, and the works of Waldspurger about them.

2. \(\theta\)-correspondence and Waldspurger involution

In this section, we recall some results of Waldspurger from [3] which analyzes rather completely the \(\theta\)-correspondence between \(\widetilde{SL}_2(F)\) and \(PGL_2(F) = SO(2, 1)\) and also between \(\widetilde{SL}_2(F)\) and \(PD^\times = SO(3)\), where \(D\) is the unique quaternion division algebra over \(F\). We will use these results repeatedly.

Now fix a non-trivial additive character \(\psi\) of \(F\). With respect to this \(\psi\), one has the \(\theta\)-correspondence between irreducible admissible genuine representations of \(\widetilde{SL}_2(F)\) and irreducible admissible representations of \(PGL_2(F)\)
\[
\operatorname{Irr}(\widetilde{SL}_2(F)) \xrightarrow{\theta(-,\psi)} \operatorname{Irr}(PGL_2(F))
\]
as well as one between irreducible admissible genuine representations of \(\widetilde{SL}_2(F)\) and irreducible admissible representations of \(PD^\times\)
\[
\operatorname{Irr}(\widetilde{SL}_2(F)) \xrightarrow{\theta(-,\psi)} \operatorname{Irr}(PD^\times).
\]
This correspondence \(\tau \mapsto \theta(\tau, \psi)\) depends on \(\psi\) and will be abbreviated to \(\tau \mapsto \theta(\tau)\) as \(\psi\) will be fixed. The \(\theta\)-correspondence between \(\widetilde{SL}_2(F)\) and \(PGL_2(F)\) gives a one-to-one mapping from the subset of irreducible admissible genuine representations of \(\widetilde{SL}_2(F)\) which have \(\psi\)-Whittaker model onto all irreducible admissible representations of \(PGL_2(F)\). Similarly, \(\theta\)-correspondence between \(\widetilde{SL}_2(F)\) and \(PD^\times\) gives a one-to-one mapping from the subset of irreducible admissible genuine representations of \(\widetilde{SL}_2(F)\) which do not have \(\psi\)-Whittaker model onto all irreducible representations of \(PD^\times\). Thus \(\theta\)-correspondence defines a bijection (which depends on the choice of \(\psi\)):
\[
\operatorname{Irr}(\widetilde{SL}_2(F)) \leftrightarrow \operatorname{Irr}(PGL_2(F)) \bigcup \operatorname{Irr}(PD^\times).
\]
Now we can describe the Waldspurger involution [3] \(W : \operatorname{Irr}(\widetilde{SL}_2(F)) \rightarrow \operatorname{Irr}(\widetilde{SL}_2(F))\) which is defined using
(1) the \(\theta\)-correspondence from \(\widetilde{SL}_2(F)\) to \(PGL_2(F)\),
(2) the \(\theta\)-correspondence from \(\widetilde{SL}_2(F)\) to \(PD^\times\) and
(3) the Jacquet-Langlands correspondence between representations of \(PGL_2(F)\) and \(PD^\times\), and makes the following diagram commutative.
This involution is defined on the set of all representations of $\widetilde{\text{SL}}_2(F)$ whose fixed points are precisely the irreducible admissible genuine representations which are not discrete series representations. Denote this involution by $\tau \mapsto \tau_W$. This involution is independent of the character $\psi$ chosen to define it.

The following theorem summarizes some of the results of Waldspurger from [3] which are relevant to our analysis. This theorem is in terms of the local $\epsilon$-factors of Jacquet-Langlands, which we will use without reviewing.

**Theorem 1.** Let $\tau$ be an irreducible admissible genuine representation of $\widetilde{\text{SL}}_2(F)$. Let $\psi$ be a non-trivial additive character of $F$. For $a \in F^\times$, let $\chi_a$ be the quadratic character of $F^\times$ defined by $\chi_a(x) = (a, x)$ where $(-,-)$ denotes the Hilbert symbol with values in $\{\pm1\}$. Both the representations $\tau$ and $\tau^a$ of $\widetilde{\text{SL}}_2(F)$ are in the domain of theta correspondence (with respect to the character $\psi$) either with $\text{PGL}_2(F)$ or with $\text{PD}^\times$ if and only if

$$\epsilon(\theta(\tau) \otimes \chi_a) = \chi_a(-1)\epsilon(\theta(\tau)),$$

and then

$$\theta(\tau^a) \cong \theta(\tau) \otimes \chi_a.$$

If $\epsilon(\theta(\tau) \otimes \chi_a) = -\chi_a(-1)\epsilon(\theta(\tau))$, and if $\theta(\tau)$ is a representation of $\text{PGL}_2(F)$, then $\theta(\tau^a)$ is a representation of $\text{PD}^\times$ and vice-versa, and

$$\theta(\tau^a) = \theta(\tau)^{\text{JL}} \otimes \chi_a.$$

### 3. Multiplicity formula on restriction from $\widetilde{\text{GL}}_2(F)$ to $\widetilde{\text{SL}}_2(F)$

Let $\pi$ be an irreducible admissible genuine representation of $\widetilde{\text{GL}}_2(F)$. Let $\mu$ be a character of $\tilde{Z}$ and $\tau$ an irreducible representation of $\widetilde{\text{SL}}_2(F)$, which are compatible, such that $\mu\tau$ appears in $\pi$ restricted to $\widetilde{\text{GL}}_2(F)_+$. We have

$$\pi|_{\text{GL}_2(F)_+} = \bigoplus_{a \in F^\times/F^\times 2} (\mu^a \tau^a)$$

where $a \in F^\times/F^\times 2$ are elements of the split torus $T \cong F^\times \times F^\times$ of the form $\text{diag}(a, 1)$ in the group $\text{GL}_2(F)$. Since the restriction of $\mu\tau$ from $\widetilde{\text{GL}}_2(F)_+$ to $\widetilde{\text{SL}}_2(F)$ is $\tau$, the multiplicity with which the representation $\tau$ appears in $\pi$, to be denoted by $m(\pi, \tau)$, is given by

$$m(\pi, \tau) = \#\{a \in F^\times/F^\times 2 : \tau^a \cong \tau\}.$$ 

**Lemma 1.** For an irreducible admissible representation $\tau$ of $\widetilde{\text{SL}}_2(F)$, and $a \in F^\times$, we have

$$\tau \cong \tau^a \iff \begin{cases} \theta(\tau) \otimes \chi_a \cong \theta(\tau), \\ \chi_a(-1) = 1. \end{cases}$$
Proof. If $\tau \cong \tau^a$, then considering the central characters on both sides, we find that $\chi_a(-1) = 1$. Further, if $\tau \cong \tau^a$, then, in particular, they both have $\theta$ lifts either to $\text{PGL}_2(F)$ or $\text{PD}^\times$, and $\theta(\tau) = \theta(\tau^a)$. Thus from Theorem 1 due to Waldspurger, we deduce the assertion in the lemma. □

**Corollary 1.** The multiplicity of $\tau$ in $\tilde{\pi}$ is given by

$$m(\tilde{\pi}, \tau) = \# \left\{ a \in F^\times / F^\times_2 : \theta(\tau) \otimes \chi_a \cong \theta(\tau) \text{ and } \chi_a(-1) = +1 \right\}.$$

**Corollary 2.** The multiplicity of $\tau$ in $\tilde{\pi}$ is either the cardinality of the $L$-packet of $\text{SL}_2(F)$ determined by $\theta(\tau)$, or half the cardinality of the $L$-packet of $\text{SL}_2(F)$ determined by $\theta(\tau)$ depending on whether all characters $\chi$ of $F^\times$ with $\theta(\tau) \otimes \chi \cong \theta(\tau)$ have the property $\chi(-1) = 1$, or not.

It is known (cf. [3]) that for a representation $\pi$ of $\text{GL}_2(F)$,

$$m(\pi) := \# \{ a \in F^\times / F^\times_2 : \pi \cong \pi \otimes \chi_a \} \in \{1, 2, 4\},$$

and is the cardinality of the $L$-packet of $\text{SL}_2(F)$ determined by $\pi$.

The condition $\chi_a(-1) = 1$ is automatic in some situations; for example, if $-1 \in F^\times_2$. Thus we get $m(\tilde{\pi}, \tau)$ to be any of the following possibilities:

$$m(\tilde{\pi}, \tau) = 1, 2 \text{ or } 4$$

for some $p$-adic field for any $p$, including $p = 2$.

**Remark.** It is a theorem of Waldspurger [3] that the multiplicity of genuine automorphic representations of $\text{SL}_2(\mathbb{A}_F)$ is one where $F$ is now a number field. Therefore unlike the case of $\text{SL}_1(D)(\mathbb{A}_F)$, $D$ a quaternion division algebra over $F$, where higher local multiplicities result in higher global multiplicities, such is not the case for $\text{SL}_2(\mathbb{A}_F)$.

4. A lemma on Waldspurger involution

We recall that by a result of Waldspurger, for an irreducible admissible genuine discrete series representation $\tau$ of $\widetilde{\text{SL}}_2(F)$, the central characters of $\tau$ and $\tau_W$ are different. The group $\text{GL}_2(F)$, or what amounts to simply $F^\times$ sitting inside $\text{GL}_2(F)$ as $\{ (\alpha, 0) : \alpha \in F^\times \}$, acts on the set of irreducible representations of $\text{SL}_2(F)$ denoted by $\tau \mapsto \tau^a$ for $a \in F^\times$. Since a similar action produces an $L$-packet for $\text{SL}_2(F)$, whereas for $\widetilde{\text{SL}}_2(F)$, one defines an $L$-packet by taking $\tau$ and $\tau_W$, we investigate in this section if it can happen that $\tau_W \cong \tau^a$ for some $a \in F^\times$ and $\tau$ a discrete series representation of $\widetilde{\text{SL}}_2(F)$.

**Lemma 2.** Let $\tau$ be a discrete series representation of $\widetilde{\text{SL}}_2(F)$. Let $\psi$ be a non-trivial additive character of $F$ such that $\tau$ has $\theta$ lift to $\text{PGL}_2(F)$ with respect to $\psi$. Then there exists $a \in F^\times$ with $\tau^a \cong \tau_W$ if and only if for $\pi = \theta(\tau, \psi)$, we have

(i) $\pi \cong \pi \otimes \chi_a$,

(ii) $\chi_a(-1) = -1$.

**Proof.** Let $\pi = \theta(\tau, \psi)$ and $\theta(\tau_W, \psi) = \pi^{\text{JL}}$, where $\pi^{\text{JL}}$ denotes the representation of $\text{PD}^\times$ which is associated to $\pi$ via the Jacquet-Langlands correspondence. From Theorem 1 it follows that if $\epsilon(\pi \otimes \chi_a) = \chi_a(-1)\epsilon(\pi)$, then $\tau^a$ lift to $\text{PGL}_2(F)$ and not to $\text{PD}^\times$ and hence $\tau^a$ cannot be isomorphic to $\tau_W$. Thus if $\tau^a$ were isomorphic
to $\tau_W$, then we must have $\epsilon(\pi \otimes \chi_a) = -\chi_a(-1)\epsilon(\pi)$. In this case, by Theorem 1, $\tau^a$ lifts to $PD^\times$ and is $\pi^{JL} \otimes \chi_a$. Therefore

$$\tau^a \cong \tau_W \iff \begin{cases} (i) & \epsilon(\pi \otimes \chi_a) = -\chi_a(-1)\epsilon(\pi), \\ (ii) & \pi^{JL} \cong \pi^{JL} \otimes \chi_a. \end{cases}$$

The equations (i) and (ii) can be combined to say that

$$\tau^a \cong \tau_W \iff \begin{cases} (i) & \pi \cong \pi \otimes \chi_a, \\ (ii) & \chi_a(-1) = -1. \end{cases}$$

This completes the proof of the lemma. \(\square\)

As a consequence of Lemma 1 and Lemma 2 together with the result of Labesse-Langlands recalled earlier regarding the cardinality of an $L$-packet of $SL_2(F)$ determined by a representation of $GL_2(F)$, we obtain:

**Corollary 3.** Let $\tau$ be an irreducible genuine discrete series representation of $SL_2(F)$. Let $m_1 = \#\{\tau^a, (\tau_W)^a \mid a \in F^\times\}$, and let $m_2$ be the cardinality of the $L$-packet of $SL_2(F)$ determined by $\theta(\tau, \psi)$. Then

$$m_1 \cdot m_2 = 2|F^\times : F^\times^2|.$$

Let $Z$ and $B$ be the group of all scalar and upper triangular matrices in $GL_2(F)$ respectively. Let $\mu_1, \mu_2$ be two characters of $F^\times$. Define a character $(\mu_1, \mu_2)$ of $B$ given by $(\begin{smallmatrix} t & 0 \\ 0 & t \end{smallmatrix}) \mapsto \mu_1(t_1)\mu_2(t_2)$. A principal series representation of $GL_2(F)$ is defined to be $PS(\mu_1, \mu_2) := \text{ind}_{B}^{GL_2(F)}(\mu_1, \mu_2)$. If $\mu_2 = \mu_1^{-1}$, then the principal series representation $PS(\mu_1, \mu_1^{-1})$ factors through $PGL_2(F)$.

Let $\pi$ be a principal series representation of $PGL_2(F)$ with $\pi \otimes \chi_a \cong \pi$. Then $\pi = PS(\mu, \mu^{-1})$ for some character $\mu$ of $F^\times$ and $\pi \otimes \chi_a \cong \pi$ implies that $PS(\mu\chi_a, \mu^{-1}\chi_a) \cong PS(\mu, \mu^{-1})$. Thus $\mu\chi_a = \mu$ or $\mu^{-1}\chi_a = \mu$. Since $\chi_a$ is a non-trivial character, the only option is to have $\mu^2 = \chi_a$, i.e., if $\pi \otimes \chi_a \cong \pi$ for $\pi$ a principal series representation, then $\chi_a(-1) = \mu^2(-1) = 1$.

**Corollary 4.** Let $\tau$ be an irreducible admissible genuine representation of $\widetilde{SL}_2(F)$ such that $\theta(\tau)$ is an irreducible principal series representation of $PGL_2(F)$. Then for $m_1 = \#\{\tau^a \mid a \in F^\times\}$, and $m_2$ the cardinality of the $L$-packet of $SL_2(F)$ determined by $\theta(\tau, \psi)$,

$$m_1 \cdot m_2 = |F^\times : F^\times^2|.$$

**REFERENCES**


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