MULTIPLE SOLUTIONS FOR AN INDEFINITE ELLIPTIC PROBLEM WITH CRITICAL GROWTH IN THE GRADIENT

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Abstract. We consider the problem
\[ (P) \quad -\Delta u = c(x)u + \mu |\nabla u|^2 + f(x), \quad u \in H^1_0(\Omega) \cap L^\infty(\Omega), \]
where \( \Omega \) is a bounded domain of \( \mathbb{R}^N \), \( N \geq 3 \), \( \mu > 0 \) and \( c, f \in L^q(\Omega) \) for some \( q > \frac{N}{2} \) with \( f \geq 0 \). Here \( c \) is allowed to change sign and we assume that \( c^+ \neq 0 \).

We show that when \( c^+ \) and \( \mu f \) are suitably small this problem has at least two positive solutions. This result contrasts with the case \( c \leq 0 \), where uniqueness holds. To show this multiplicity result we first transform \((P)\) into a semilinear problem having a variational structure. Then we are led to the search of two critical points for a functional whose superquadratic part is indefinite in sign and has a so-called slow growth at infinity. The key point is to show that the Palais-Smale condition holds.

1. Introduction

Let \( \Omega \) be a bounded domain of \( \mathbb{R}^N \) with \( N \geq 3 \). In this paper we are concerned with the boundary value problem
\[ (P) \quad -\Delta u = c(x)u + \mu |\nabla u|^2 + f(x), \quad u \in H^1_0(\Omega) \cap L^\infty(\Omega), \]
where
\[ (H) \quad \mu > 0, \quad f \geq 0 \quad \text{and} \quad c, f \in L^q(\Omega) \quad \text{for some} \quad q > \frac{N}{2}. \]

Quasilinear elliptic equations with a gradient dependence up to the critical growth \( |\nabla u|^2 \) were first studied by Boccardo, Murat and Puel in the 1980’s [12,14] and have been an active field of research until now; see for example [2,18,19]. To situate our problem we underline that we are interested in bounded solutions. The main goal of this paper is to carry on the study of non-uniqueness of solutions for such problems, of which \( (P) \) is a prototype.

The sign of \( c \) plays in \( (P) \) a central role regarding uniqueness, as well as existence, of bounded solutions. We refer to [20] for a heuristic discussion on the influence of the sign of \( c \) on the nature of the problem. The case \( c \leq -\alpha_0 \) a.e. in \( \Omega \) for some \( \alpha_0 > 0 \) is referred to as the coercive case. In this case, the existence of solutions holds under very general assumptions and it was shown in [9,10] (see also [8,11])...
that there is a unique bounded solution. When one just requires \( c \leq 0 \) (in particular when \( c \equiv 0 \)) the situation is already more complex. The fact that restrictions on the data are necessary for (P) to have a solution was first observed in \([16,17]\). Concerning uniqueness, some partial results are given in \([9,10]\), but it was only in [6] that uniqueness of bounded solutions was established under the mere condition \( c \leq 0 \). See also [7] for an extension to a larger class of problems.

The case \( c \geq 0 \) started to be studied only recently; surely in part because it was not accessible by the methods traditionally used in the coercive case. In [20] it was shown that when \( c \geq 0 \) and \( c, \mu \) and \( f \) are sufficiently small in an appropriate sense, (P) has two solutions. See also [13,24] for related results. Note that the case where \( \mu \) is allowed to be non-constant was treated in [6] leading also, when \( c \geq 0 \) and under appropriate conditions, to the existence of two bounded solutions.

In view of these results it remained to analyse the case where \( c \) is allowed to change sign, which is the aim of the present paper. Roughly speaking we shall show that the uniqueness is lost as soon as \( c^+ \neq 0 \), where \( c^+ = \max\{0,c\} \); see Theorem 1.1.

We first observe that (P) is equivalent to

\[
(P') \quad -\Delta w = c(x) w + |\nabla w|^2 + \mu f(x), \quad w \in H^1_0(\Omega) \cap L^\infty(\Omega).
\]

Indeed, it is easy to check that \( u \) is a solution of (P) if and only if \( w = \mu u \) is a solution of \((P')\). Now, we use the change of variable

\[
v = e^w - 1,
\]

which goes back to [21] and rids the gradient term of \((P')\), reducing it to a semilinear problem with a variational structure; namely,

\[
(Q) \quad -\Delta v - (c(x) + \mu f(x)) v = c(x) g(v) + \mu f(x), \quad v \in H^1_0(\Omega) \cap L^\infty(\Omega),
\]

where

\[
g(s) = \begin{cases} 
(1 + s) \ln(1 + s) - s & \text{if } s \geq 0, \\
0 & \text{if } s \leq 0.
\end{cases}
\]

We shall prove in Lemma 2.1 that if \( v \) is a non-negative solution of \((Q)\), then \( w \) defined by \((1.1)\) is a non-negative (and therefore positive, by Harnack inequality) solution of \((P')\). Solutions of \((Q)\) will be obtained as critical points of the functional

\[
I(v) = \frac{1}{2} \int_\Omega \left[ |\nabla v|^2 - [c(x) + \mu f(x)](v^+)^2 \right] - \int_\Omega c(x) G(v^+) - \mu \int_\Omega f(x) v
\]

defined on \( H^1_0(\Omega) \) and where \( G(s) = \int_0^s g(t) \, dt \). Note that since \( f \geq 0 \), critical points of \( I \) are necessarily non-negative; see Lemma 2.1 Since \( g \) behaves essentially as \( s \ln s \) for \( s \) large, the superquadratic part of \( I \) has at infinity a growth which is usually referred to as a \textit{slowly superlinear growth}.

To obtain two critical points we start following the strategy used in [20]. Note that if the positive part of \( c + \mu f \) is not ‘too large’ in a suitable sense (cf. Lemma 2.2), then

\[
\int_\Omega \left[ |\nabla v|^2 - [c(x) + \mu f(x)](v^+)^2 \right]
\]

is coercive. Moreover, as \( g \) is superlinear, we shall prove that \( I \) takes positive values on a sphere \( \|v\| = \rho \) if either \( c \) or \( \mu f \) is sufficiently small. Moreover it is easily seen that since \( f \neq 0 \), \( I \) takes negative values in the ball \( B(0, \rho) \). Finally, since \( c^+ \neq 0 \), it is possible to show that \( I \) takes a negative value at some point \( v_0 \) outside of
the ball \( B(0, \rho) \). Thus \( I \) has a mountain-pass geometry and it is reasonable to search for a first critical point as a minimizer of \( I \) in \( B(0, \rho) \) and a second one at the mountain-pass level. The existence of a minimizer will follow from a standard lower semicontinuity argument, whereas in the proof of the existence of a mountain-pass critical point we will face the difficulty of showing that Palais-Smale sequences are bounded.

We recall that the Palais-Smale condition holds for \( I \) if any sequence \((u_n) \subset H^1_0(\Omega)\) such that \((I(u_n))) \subset \mathbb{R}\) is bounded and \( ||I'(u_n)||_s \to 0\) admits a convergent subsequence. The boundedness of such sequences proves to be a delicate issue due to the fact that \( c \) is sign-changing and \( g \) has a slow growth at infinity. In particular \( g \) does not satisfy an Ambrosetti-Rabinowitz type condition. Let us recall that a non-linearity \( f \) is said to satisfy the Ambrosetti-Rabinowitz condition if

\[(\text{AR}) \quad \text{There exist } \theta > 2 \text{ and } s_1 > 0 \text{ such that } 0 < \theta F(s) \leq sf(s) \quad \forall s \geq s_1, \]

where \( F(s) = \int_0^s f(t) \, dt \). This condition is known to be central when proving that Palais-Smale sequences are bounded. When the domain \( \Omega \subset \mathbb{R}^N \) is bounded and the non-linearity is subcritical, the boundedness leads directly to the strong convergence of a subsequence.

In the case where the superquadratic term is positive, many efforts have been made to weaken the condition \((\text{AR})\). However, to the best of our knowledge, this issue has not been considered for functionals of the type

\[ J(u) = \int_{\Omega} \left( \frac{1}{2} \nabla u^2 - c(x) F(u) \right), \quad u \in H^1_0(\Omega) \]

when \( c \) changes sign and \( f \) is a superlinear function not satisfying \((\text{AR})\). A typical example of such a non-linearity is \( f(s) = s \ln(s+1) \).

When \( f(s) = s^{p-1} \) with \( p \in [2, 2^*) \), using the homogeneity of \( f \) it is straightforward that \( J \) satisfies the Palais-Smale condition. When \( f \) is not powerlike, this issue becomes delicate, as shown in [5] (see also [4]), where the authors assume that \( f \) is superlinear and asymptotically powerlike at infinity, i.e.,

\[(\mathcal{G}) \quad \text{There exists } p > 2 \text{ such that } \lim_{s \to \infty} \frac{f(s)}{s^{p-1}} = 1. \]

Note that this condition implies \((\text{AR})\). Furthermore, in [5] one needs to assume the so-called thick zero set condition on \( c \in C(\overline{\Omega}) \):

\[(\text{AT}) \quad (\Omega_+) \cap (\overline{\Omega_-}) = \emptyset, \]

where

\[ \Omega_+ := \{ x \in \Omega; \ c(x) > 0 \} \quad \text{and} \quad \Omega_- := \{ x \in \Omega; \ c(x) < 0 \}. \]

In [23], still under \((\mathcal{G})\), the authors were able to remove \((\text{AT})\), but at the expense of some alternative strong conditions on \( c \).

In our problem we prove that the Palais-Smale condition is satisfied without assuming \((\text{AT})\) nor any special condition on \( c \). Given \( V \in L^q(\Omega) \), with \( q > \frac{N}{2} \), we denote by \( \lambda_1(V) = \lambda_1(V, \Omega) \) the first eigenvalue of the problem

\[
\begin{cases}
-\Delta u + V(x)u = \mu u & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]
Let us recall that \( \lambda_1(V) \) is given by
\[
\lambda_1(V) = \inf \left\{ \int_\Omega \left( |\nabla v|^2 + V(x)v^2 \right) ; \ u \in H^1_0(\Omega), \ \|u\|_2 = 1 \right\}.
\]
It is well known that \( \lambda_1(V) \) is simple, so that it is achieved by a unique \( \varphi_1 > 0 \) such that \( \|\varphi_1\|_2 = 1 \); cf. [22].

Our main result is the following:

**Theorem 1.1.** Assume \((\mathcal{H})\) and \( c^+ \not\equiv 0 \). Then \((P)\) has two positive solutions if either one of the following conditions hold:

1. \( \lambda_1(-\mu f) > 0 \) and \( \|c^+\|_q < K \), where \( K \) is a constant depending on \( f \) and \( \mu \).
2. \( \lambda_1(-c) > 0 \) and \( \|\mu f\|_q < K \), where \( K \) is a constant depending on \( c \).

**Remark 1.2.** In [20, Theorem 2], assuming \( c \not\equiv 0 \), it is proved that if
\[
(1.3) \quad \|\mu f\|_N^\frac{N}{q} < C_N,
\]
where \( C_N \) denotes the best Sobolev constant for the embedding \( H^1_0(\Omega) \subset L^2^* (\Omega) \), then there exists \( \bar{c} > 0 \) such that \((P)\) has at least two bounded solutions if \( \|c\|_q < \bar{c} \).

We observe that under \((1.3)\) we have \( \lambda_1(-c) > 0 \). Thus Theorem 1.1 (2) is consistent with [20, Theorem 2]. We point out however that \( f \) is allowed to be sign-changing in [20].

In [6] (see Corollary 3.2 and Remark 3.2) it is shown that when \( c \equiv 0 \), \((P)\) has a solution if and only if \( \lambda_1(-\mu f) > 0 \). We now complement this result.

**Lemma 1.3.** Assume \((\mathcal{H})\).

1. If \( c \geq 0 \), then \( \lambda_1(-c - \mu f) > 0 \) is necessary for \((P)\) to have a non-negative solution.
2. \( \lambda_1(-c) > 0 \) is necessary for \((P)\) to have a non-negative solution and under this condition every solution of \((P)\) is non-negative.

**Remark 1.4.** As far as non-negative solutions are concerned, Lemma 1.3 (1) shows that when \( c \geq 0 \) the condition \( \lambda_1(-c - \mu f) > 0 \) is necessary in Theorem 1.1. However, other kinds of solutions of \((P)\), namely negative or sign-changing solutions, may exist if \( \lambda_1(-c - \mu f) \leq 0 \). See [15] in this direction.

This paper is organized as follows. In Section 2 we prove some preliminary results and show that the functional \( I \) has the geometry described above. Section 3 is devoted to the Palais-Smale condition for \( I \). Finally in Section 4 we prove Theorem 1.1 and Lemma 1.3. Also in Remark 1.1 we discuss the necessity of some assumptions in Theorem 1.1.

### 1.1. Notation.

- The Lebesgue norm in \( L^r(\Omega) \) will be denoted by \( \| \cdot \|_r \) and the usual norm of \( H^1_0(\Omega) \) by \( \| \cdot \| \), i.e., \( \|u\| = \|\nabla u\|_2 \). The Holder conjugate of \( r \) is denoted by \( r' \).
- The weak convergence is denoted by \( \rightharpoonup \).
- The positive and negative parts of a function \( u \) are defined by \( u^\pm := \max\{\pm u, 0\} \).
- We denote by \( B(0,R) \) the ball of radius \( R \) centered at 0 in \( H^1_0(\Omega) \).
2. Preliminaries

Lemma 2.1. Assume (H).

1) If \( v \) is a non-negative solution of (Q), then \( w = \ln(1 + v) \) is a non-negative solution of \((P')\). Similarly if \( w \) is a non-negative solution of \((P')\), then \( v \) given by \( (\ref{1.1}) \) is a non-negative solution of (Q).

2) If \( v \) is a critical point of \( I \), then \( v \) is a non-negative solution of (Q).

3) If \( u \) is a non-negative solution of (P), then \( u \) is positive.

Proof. Let \( v \geq 0 \) be a solution of (Q). From the expression of \( g \) it is seen that \( v \) solves

\[
-\Delta v = c(x)(1 + v)\ln(1 + v) + \mu f(x)(1 + v). \tag{2.1}
\]

Let \( w = \ln(1 + v) \), i.e., \( e^w = 1 + v \). Since \( v \geq 0 \) and \( \nabla w = \frac{\nabla v}{1+v} \), one may easily see that \( w \in H^1_0(\Omega) \). If \( \phi \in H^1_0(\Omega) \), then \( \psi = \frac{\phi}{1+v} \in H^1_0(\Omega) \), so that \( (\ref{2.1}) \) provides

\[
\int_\Omega \nabla v \nabla \psi = \int_\Omega c(x)\psi(1+v)\ln(1+v) + \mu \int_\Omega f(x)\psi(1+v)
\]

(2.2)

Now, from \( \nabla v = e^w \nabla w \) and \( \nabla \psi = \frac{\nabla \phi}{1+v} - \frac{\phi \nabla v}{(1+v)^2} \), we get

\[
\int_\Omega \nabla v \nabla \psi = \int_\Omega e^w \nabla w \left( \frac{\nabla \phi}{1+v} - \frac{\phi \nabla v}{(1+v)^2} \right) = \int_\Omega \nabla w \left( \nabla \phi - \frac{\phi \nabla v}{1+v} \right)
\]

= \int_\Omega \nabla w \left( \nabla \phi - \frac{\phi e^w \nabla w}{1+v} \right) = \int_\Omega (\nabla w \nabla \phi - |\nabla w|^2 \phi).

Furthermore, we have

\[
\int_\Omega c(x)\phi \ln(1+v) = \int_\Omega c(x)w\phi,
\]

so we deduce from (2.2) that \( u \) is a solution of \((P')\). By similar arguments we prove the reverse statement. This proves (1).

To prove (2), let \( v \) be a critical point of \( I \). Then

\[
\int_\Omega [\nabla v \nabla \varphi - (c(x) + \mu f(x))v^+ \varphi] - \int_\Omega c(x)g(v^+)\varphi - \mu \int_\Omega f(x)\varphi = 0 \tag{2.3}
\]

for all \( \varphi \in H^1_0(\Omega) \). Taking \( \varphi = -v^- \) we get

\[
\int_\Omega |\nabla v^-|^2 + \mu \int_\Omega f(x)v^- = 0.
\]

Since \( f \geq 0 \), we get

\[
\int_\Omega |\nabla v^-|^2 \leq 0
\]

and it follows that \( v^- \equiv 0 \), i.e., \( v \geq 0 \). The proof that \( v \in L^\infty(\Omega) \) can be found in [20, Lemma 13], so we omit it.

Finally, if \( u \geq 0 \) is a solution of (P), then, since \( \mu > 0 \) and \( f \geq 0 \), \( u \) is a bounded weak supersolution of

\[
-\Delta u = c(x)u, \quad u \in H^1_0(\Omega).
\]

By a standard argument relying on the Harnack inequality (see [25, Theorem 1.2]) we have either \( u \equiv 0 \) or \( u > 0 \). Since \( f \geq 0 \), we get \( u > 0 \). \( \square \)
We shall now prove that when \( \lambda_1(-c - \mu f) > 0 \) the functional \( I \) takes positive values on a sphere centered at the origin if either \( \|c^+\|_q \) or \( \|\mu f\|_q \) is small enough.

**Lemma 2.2.** Let \( V \in L^q(\Omega) \), with \( q > \frac{N}{2} \). If \( \lambda_1(V) > 0 \), then there exists \( K_1 > 0 \) such that

\[
\int_{\Omega} (|\nabla v|^2 + V(x)(v^+)^2) \geq K_1 \|v\|^2 \quad \forall v \in H_0^1(\Omega).
\]

**Proof.** Let us first prove that there exists a constant \( K_1 > 0 \) such that

\[
\int_{\Omega} (|\nabla v|^2 + V(x)v^2) \geq K_1 \|v\|^2 \quad \forall v \in H_0^1(\Omega).
\]

Indeed, assume by contradiction that there is a sequence \( (v_n) \subset H_0^1(\Omega) \) such that

\[
\int_{\Omega} (|\nabla v_n|^2 + V(x)(v_n)^2) \leq \frac{\|v_n\|^2}{n}.
\]

Setting \( w_n = \frac{v_n}{\|v_n\|} \) we may assume that, up to a subsequence,

\[
w_n \rightharpoonup w_0 \text{ in } H_0^1(\Omega) \quad \text{and} \quad w_n \to w_0 \text{ in } L^r(\Omega) \quad \text{for } r \in [1, 2^*).
\]

In particular since \( q > \frac{N}{2} \) we have that \( w_n \to w_0 \) in \( L^{2q'}(\Omega) \). Thus from

\[
\int_{\Omega} (|\nabla w_n|^2 + V(x)(w_n)^2) \leq \frac{1}{n}
\]

it follows that

\[
\int_{\Omega} (|\nabla w_0|^2 + V(x)(w_0)^2) \leq 0.
\]

We claim that \( w_0 \not\equiv 0 \). Indeed, if \( w_0 \equiv 0 \), then \( w_n \rightharpoonup 0 \) in \( L^{2q'}(\Omega) \) and \( (2.6) \) yields \( w_n \to 0 \) in \( H_0^1(\Omega) \), which is impossible since \( \|w_n\| = 1 \). Hence \( w_0 \not\equiv 0 \) and consequently \( (2.7) \) provides \( \lambda_1(V) \leq 0 \), which contradicts our assumption. Thus \( (2.5) \) is proved. Finally, we may assume that \( K_1 \leq 1 \), so that

\[
\int_{\Omega} (|\nabla v|^2 + V(x)(v^+)^2) = \int_{\Omega} |\nabla v^-|^2 + \int_{\Omega} (|\nabla v^+|^2 + V(x)(v^+)^2) \geq \|v^-\|^2 + K_1 \|v^+\|^2 \geq K_1 \|v\|^2.
\]

We are now ready to prove that \( I \) has the appropriate geometry. Note that \( g \) given by \( (1.2) \) satisfies

\[
\lim_{s \to 0} \frac{g(s)}{s^p} = \lim_{s \to \infty} \frac{g(s)}{s^p} = 0
\]

if \( p \in (1, 2) \). As a consequence, there exists a constant \( C > 0 \) such that

\[
(2.8) \quad 0 \leq G(s) \leq Cs^{p+1}, \quad \forall s \in \mathbb{R}.
\]

**Proposition 2.3.** Assume that \( \lambda_1(-c - \mu f) > 0 \). Given \( R > 0 \) sufficiently large, there exist \( K, M > 0 \) depending on \( R \) and such that:

1. If \( \|c^+\|_q < K \), then \( I(v) \geq M \) for every \( v \in H_0^1(\Omega) \) with \( \|v\| = R \).
2. If \( \|\mu f\|_q < K \), then \( I(v) \geq M \) for every \( v \in H_0^1(\Omega) \) with \( \|v\| = R^{-1} \).
Proposition 3.1. If $\lambda_1 (-c - \mu f) > 0$, by Lemma 2.2 there exists $K_1 > 0$ such that

$$\int_{\Omega} \left( |\nabla v|^2 - |c(x) + \mu f(x)(v^+)^2 \right) \geq K_1 \|v\|^2 \quad \forall v \in H^1_0(\Omega).$$

Let $p \in (1, 2)$. By (2.8) we have

$$I(v) \geq K_1 \|v\|^2 - C_1 \|c^+\|_q \|v\|^{p+1} - C_2 \|\mu f\|_q \|v\|$$

for some $C_1, C_2 > 0$. If $\|v\| = R$ and $\|c^+\|_q \leq R^{-\beta}$, with $\beta > p - 1$, then

$$I(v) \geq K_1 R^2 - C_1 R^{p+1-\beta} - C_2 \mu R \geq R$$

for $R$ sufficiently large. Thus (1) holds with $K = R^{-\beta}$ and $M = R$.

In a similar way, if now $\|v\| = R^{-1}$ and $\|\mu f\|_q \leq R^{-\beta}$, with $\beta > 1$, then

$$I(v) \geq K_1 R^{-2} - C_1 \|c^+\|_q R^{-(p+1)} - C_2 R^{-\beta-1} \geq R^{-3}$$

for $R$ sufficiently large. Hence we may take $K = R^{-\beta}$ and $M = R^{-3}$ to get (2). □

3. The Palais-Smale condition

We set

$$\alpha_c = \inf \left\{ \int_{\Omega} \left( |\nabla u|^2 - \mu f(x)(u^+)^2 \right) ; u \in H^1_0(\Omega), \|u\|_2 = 1, cu^+ \equiv 0 \right\}.$$

In the next proposition, we shall use an explicit expression of $G$; namely,

$$G(s) = \frac{s^2}{2} \ln(s + 1) - \frac{3}{4} s^2 + s \ln(s + 1) - \frac{s}{2} + \frac{1}{2} \ln(s + 1)$$

for $s > 0$.

Proposition 3.1. If $\alpha_c > 0$, then $I$ satisfies the Palais-Smale condition.

Proof. Let $(u_n)$ be a Palais-Smale sequence for $I$ at the level $d \in \mathbb{R}$, i.e.,

$$I(u_n) \to d \quad \text{and} \quad \|I'(u_n)\|_* \to 0.$$

From (3.2) we have

$$\frac{1}{2} \int_{\Omega} \left[ |\nabla u_n|^2 - (c(x) + \mu f(x))(u_n^+)^2 \right] - \int_{\Omega} c(x) G(u_n^+) - \mu \int_{\Omega} f(x) u_n = d + o(1)$$

and

$$\left| \int_{\Omega} \left[ |\nabla u_n \nabla \varphi - (c(x) + \mu f(x))u_n^+ \varphi \right] - \int_{\Omega} c(x) g(u_n^+) \varphi - \mu \int_{\Omega} f(x) \varphi \right| \leq \varepsilon_n \|\varphi\|$$

for some sequence $\varepsilon_n \to 0$ and for every $\varphi \in H^1_0(\Omega)$. In particular, we have

$$|I'(u_n), u_n)| \leq \varepsilon_n \|u_n\|.$$  

Let us assume that $\|u_n\| \to \infty$ and set $v_n = \frac{u_n}{\|u_n\|}$. Up to a subsequence, we have $v_n \to v_0$ in $H^1_0(\Omega)$, $v_n \to v_0$ in $L^r(\Omega)$, $\forall r \in [1, 2^*)$, and $v_n \to v_0$ a.e. in $\Omega$.

We claim that $cv_0^+ \equiv 0$. Indeed, from (3.4) we have, using the convergences above,

$$\int c(x) g(u_n^+) \|u_n\| \varphi = \int \left[ |\nabla v_0 \nabla \varphi - (c(x) + \mu f(x))v_0^+ \varphi \right] + o(1) < \infty,$$
for every \( \varphi \in H^1_0(\Omega) \). If \( cv_0^+ \neq 0 \), then we may choose \( \varphi \in H^1_0(\Omega) \) and a measurable subset \( \Omega_\varphi \subset \Omega \) such that
\[
|\Omega_\varphi| > 0, \quad cv_0^+ \varphi > 0 \text{ on } \Omega_\varphi \subset \Omega, \quad \text{and } cv_0^+ \varphi = 0 \text{ on } \Omega \setminus \Omega_\varphi.
\]
Now, using that \( \lim_{s \to \infty} \frac{g(s)}{s} = \infty \), we have
\[
\liminf c(x) \frac{g(u_n^+)}{\|u_n\|} \varphi = \liminf c(x)\|u_n\|v_n^+ \varphi = +\infty \quad \text{on } \Omega_\varphi.
\]
Fatou's lemma then yields a contradiction with (3.6). Therefore \( cv_0^+ \equiv 0 \). On the other hand, taking \( \varphi = v_0 \) in (3.4) and dividing it by \( \|u_n\| \) we get
\[
\int_\Omega [\nabla v_n \cdot \nabla v_0 - (c(x) + \mu f(x))v_n^+ v_0] \to 0,
\]
so that, using \( v_n \to v_0 \) in \( H^1_0(\Omega) \) and \( cv_0^+ \equiv 0 \), we get
\[
\int_\Omega [\|\nabla v_0\|^2 - \mu f(x)(v_0^+)^2] = 0.
\]
Thus \( v_0 \equiv 0 \) (otherwise \( \alpha_c \leq 0 \)). Now from (3.4) we have, taking \( \varphi = u_n \) and using the definition (1.2) of \( g \),
\[
(3.7) \quad \left| \int_\Omega (\|\nabla u_n\|^2 - \mu f(x))(u_n^+)^2 - \int_\Omega c(x)(1 + u_n^+) \ln(1 + u_n^+) u_n^+ - \mu \int_\Omega f(x) u_n^+ \right| \leq \varepsilon_n \|u_n\|.
\]
Dividing by \( \|u_n\|^2 \) and using that \( v_n \to 0 \) in \( L^r(\Omega) \), \( \forall r \in [1, 2*) \) we get
\[
1 - \int_\Omega c(x)(v_n^+)^2 \ln(1 + \|u_n\|v_n^+) \to 0.
\]
Now, using the property \( \ln(st) = \ln s + \ln t \), it follows that
\[
1 - \ln(\|u_n\|) \int_\Omega c(x)(v_n^+)^2 - \int_\Omega c(x)(v_n^+)^2 \ln \left( v_n^+ + \frac{1}{\|u_n\|} \right) \to 0.
\]
We claim that
\[
(3.8) \quad \ln(\|u_n\|) \int_\Omega c(x)(v_n^+)^2 \to 0.
\]
In that case we would get
\[
\int_\Omega c(x)(v_n^+)^2 \ln \left( v_n^+ + \frac{1}{\|u_n\|} \right) \to 1,
\]
which clearly contradicts the fact that \( v_0 = 0 \). To prove (3.8) we define for every \( s > 0 \)
\[
H(s) = \frac{1}{2}g(s)s - G(s).
\]
From (1.2) and (3.1) it follows that
\[
(3.9) \quad H(s) = \frac{s^2}{4} - s \ln(s + 1) + \frac{s}{2} - \frac{1}{2} \ln(1 + s).
\]
From (3.5) we get
\[
I(u_n) - \frac{1}{2} \langle I'(u_n), u_n \rangle = c + \varepsilon_n \|u_n\| + o(1),
\]
which leads, using the definition of \( H \), to
\[
\int_{\Omega} c(x) H(u_n) - \frac{\mu}{2} \int_{\Omega} f(x) u_n = c + \varepsilon_n \| u_n \| + o(1).
\]
(3.10)

Now, combining (3.9) and (3.10), we obtain
\[
\frac{1}{4} \int_{\Omega} c(x) (u_n^+)^2 = c + \varepsilon_n \| u_n \| + \frac{1}{2} \int_{\Omega} c(x) u_n^+ - \int_{\Omega} c(x) u_n^+ \ln(1 + u_n^+)
\]
\[
+ \frac{1}{2} \int_{\Omega} c(x) \ln(1 + u_n^+) + \frac{\mu}{2} \int_{\Omega} f(x) u_n + o(1).
\]
Hence
\[
\ln(\| u_n \|) \int_{\Omega} c(x) (u_n^+)^2 = \frac{4 \ln(\| u_n \|)}{\| u_n \|^2} (c + \varepsilon_n \| u_n \|) + \frac{1}{2} \int_{\Omega} c(x) u_n^+ - \int_{\Omega} c(x) u_n^+ \ln(1 + u_n^+)
\]
\[
+ \frac{1}{2} \int_{\Omega} c(x) \ln(1 + u_n^+) + \frac{\mu}{2} \int_{\Omega} f(x) u_n + o(1) \rightarrow 0.
\]
Thus (3.8) is proved and we reach a contradiction. Therefore \( (u_n) \) must be bounded and, up to subsequence, we have \( u_n \rightharpoonup u_0 \) in \( H^1_0(\Omega) \) and \( u_n \rightarrow u_0 \) in \( L^p(\Omega) \) for \( p \in [1, 2^*) \). At this point the strong convergence follows in a standard way. We refer to [20] Lemma 11 for a proof.

**Corollary 3.2.** If \( \lambda_1(-c - \mu f) > 0 \), then \( I \) satisfies the Palais-Smale condition.

**Proof.** Let \( \| u \|_2 = 1 \) with \( cu^+ \equiv 0 \). Since \( \lambda_1(-c - \mu f) > 0 \), by Lemma 2.2 there is a constant \( K_1 > 0 \) such that
\[
\int_{\Omega} (|\nabla u|^2 - \mu f(x) (u^+)^2) = \int_{\Omega} (|\nabla u|^2 - (c + \mu f(x))(u^+)^2)
\]
\[
\geq K_1 \| u \|^2 \geq SK_1 \| u \|^2 = SK_1 > 0,
\]
where \( S \) is the best Sobolev constant for the embedding \( H^1_0(\Omega) \subset L^2(\Omega) \). Thus \( \alpha > 0 \) and by Proposition 3.1 we get the conclusion.

\[
4. \text{Proof of Theorem 1.1 and Lemma 1.3}
\]

We are now ready to prove our main results.

**Proof of Theorem 1.1** First of all, we fix \( K > 0 \) such that \( \lambda_1(-c - \mu f) > 0 \) if either \( \lambda_1(-\mu f) > 0 \) and \( c^+ \| f \|_q < K \) or \( \lambda_1(-c) > 0 \) and \( \| f \|_q < K \). This is possible in view of the continuity of \( \lambda_1(V) \) with respect to \( V \in L^q(\Omega) \). Decreasing \( K \) if necessary, we fix \( R \) sufficiently large so that, by Proposition 2.3 if \( c^+ \| f \|_q < K \) (respect. \( \| f \|_q < K \) then \( I(v) \geq M > 0 \) for \( \| v \| = R \) (respect. \( \| v \| = R^{-1} \)). We set \( \rho = R \) if \( c^+ \| f \|_q < K \) and \( \rho = R^{-1} \) if \( \| f \|_q < K \). It is easily seen that if \( f \neq 0 \), then \( I \) takes negative values in the ball \( B(0, \rho) \). Therefore, by weak lower semi-continuity, we infer that if either \( c^+ \| f \|_q < K \) or \( \| f \|_q < K \), then the infimum of \( I \) in \( B(0, \rho) \) is achieved by some \( v_0 \neq 0 \), which is a critical point of \( I \). Furthermore, since \( G(s)/s^2 \rightarrow \infty \) as \( s \rightarrow \infty \), if \( v \in H^1_0(\Omega) \) is such that \( \int_{\Omega} c(x) G(v^+) > 0 \), then \( I(tv) \rightarrow -\infty \) as \( t \rightarrow \infty \). We fix \( t > 0 \) and \( v \) such that \( v_0 = tv \) satisfies \( \| v_0 \| > \rho \) and \( I(v_0) < 0 \). Now let
\[
\Gamma := \{ \gamma \in C([0, 1], H^1_0(\Omega)); \gamma(0) = 0, \gamma(1) = v_0 \}
\]
and
\[
d := \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} I(\gamma(t)).
\]
Since $I$ satisfies the Palais-Smale condition, by the mountain-pass theorem it is straightforward that $I$ has a critical point $w_1$, which, by Proposition 2.3 satisfies $I(w_1) = d > 0$. In particular, we have $w_0 \neq w_1$. Finally, from Lemma 2.1 we know that these two critical points provide two positive solutions of $(P')$, and consequently, two positive solutions of $(P)$.

Proof of Lemma 1.3  By Lemma 2.1 we know that if $u \geq 0$ is a solution of $(P)$, then $u$ is positive so that $w = \mu u$ is a positive solution of $(P')$. Thus $v$ given by $1.1$ is a positive solution of $(Q)$. Taking $\phi > 0$, the first positive eigenfunction associated to $\lambda_1(-c - \mu f)$, as test function and using that $g \geq 0$ on $\mathbb{R}$ we obtain

$$ \int_\Omega (\nabla v \nabla \phi - c(x)v\phi - \mu f(x)v\phi) = \int_\Omega (c(x)g(v)\phi + \mu f(x)\phi) > 0, $$

so that

$$ \lambda_1(-c - \mu f) \int_\Omega v\phi > 0. $$

Thus $\lambda_1(-c - \mu f) > 0$.

Similarly, let $\varphi > 0$ be an eigenfunction associated to $\lambda_1(-c)$ and assume that $u \geq 0$ is a solution of $(P)$. Taking $\varphi > 0$ as test function we get

$$ \int_\Omega (\nabla u \nabla \varphi - c(x)u\varphi) = \int_\Omega (\mu |\nabla u|^2 \varphi + f(x)\varphi) > 0. $$

Thus

$$ \lambda_1(-c) \int_\Omega u\varphi > 0, $$

so that $\lambda_1(-c) > 0$. Finally, let $u$ be a solution of $(P)$. Using $u^-$ as test function in $(P)$, we obtain

$$ - \int_\Omega (|\nabla u^-|^2 - c(x)|u^-|^2) = \int_\Omega (\mu |\nabla u^-|^2 + f(x)u^-) \geq 0. $$

Hence

$$ \int_\Omega (|\nabla u^-|^2 - c(x)|u^-|^2) \leq 0, $$

so that under the condition $\lambda_1(-c) > 0$ we get $u^- \equiv 0$, i.e., $u \geq 0$. 

Our last result shows that when $\lambda_1(-c) > 0$ a restriction on the size of $\mu f$ is necessary in Theorem 1.1.

Remark 4.1. Assume $(H)$, $\lambda_1(-c) > 0$, and $c \geq 0$ in some open set $\Omega_0 \subset \Omega$. Then there exist an $R > 0$ and an $f \in L^q(\Omega)$ with $\|\mu f\|_q = R$ such that $(P)$ has no non-negative solutions.

Proof. Equivalently we shall prove that $(P')$ has no non-negative solutions. We choose $\phi \in C_0^\infty(\Omega_0)$ and $f \in L^q(\Omega)$ such that $f > 0$ on sup $\phi$. In particular we have

$$ \int_\Omega f(x)\phi^2 > 0. $$

By Cauchy-Schwartz inequality we have

$$ \int_\Omega \nabla u \nabla (\phi^2) = \int_\Omega 2\phi \nabla u \nabla \phi \leq \int_\Omega |\nabla \phi|^2 + |\nabla u|^2 \phi^2. $$
Now assume that \( (P') \) has a non-negative solution. Using \( \phi^2 \) as test function in \( (P') \) and (4.2) we get
\[
\int_\Omega |\nabla \phi|^2 \geq \int_\Omega c(x)u\phi^2 + \mu \int_\Omega f(x)\phi^2 \geq \mu \int_\Omega f(x)\phi^2.
\]
Because of (4.1) we get a contradiction for \( \mu > 0 \) large enough. \( \square \)

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