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REMARKS ON THE OBRECHKOFF INEQUALITY

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ABSTRACT. Let u be the logarithmic potential of a probability measure μ in the plane that satisfies

$$u(z) = u(\overline{z}), \quad u(z) \leq u(|z|), \quad z \in \mathbb{C},$$

and $m(t) = \mu\{z \in \mathbb{C}^* : |\operatorname{Arg} z| \leq t\}$. Then

$$\frac{1}{a}\int_0^a m(t)dt \leq \frac{a}{2\pi},$$

for every $a \in (0,\pi)$. This improves and generalizes a result of Obrechkoff on zeros of polynomials with positive coefficients.

1. Introduction

Distribution of zeros of polynomials with positive coefficients is an old subject going back to Poincaré [6]. For some recent results we mention [1] and references there.

Obrechkoff [5] proved that for every polynomial P of degree d with non-negative coefficients, and every $\alpha \in (0, \pi/2)$, the number of roots in the sector $\{z \in \mathbb{C}^* : |\operatorname{Arg} z| \leq \alpha\}$ is at most $2\alpha d/\pi$.

A general question about distribution of roots of polynomials with non-negative coefficients was asked by Subhro Ghosh and Ofer Zeitouni [7] in connection with their research on the large deviation theorems for zeros of random polynomials [3].

For each polynomial of degree d, we consider the *empirical measure* which is a probability measure in the plane consisting of atoms of charge m/d at every root of multiplicity m. The question of Ghosh and Zeitouni was to describe the closure of empirical measures of polynomials with positive coefficients.

Obrechkoff's inequality implies that every measure μ in this closure must satisfy

$$\mu\{z \in \mathbb{C}^* : |\operatorname{Arg} z| \leq \alpha\} \leq \frac{2\alpha}{\pi}, \tag{1}$$

for every $\alpha \in (0, \pi/2)$.

A complete description of the closure was given in [2]. It is evident that every polynomial with non-negative coefficients satisfies

$$|P(z)| \leq P(|z|),$$

and that the empirical measure of P is symmetric with respect to the real axis.

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For every finite measure μ in the plane, we define the potential

$$u_\mu(z) = \int_{|\zeta|\leq 1} \log|z-\zeta| d\mu + \int_{|z|>1} \log|1-z/\zeta|\, d\mu. \tag{2}$$

Theorem A ([2]). *A measure μ belongs to the closure of empirical measures of polynomials with positive coefficients if and only if $\mu(\mathbb{C}) \leq 1$, μ is symmetric with respect to the real axis, and*

$$u_\mu(z) \leq u_\mu(|z|), \quad z \in \mathbb{C}. \tag{3}$$

Theorem A is proved by approximation of arbitrary potential satisfying (3) and $u(z) = u(\overline{z})$ by potentials of the form $\log|P|/\deg P$, where P is a polynomial with positive coefficients.

Combining Theorem A with Obrechkoff's inequality, one concludes that for every finite measure μ, symmetric with respect to the real line, condition (3) implies (1). The proof of theorem A is complicated, and it is desirable to obtain a direct potential-theoretic proof of the implication (3)→(1). Such a proof will be given in this paper. In fact we will prove a stronger statement.

Theorem 1. *Let μ be a probability measure in the plane, symmetric with respect to the real line, whose potential* (2) *satisfies* (3)*. Then the function*

$$m(t) = \mu\{z \in \mathbb{C}^* : 0 \leq |\operatorname{Arg} z| \leq t\} \tag{4}$$

satisfies

$$\frac{1}{a}\int_0^a m(t)\, dt \leq \frac{a}{2\pi}, \quad 0 \leq a \leq \pi. \tag{5}$$

For the uniform distribution on the unit circle, we have $m(t) = t/\pi$, and equality holds in (5) for all a. Obrechkoff's inequality (1) is an immediate corollary of (5): setting $a = 2\alpha$, we obtain

$$m(\alpha) \leq \frac{1}{\alpha}\int_\alpha^{2\alpha} m(t)dt \leq \frac{2}{a}\int_0^a m(t)dt \leq \frac{a}{\pi} = \frac{2\alpha}{\pi}. \tag{6}$$

Next we discuss the possibility of equality in (1). For the polynomial $P(z) = z^d + 1$ with non-negative coefficients and $\alpha = \pi/d$, we have equality in (1). Thus (1) is exact for each α of the form $\pi/d,\ d = 2, 3, 4, 5, \ldots$. The second result of this paper is that in fact (1) is best possible for all α. For each $\alpha \in (0, \pi/2)$ we will find a probability measure μ symmetric with respect to the real axis, satisfying (3) and such that equality holds in (1). Then it follows from Theorem A that the right hand side of (1) cannot be replaced by a smaller number if the resulting inequality must hold for empirical measures of all polynomials with non-negative coefficients.

2. Proof of Theorem 1

Without loss of generality we assume that the closed support of μ is bounded and does not contain 0: it was shown in [2] that arbitrary finite measure satisfying (3) can be approximated by a measure with such a support which also satisfies (3).

Then it is sufficient to consider a potential of the form

$$u(z) := \int_{\mathbb{C}} \log|1 - z/\zeta|\, d\mu(\zeta),$$

which differs from (2) by an additive constant, and hence, also satisfies (3).

For a fixed $\rho \in (0,1)$, consider the function

$$v_\rho(z) = \int_0^\infty u(z/t)t^{\rho-1}dt.$$

This function is subharmonic and homogeneous,

$$v_\rho(\lambda z) = \lambda^\rho v_\rho(z), \quad \text{for every} \quad \lambda > 0,$$

therefore it has the form

$$v_\rho(re^{i\theta}) = r^\rho h_\rho(\theta). \tag{7}$$

To relate h with μ, we need the integral

$$\int_0^\infty \log\left|1 - \frac{z}{t}\right| t^{\rho-1}dt = c_\rho r^\rho \cos\rho(\theta-\pi), \quad z = re^{i\theta},\ 0 \le \theta \le 2\pi, \tag{8}$$

where $c_\rho = \pi/(\rho \sin \pi\rho)$. To see that (8) holds, we notice that the left hand side is a homogeneous subharmonic function of degree ρ, so it has the form (7). This function is harmonic in the complement of the positive ray, so it has the form as in the right hand side of (8). To find c_ρ we plug $z = -1$ into (8).

Let us define ϕ_ρ as the 2π-periodic extension of $\cos\rho(\theta-\pi)$, $0 \le \theta \le 2\pi$. Then we have

$$v_\rho(re^{i\theta}) = \int_0^\infty \int_{\mathbb{C}} \log\left|1 - \frac{re^{i\theta}}{t\zeta}\right| d\mu(\zeta)t^{\rho-1}dt = c_\rho r^\rho \int_{\mathbb{C}} \phi_\rho(\theta - \arg\zeta)\frac{d\mu(\zeta)}{|\zeta|^\rho}.$$

Comparing this with (7), we obtain

$$h_\rho(\theta) = \int_{\mathbb{T}} \phi_\rho(\theta - t)d\nu_\rho(t), \tag{9}$$

where $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ is the unit circle, and

$$\nu_\rho(E) = c_\rho \int_{\arg\zeta\in E} \frac{d\mu(\zeta)}{|\zeta|^\rho}, \tag{10}$$

for every Borel set $E \subset \mathbb{T}$. When $\rho \to 0$, $\nu_\rho/c_\rho \to \nu_0$, where ν_0 is proportional to the radial projection of the measure μ, so $m(t) = \nu_0[-t,t]$.

Inequality (3) and symmetry $u(z) = u(\overline{z})$ imply that

$$2h_\rho(0) - h_\rho(a) - h_\rho(-a) \ge 0, \quad a \in [0,\pi]. \tag{11}$$

Using the expression (9), we conclude that

$$\int_{\mathbb{T}} J_\rho(t)d\nu_\rho(t) \ge 0,$$

where

$$J_\rho(t) = 2\phi_\rho(t) - \phi_\rho(t-a) - \phi_\rho(t+a).$$

Now we divide by ρ^2 and pass to the limit $\rho \to 0$, using $\cos t \sim 1 - t^2/2$. A simple direct computation shows that $J_\rho/\rho^2 \to J$, where

$$J(t) = \begin{cases} 2\pi|t| - 2\pi a + a^2, & |t| \le a, \\ a^2, & a < |t| \le \pi. \end{cases}$$

Notice that J_ρ is 2π-periodic and satisfies the distributional equation

$$J_\rho'' + \rho^2 J = 2\rho\sin\pi\rho(2\delta_0 - \delta_a - \delta_{-a}),$$

where δ's are 2π-periodic delta functions. Therefore J is a 2π-periodic solution with zero average of the distributional equation

$$J'' = 2\pi(2\delta_0 - \delta_{-a} - \delta_a),$$

where the δ's are the 2π-periodic delta functions. This property defines J uniquely, which permits us to write it without any computation.

We conclude that

$$\int_0^\pi J(t)\, d\nu_0(t) \geq 0, \quad \text{and thus} \quad \int_0^\pi J(t)\, dm(t) \geq 0.$$

Integrating the last integral by parts, we obtain

$$2\pi \int_0^a m(t)\, dt \leq J(\pi)m(\pi) = a^2,$$

which is equivalent to (5).

3. Example

In this section, for any given $\alpha \in (0, \pi/2)$, we construct a probability measure μ symmetric with respect to the real line, and satisfying (3), such that Obrechkoff's inequality (1) holds with equality.

Inequalities (6) suggest that the sectors $|\mathrm{Arg}\, z| < \alpha$ and $|\mathrm{Arg}\, z| \in (\alpha, 2\alpha)$ must be free of the measure.

Potential

$$u(z) := \log|z^2 + 1|$$

satisfies (3), and its total Riesz' measure equals 2. Take $\alpha \in (0, \pi/2)$ and define the function

$$u_\alpha(z) := \begin{cases} u(z^{\pi/(2\alpha)}), & |\mathrm{Arg}\,(z)| < 2\alpha, \\ u(|z|^{\pi/(2\alpha)}), & \text{otherwise.} \end{cases}$$

This function is subharmonic (its Laplacian will be computed below). It is clear that u_α satisfies (3). Let λ_α be the Riesz' measure of u_α. One should notice that λ_α is supported on the set

$$\{z : |\mathrm{Arg}\,(z)| \geq 2\alpha\} \cup \{e^{i\alpha}\} \cup \{e^{i\alpha}\}\,.$$

Notice that $\lambda_\alpha\{e^{\pm i\alpha}\} = 1$, and λ_α is absolutely continuous on $\{z : |\mathrm{Arg}\,(z)| \geq 2\alpha\}$ with respect to the plane Lebesgue measure, and its density is

$$\rho_\alpha = \frac{1}{2\pi}\Delta u_\alpha.$$

Since $u_\alpha(e^{i\theta})$ does not depend on θ for $|\theta| \in (2\alpha, \pi)$, we compute the Laplacian Δu_α in polar coordinates ($z = re^{i\phi}$) as follows:

$$\begin{aligned}\rho_\alpha(r^{i\phi}) &= \frac{1}{2\pi r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\alpha}{\partial r}\right) = \frac{1}{2\pi r}\frac{d}{dr}\left(r\frac{d}{dr}\log(1 + r^{\pi/\alpha})\right) \\ &= \frac{1}{2\alpha r}\frac{d}{dr}\left(\frac{r^{\pi/\alpha}}{1 + r^{\pi/\alpha}}\right).\end{aligned}$$

Thus,

$$\lambda_\alpha\{z : |\mathrm{Arg}\,(z)| \geq 2\alpha\}) = \frac{(2\pi - 4\alpha)}{2\alpha}\int_0^\infty r\rho_\alpha(r)\, dr = \frac{\pi - 2\alpha}{\alpha}\,,$$

and

$$\lambda_\alpha\{\mathbb{C}\} = 2 + \frac{\pi - 2\alpha}{\alpha} = \frac{\pi}{\alpha}\,.$$

Then we define normalized measure $\mu_\alpha := \lambda_\alpha/\lambda_\alpha(\mathbb{C})$, and

$$\mu_\alpha\{e^{\pm i\alpha}\} = \frac{\alpha}{\pi}.$$

So the measure μ_α satisfies the equation

$$\mu_\alpha\{|\mathrm{Arg}\,(z)| \le \alpha\} = \frac{2\alpha}{\pi}.$$

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