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FREDHOLMNESS VS. SPECTRAL DISCRETENESS FOR FIRST-ORDER DIFFERENTIAL OPERATORS

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Abstract. It is shown that for essentially self-adjoint first-order differential operators D, acting on sections of bundles over complete (non-compact) manifolds, Fredholmness vs. Spectral Discreteness is the same as '$\exists c > 0$, D is c-invertible at infinity' vs. '$\forall c > 0$, D is c-invertible at infinity'. An application involving the spectral theory of electromagnetic Dirac operators is then given.

0. Introduction

It has been known since Hermann Weyl [W] that the self-adjoint operators T with discrete spectrum are precisely those for which $T - \lambda I$ is a Fredholm operator for every $\lambda \in \mathbf{R}$. Now, for first-order differential operators D acting on sections of bundles S over manifolds M we have shown [A] that Fredholmness is equivalent to the existence of a certain c-invertibility, $c > 0$, of D at infinity (i.e., outside requisite compact subsets of M).

The purpose of this note, which should have been written twenty years ago as a companion to [A], is to emphasize that in every respect, including the ease of verifiability, Fredholmness vs. Spectral Discreteness for D is the same as '$\exists c > 0$, D is c-invertible at infinity' vs. '$\forall c > 0$, D is c-invertible at infinity'.

We now briefly recall the set-up of [A]. Let S be a Hermitian vector bundle over a complete (non-compact) oriented smooth-C^∞ Riemannian manifold M. The inner product and the norm on $L^2(S)$, the Hilbert space of L^2-integrable sections of S, will be denoted by $(\cdot,\cdot)$, respectively $||\cdot||$. They are induced as usual by the pointwise inner product $\langle\cdot,\cdot\rangle$ on S and the canonical volume form $dvol$ on M.

Let D be a formally self-adjoint elliptic first-order differential operator on $C_0^\infty(S)$, the space of compactly supported smooth sections of S. We assume that D is also essentially self-adjoint, i.e., its minimal (graph) and maximal (distributional) closures in the L^2-space coincide. The unique closed extension of D to the L^2-space will also be denoted by D. Its domain, the first Sobolev space $W^1(S)$, consists in sections $s \in L^2(S)$ such that the distributional image Ds is in $L^2(S)$. $W^1(S)$ is a Hilbert space equipped with the inner product $(\cdot,\cdot)_1 = (\cdot,\cdot) + (D\cdot, D\cdot)$. It is also the completion of $C_0^\infty(S)$ with respect to the norm $||\cdot||_1$. Similarly the second Sobolev space $W^2(S)$ is the completion of $C_0^\infty(S)$ with respect to the norm $||\cdot||_2$ induced by the inner product $(\cdot,\cdot)_2 = (\cdot,\cdot)+(D\cdot, D\cdot)+(D^2\cdot, D^2\cdot)$. $W^2(S)$ is also the

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minimal L^2-closure of D^2 with domain $C_0^\infty(S)$. We ignore whether D^2 is essentially self-adjoint, although this happens often. The following symbol formula holds for D:

$$D(fs) = \sigma(df)s + fD(s), \quad f \in C^\infty(M),\ s \in C^\infty(S), \tag{1}$$

where $\sigma(\xi) \in End(S_x)$, $x \in M$, $\xi \in T_x^*M$ is the principal symbol map.

Such operators appear frequently in geometry and mathematical physics. For instance the Euclidean electromagnetic Dirac operators, and more generally the generalized Dirac operators introduced by Gromov and Lawson [GL] or the operators with 'infinite propagation time' introduced by Chernoff [C] are all of this type.

One of the key results of [A] is this:

Theorem F. *The following statements about essentially self-adjoint first-order elliptic differential operators are equivalent:*

(F_1) *The bounded operator $D : W^1(S) \to L^2(S)$ is a Fredholm operator, i.e., it has closed range and finite dimensional kernel and cokernel.*

(F_2) *There is a constant $c > 0$, for which there are a bounded positive (in the L^2-sense) operator $P : W^2(S) \to L^2(S)$, a bundle morphism $R \in End(S)$, and a compact subset $K \Subset M$, such that pointwise on $S|_{M\setminus K}$, $R \geq c$ and on $W^2(S)$,*

$$D^2 = P + R.$$

(F_3) *There is a constant $c > 0$ for which there is a compact subset $K \Subset M$ such that*

$$||D(s)|| \geq c||s||, \quad s \in W^1(S),\ \mathrm{supp}(s) \cap K = \emptyset.$$

Clearly, in (F_2) and (F_3) the respective Sobolev spaces can be replaced by $C_0^\infty(S)$.

1. The main result

In stating our main result, all we need to do is replace in Theorem F Fredholmness by Spectral Discreteness in (F_1), and the existential quantifier $\exists c > 0$ by the universal quantifier $\forall c > 0$ in (F_2) and (F_3). For the sake of completeness, we rewrite the statement of the new theorem, and then proceed to prove it.

Theorem SD. *The following statements about essentially self-adjoint first-order elliptic differential operators are equivalent:*

(SD_1) *As a self-adjoint unbounded operator on $L^2(S)$, D has discrete spectrum, i.e., the spectrum of D consists in a denumerable set of isolated (real) eigenvalues of finite multiplicity.*

(SD_2) *For every constant $c > 0$, there are a bounded positive (in the L^2-sense) operator $P : W^2(S) \to L^2(S)$, a bundle morphism $R \in End(S)$, and a compact subset $K \Subset M$, such that pointwise on $S|_{M\setminus K}$, $R \geq c$ and on $W^2(S)$,*

$$D^2 = P + R.$$

(SD_3) *For every constant $c > 0$ there is a compact subset $K \Subset M$ such that*

$$||D(s)|| \geq c||s||, \quad s \in W^1(S),\ \mathrm{supp}(s) \cap K = \emptyset.$$

Proof of Theorem SD. $(SD_1) \Longrightarrow (SD_2)$. Let $(K_n)_{n=1}^{\infty}$ be an exhaustion of M by compact subsets, i.e., for every n, $K_n \subset \mathrm{int}(K_{n+1})$ and $\bigcup_{n=1}^{\infty} K_n = M$. Next, for every n let $f_n \in C^{\infty}(M)$ be a real-valued function such that $-1 \leq f_n \leq 1$ and

$$f_n = \begin{cases} 1 & \text{on } K_n, \\ -1 & \text{on } M \setminus K_{n+1}. \end{cases}$$

For a fixed $c > 0$ it is obvious that multiplication on S by $-cf_n$ provides a bundle morphism $\geq c$ on $M \setminus K_{n+1}$, and we claim that there is an n such that $P := D^2 + cf_n$ is a bounded positive operator on $W^2(S)$. Assume *not*. Then for every n there is a section $s_n \in W^2(S)$, $||s_n|| = 1$, such that

$$\big((D^2 + cf_n)s_n, s_n\big) < 0. \tag{2}$$

Since the spectrum of D is discrete we can write, if $\{(\lambda, \phi_\lambda) | \lambda \in Spec(D)\}$ is the spectral resolution of D, by virtue of the spectral theorem,

$$L^2(S) = H \oplus H^{\perp}, \tag{3}$$

where, as L^2-direct sums,

$$H = \bigoplus_{\lambda \in Spec(D), |\lambda| \leq \sqrt{c}} \mathbf{C}\phi_\lambda \qquad H^{\perp} = \bigoplus_{\lambda \in Spec(D), |\lambda| > \sqrt{c}} \mathbf{C}\phi_\lambda.$$

Clearly H is finite dimensional, $H \subset W^2(S)$, and D^2 leaves H invariant. Also $D^2\left(H^{\perp} \cap W^2(S)\right) \subset H^{\perp}$, and moreover there is $\epsilon > 0$ such that $\left(D^2 s, s\right) \geq (c + \epsilon)||s||^2$, $s \in H^{\perp} \cap W^2(S)$ which further gives

$$\big((D^2 + cf_n)s, s\big) \geq \epsilon ||s||^2, \quad s \in H^{\perp} \cap W^2(S). \tag{4}$$

Using the orthogonal decomposition (3) we can find $h_n \in H$, $\sigma_n \in H^{\perp}$ such that $s_n = h_n + \sigma_n$ for every n; $\sigma_n \in W^2(S)$ since $H \subset W^2(S)$, and $||h_n||^2 + ||\sigma_n||^2 = 1$. Being a bounded sequence in a finite dimensional vector space $(h_n)_n$ admits an L^2-convergent subsequence, assumed to be itself. Denoting its limit by $h \in H$, then clearly $||h|| \leq 1$. We now have for every n,

$$\begin{aligned}
\big((D^2 + cf_n)s_n, s_n\big) &= \big((D^2 + cf_n)\sigma_n, \sigma_n\big) + \big((D^2 + cf_n)h_n, \sigma_n\big) \\
&\quad + \big((D^2 + cf_n)\sigma_n, h_n\big) + \big((D^2 + cf_n)h_n, h_n\big) \\
&= \big((D^2 + cf_n)\sigma_n, \sigma_n\big) + 2c\mathrm{Re}\,(f_n h_n, \sigma_n) + ||Dh_n||^2 + c(f_n h_n, h_n) \\
&\geq \epsilon ||\sigma_n||^2 + 2c\mathrm{Re}\,(f_n h_n, \sigma_n) + c(f_n h_n, h_n),
\end{aligned}$$

which leads to

$$\big((D^2 + cf_n)s_n, s_n\big) \geq \epsilon(1 - ||h_n||^2) + 2c\mathrm{Re}\,(f_n h_n, \sigma_n) + c(f_n h_n, h_n). \tag{5}$$

To get the desired contradiction notice that as $n \to \infty$, $(f_n h_n, \sigma_n)$ converges to 0 and $(f_n h_n, h_n)$ converges to $||h||^2$. These facts follow by standard applications of the Cauchy-Schwarz Inequality and the Lebesgue Dominated Convergence Theorem in $L^2(S)$, given that $h_n \perp \sigma_n$, $(h_n)_n$ is L^2-convergent, $(\sigma_n)_n$ is L^2-bounded, and f_n converges to 1, pointwise on M. Taking then the limit in equation (5) we see that

$$\limsup_{n \to \infty} \big((D^2 + cf_n)s_n, s_n\big) \geq \epsilon(1 - ||h||^2) + c||h||^2 > 0,$$

a contradiction of (2).

$(SD_2) \Longrightarrow (SD_3)$. Fix $c > 0$. By hypothesis there is a decomposition $D^2 = P + R$ on $W^2(S)$, where P is a bounded positive operator and $R \in End(S)$ is such that

$R \geq c^2$, outside some compact subset $L \Subset M$. Let $s \in W^2(S)$, $\text{supp}(s) \cap L = \emptyset$. Then

$$||Ds||^2 = (D^2s, s) = (Ps, s) + (Rs, s) \geq (Rs, s) \geq c^2||s||^2. \tag{6}$$

Now choose $K \Subset M$ such that $L \subset \text{int}(K)$. Then (6) proves (SD_3) since every section in $W^1(S)$ with support in $M \setminus K$ can be approximated in the $|| \cdot ||_1$ norm by sections in $W^2(S)$ with support in $M \setminus L$, in fact by smooth such sections.

$(SD_3) \implies (SD_1)$. Notice that for every $\lambda \in \mathbf{R}$, $D - \lambda I$ is again an essentially self-adjoint operator with domain $W^1(S)$. Since for every self-adjoint Fredholm operator 0 is at most an isolated point of its spectrum (eigenvalue of finite multiplicity) ([W]) to show that D has discrete spectrum it suffices to show that the family $\{D - \lambda I | \lambda \in \mathbf{R}\}$ consists only in Fredholm operators. To this end fix $\lambda \in \mathbf{R}$. By hypothesis there is a compact subset $K \Subset M$ such that if $s \in W^1(S)$, $\text{supp}(s) \cap K = \emptyset$, then $||Ds|| \geq (|\lambda| + 1)||s||$. Thus for such sections s,

$$||(D - \lambda I)s|| = ||Ds - \lambda s|| \geq ||Ds|| - |\lambda|||s|| \geq ||s||.$$

This fulfills the statement (F_3) in the case of the operator $D - \lambda I$, for the constant $c = 1$ and the compact subset K, and so by Theorem F, $D - \lambda I$ is a Fredholm operator. □

2. Remarks

Few remarks, ranging from common sense to deep, are in place now. In fact, the last one is an interesting result in its own right, so we prefer to state it as an independent proposition.

(R_1) Just as with Fredholmness, the respective Sobolev spaces in (SD_2) and (SD_3) can be replaced by $C_0^\infty(S)$.

This is obvious, given the density of $C^\infty(S)$ in those spaces, plus the fact that if $c > 0$ is an invertibility constant which works on some compact subset $K \Subset M$, then c also works on any other compact subset L, $K \subset \text{int}(L)$.

(R_2) Fredholmness and Spectral Discreteness are qualities invariant under perturbations of differential operators on compact subsets of M.

This is plain, from (F_2) or (F_3), and (SD_2) or (SD_3), given the local character of differential operators and (R_1).

(R_3) It is well known that the spectral theories of electromagnetic Dirac operators (the main examples of first-order differential operators considered in this note) and Schrödinger operators (second-order differential operators) are quite different.

In the simplest implementation of this comparison, let $M = \mathbf{R}$ with coordinate x, $S = \mathbf{R} \times \mathbf{C}$, and let $V \in C^\infty(\mathbf{R})$ be a real-valued function. Then $D = \frac{1}{i}\frac{d}{dx} + V(x)$ has no eigenvalues, as for $\lambda \in \mathbf{R}$ the only non-trivial smooth solutions of the equation $Ds = \lambda s$ are $\phi(x) = Ce^{i\left(\lambda x - \int_0^x V(t)\, dt\right)}$, $C \neq 0$, which are not in $L^2(S)$.

By contrast, the Schrödinger operator $U = -\frac{d^2}{dx^2} + V(x)$ has, as an unbounded essentially self-adjoint operator in $L^2(S)$, discrete spectrum if and only if for every $\alpha > 0$, $\lim_{|x| \to \infty} \int_x^{\alpha + x} |V(t)|\, dt = \infty$, by work of Molčanov [M]. In particular, this is the case if $\lim_{|x| \to \infty} |V(x)| = \infty$, as classically proved by Weyl and Titchmarsh [W, T].

(R_4) It appears that an operator D as those considered in this note has discrete spectrum if and only if there is a real-valued function $f \in C^\infty(M)$, $\lim_{M \ni x \to \infty} f(x) = \infty$, such that for $s \in C_0^\infty(S)$,

$$||Ds||^2 \geq (fs, s).$$

The 'if' part is simple, and the 'only if' part is likely to be true too, however we can validate it only for operators D with 'sub-linear propagation speed', a concept we proceed to describe now:

The speed of propagation at $x \in M$ associated to D is ([C])

$$v(x) := \sup \{ |\sigma(\xi)s| \mid \xi \in T_x^*M, |\xi| = 1, s \in S_x, |s| = 1 \}, \tag{7}$$

where σ is the principal symbol map of D. Since D is elliptic, $v(x)$ does not vanish. Now fix a point $x_0 \in M$ and let $\delta : M \to [0, \infty)$ be a regularization of the function $\text{dist}(x, x_0)$ such that $|d\delta| \leq 2$, uniformly on M. Then $B_r := \{x \in M \mid \delta(x) \leq r\}$ is approximately the ball of radius r centered at x_0. Completeness of M implies that B_r is compact for every $r > 0$. Chernoff [C] has shown that if D has infinite propagation time, i.e., $\int_1^\infty 1/v_r dr = \infty$, where $v_r := \sup\{v(x) | x \in B_r\}$, then D is essentially self-adjoint. This is clearly the case if there are positive constants α and β such that

$$v_r \leq \alpha r + \beta, \quad r > 0, \tag{8}$$

in which case we say that D has sub-linear propagation speed. The most familiar cases, when D has constant propagation speed ($v(x)$ is constant for $x \in M$), or finite propagation speed ($\sup\{v(x) | x \in M\} < \infty$) are certainly contained in (8).

Proposition. *Let D be an essentially self-adjoint first-order elliptic differential operator with sub-linear propagation speed. Then D has discrete spectrum if and only if there is a real-valued function $f \in C^\infty(M)$, $\lim_{M \ni x \to \infty} f(x) = \infty$, such that for $s \in C_0^\infty(S)$,*

$$||Ds||^2 \geq (fs, s), \quad \textit{that is if} \quad D^2 - f \quad \textit{is} \quad L^2\textit{– positive on} \quad C_0^\infty(S). \tag{9}$$

Proof. The 'if' part is obvious by (SD_3), given that $\lim_{M \ni x \to \infty} f(x) = \infty$ means that for every $c > 0$ there is a compact subset $K \Subset M$ such that $f(x) \geq c^2$, $x \in M \setminus K$.

As for the 'only if' part, the existence of an f as in (9) eventually reflects, via partitions of unity, the fact that for any exhaustion $(K_n)_n$ of M by compact subsets, in particular for the exhaustion $(B_{2^n})_n$, Spectral Discreteness is equivalent to the non-decreasing sequence of non-negative real numbers,

$$\rho_n = \inf \left\{ \frac{||Ds||}{||s||} \mid s \in W^1(S), s \neq 0, \text{supp}(s) \cap K_n = \emptyset \right\}, \tag{10}$$

being convergent to ∞.

To see this we now introduce, as in [I], two real-valued smooth functions on $\mathbf{R}$, χ_0, χ_1, such that $\text{supp}(\chi_0) \subset (-\infty, 2)$, $|\chi_0'| \leq 2$ on $\mathbf{R}$, $\text{supp}(\chi_1) \subset (1, \infty)$, $|\chi_1'| \leq 2$ on $\mathbf{R}$, and $\chi_0^2(t) + \chi_1^2(t) = 1$ for all $t \in \mathbf{R}$. Then the sequence of real-valued functions $f_n \in C_0^\infty(M)$, $n \geq 0$, defined by

$$f_0(x) = \chi_0(\delta(x)),$$

$$f_n(x) = \begin{cases} \chi_1\left(\frac{\delta(x)}{2^{n-1}}\right), & \text{if } \delta(x) \leq 2^n \\ \chi_0\left(\frac{\delta(x)}{2^n}\right), & \text{if } \delta(x) \geq 2^n \end{cases}, \quad \text{for } n \geq 1,$$

has the following properties:

$$
(11)\qquad
\begin{aligned}
&\operatorname{supp}(f_0) \subset \operatorname{int}(B_2), \quad |df_0(x)| \le 4,\ x \in M,\\
&\operatorname{supp}(f_n) \subset \operatorname{int}(B_{2^{n+1}}) \setminus B_{2^{n-1}}, \quad |df_n(x)| \le \frac{4}{2^{n-1}},\ x \in M, \quad n \ge 1,\\
&\sum_{n=0}^{\infty} f_n^2(x) = 1, \quad x \in M.
\end{aligned}
$$

For $h \in C^\infty(M)$ real-valued and $s \in C_0^\infty(S)$ the principal symbol formula (1) gives

$$
\begin{aligned}
\operatorname{Re}\big(D(h^2 s), Ds\big) &= \operatorname{Re}(2h\sigma(dh)s, Ds) + ||hDs||^2\\
&= 2\operatorname{Re}(h\sigma(dh)s, Ds) + ||D(hs)||^2 - ||\sigma(dh)s||^2\\
&\quad - (\sigma(dh)s, hDs) - (hDs, \sigma(dh)s)\\
&= ||D(hs)||^2 - ||\sigma(dh)s||^2.
\end{aligned}
$$

As a result, for $s \in C_0^\infty(S)$

$$
\begin{aligned}
||Ds||^2 &= (D(\sum_{n=0}^{\infty} f_n^2)s, Ds) = \sum_{n=0}^{\infty} \operatorname{Re}\big(D(f_n^2 s), Ds\big)\\
&= \sum_{n=0}^{\infty}\big(||D(f_n s)||^2 - ||\sigma(df_n)s||^2\big) = \sum_{n=0}^{\infty} ||D(f_n s)||^2 - \sum_{n=0}^{\infty} ||\sigma(df_n)s||^2\\
&\ge \sum_{n=1}^{\infty} \rho_{n-1}^2 ||f_n s||^2 - \sum_{n=0}^{\infty} (\alpha 2^{n+1} + \beta)^2 ||\,|df_n| s||^2,
\end{aligned}
$$

where $(\rho_n)_n$ is the unbounded sequence of non-negative real numbers assigned by (10) to the exhaustion $(B_{2^n})_n$ of M and the factors $\alpha 2^n + \beta$ are supplied by the sub-linear propagation property of D given by (8).

Setting now

$$
f(x) := \sum_{n=1}^{\infty} \rho_{n-1}^2 f_n^2(x) - \sum_{n=0}^{\infty} (\alpha 2^{n+1} + \beta)^2 |df_n|^2(x),
$$

we see by (10) and (11) that $f \in C^\infty(M)$, $\lim_{M \ni x \to \infty} f(x) = \infty$, and ultimately, $||Ds||^2 \ge (fs, s)$, $s \in C_0^\infty(S)$, since

$$
\sum_{n=1}^{\infty} \rho_{n-1}^2 ||f_n s||^2 - \sum_{n=0}^{\infty} (\alpha 2^{n+1} + \beta)^2 ||\,|df_n| s||^2 = (fs, s).
$$

□

3. An application

We conclude this note with an application to the spectral theory of electromagnetic Dirac operators, in the spirit of [S]. However, in order to state it one needs a little background.

Let $Cl(M)$ be the real Clifford bundle of algebras induced by the tangent bundle TM and the Riemannian metric on M. Via the canonical embedding $TM \subset Cl(M)$ the Riemannian metric and Levi-Cività connection extend from TM to $Cl(M)$ in such a way that the extended connection, ∇^{LC}, preserves the metric and acts as a derivation.

Assume now that the bundle S over M is a (generalized) Dirac bundle ([GL]), i.e., it is a complex bundle of left modules over the bundle of algebras $Cl(M)$, furnished with a Hermitian metric and a metric connection ∇ such that

i) The action on S by unit vectors in $TM \subset Cl(M)$ is a pointwise isometry.

ii) The connection ∇ is compatible with the Clifford multiplication, in the sense that for local sections e in TM, ϕ in $Cl(M)$, and s in S, we have

$$\nabla_e(\phi \cdot s) = \left(\nabla_e^{LC}\phi\right) \cdot s + \phi \cdot (\nabla_e s)\,.$$

Above, the '$\cdot$' indicates the action of $Cl(M)$ on S. Any Dirac bundle S generates a distinguished first-order differential operator $\not{D}$: $C^\infty(S) \to C^\infty(S)$, the generalized Dirac operator, defined as follows: If $\mu : T^*M \otimes S \to S$ denotes the restriction to T^*M (metrically identified with TM) of the Clifford action $\cdot$ of $Cl(M)$ on S, then $\not{D} = \mu \circ \nabla$. Locally, $\not{D}$ admits the representation

$$\not{D} = \sum_{j=1}^{p} e_j \cdot \nabla_{e_j},$$

where $p = \dim M$ and $(e_1, e_2, \ldots, e_p)$ is any local orthonormal frame in TM.

For generalized Dirac operators the principal symbol formula (1) becomes

$$\not{D}(fs) = \text{grad} f \cdot s + f\not{D}(s), \quad f \in C^\infty(M),\ s \in C^\infty(S),$$

and so they have constant propagation speed 1. Since M is complete, $\not{D}$ with domain $C_0^\infty(S)$ is an essentially self-adjoint elliptic first-order differential operator in $L^2(S)$ ([C, GL]).

For the square of a generalized Dirac operator $\not{D}^2$ the following Bochner-Weitzenböck formula holds true ([GL]),

$$\not{D}^2 = \nabla^*\nabla + \mathcal{R}, \tag{12}$$

where $\mathcal{R}$ is the Hermitian curvature bundle morphism acting on S according to the formula

$$\mathcal{R} = \sum_{j<k} e_j e_k \cdot R_{e_j,e_k}, \quad R_{e_j,e_k} = [\nabla_{e_j}, \nabla_{e_k}] - \nabla_{[e_j,e_k]}. \tag{13}$$

Assume further that $\not{D}$ is supersymmetric in the sense that there is a Hermitian involution $\epsilon \in End(S)$ such that $v \cdot \epsilon + \epsilon v\cdot = 0$, $v \in TM$, and $\nabla\epsilon = \epsilon\nabla$. Thus $S = S^+ \oplus S^-$, $S^\pm = \{\epsilon = \pm 1\}$ and then Clifford multiplication by sections of TM and $\not{D}$ interchange $S^\pm$ while ∇ and $\mathcal{R}$ preserve $S^\pm$. The supersymmetric version of the Bochner-Weitzenböck formula (12) is

$$\begin{aligned} \not{D}^-\,\not{D}^+ &= \left(\nabla^+\right)^*\nabla^+ + \mathcal{R}^+, \quad \text{on } C^\infty(S^+),\\ \not{D}^+\,\not{D}^- &= \left(\nabla^-\right)^*\nabla^- + \mathcal{R}^-, \quad \text{on } C^\infty(S^-), \end{aligned} \tag{14}$$

where $\not{D}^\pm$, $\nabla^\pm$, and $\mathcal{R}^\pm$ represent the restrictions of $\not{D}$, ∇, and $\mathcal{R}$ to $S^\pm$, respectively.

An important example of supersymmetric Dirac operator is the classical Dirac operator $\not{\partial}$ on an *even*-dimensional *spin* manifold M ([GL]). Here the involution ϵ is provided by the Clifford volume form $i^{\frac{p}{2}} e_1 e_2 \cdots e_p$, and the curvature bundle morphism $\mathcal{R}$ equals $\frac{k}{4}$, where k is the scalar curvature function of M. So is more generally the magnetic classical Dirac operator $\partial\!\!\!/_a$, obtained by coupling $\partial\!\!\!/$ with a magnetic potential a, a smooth real 1-form $a \in \Omega^1(M, \mathbf{R})$, in which case $\mathcal{R} = \frac{k}{4} + i\rho^a\cdot$, where ρ^a is the global section of $Cl(M)$ given by

$$\rho^a = \sum_{j<k} R^a_{e_j,e_k} e_j e_k, \quad R^a_{e_j,e_k} = e_j(a(e_k)) - e_k(a(e_j)) - a([e_j, e_k]). \tag{15}$$

Under the standard linear isometry $\Lambda(T^*M) \simeq Cl(M)$, ρ^a is the image of the real 2-form $B = da \in \Omega^2(M, \mathbf{R})$, the magnetic field associated to the potential a.

Finally, for an electric potential V, i.e., a real-valued function in $C^\infty(M)$, the electromagnetic classical Dirac operator $\partial\!\!\!/_a + V$ is a fundamental example of an operator D as considered in this note.

Application. Let $D\!\!\!\!/$ be a supersymmetric generalized Dirac operator as described above, with Bochner-Weitzenböck curvature bundle morphism $\mathcal{R} = \mathcal{R}^+ \oplus \mathcal{R}^- \in End(S^+ \oplus S^-)$. Let $V \in C^\infty(M)$ be a real-valued electric potential. Assume that the following three hypotheses hold:

(A_1) $\lim_{x\to\infty} |V(x)| = \infty$.

(A_2) $\lim_{x\to\infty} \frac{|dV|}{V^2} = 0$.

(A_3) Either $\lim_{x\to\infty} \frac{\mathcal{R}^+(x)}{V^2(x)} > 3$ or $\lim_{x\to\infty} \frac{\mathcal{R}^-(x)}{V^2(x)} > 3$.

Then as an essentially self-adjoint operator in the L^2-space the operator $D\!\!\!\!/ + V$ has discrete spectrum.

Proof. Assume that $\lim_{x\to\infty} \frac{\mathcal{R}^+(x)}{V^2(x)} > 3$, the proof if the other option in (A_3) happened being similar. Fix $c > 0$. There is a compact subset $K \Subset M$ and a constant $0 < \delta < 1$ such that for $x \in M \setminus K$, $V^2(x) \geq \frac{2c^2}{\delta}$, $|dV(x)| \leq \frac{\delta}{4} V^2(x)$, and $\mathcal{R}^+(x) \geq (3 + \delta) V^2(x)$.

If $s \in C_0^\infty(S)$, $\text{supp}(s) \cap K = \emptyset$, we have

$$\begin{aligned}
||(D\!\!\!\!/ + V)s||^2 &= ||D\!\!\!\!/ s||^2 + 2\text{Re}(D\!\!\!\!/ s, Vs) + ||Vs||^2 = ||D\!\!\!\!/^+ s^+||^2 + ||D\!\!\!\!/^- s^-||^2 \\
&\quad + 2\text{Re}(D\!\!\!\!/^- s^-, Vs^+) + 2\text{Re}(D\!\!\!\!/^+ s^+, Vs^-) + ||Vs^+||^2 + ||Vs^-||^2 \\
&= ||D\!\!\!\!/^+ s^+||^2 + ||D\!\!\!\!/^- s^-||^2 + 4\text{Re}(D\!\!\!\!/^- s^-, Vs^+) - 2\text{Re}(s^+, \text{grad}V \cdot s^-) \\
&\quad + ||Vs^+||^2 + ||Vs^-||^2 \geq ||D\!\!\!\!/^+ s^+||^2 + ||D\!\!\!\!/^- s^-||^2 - 4||D\!\!\!\!/^- s^-||\,||Vs^+|| \\
&\quad - 2||Vs^+||\,||\frac{\text{grad}V}{V^2} \cdot Vs^-|| + ||Vs^+||^2 + ||Vs^-||^2 \geq ||D\!\!\!\!/^+ s^+||^2 \\
&\quad + \left(||D\!\!\!\!/^- s^-||^2 - 2||Vs^+||\right)^2 - 2||Vs^+||\,||\frac{|dV|}{V^2} Vs^-|| - 3||Vs^+||^2 + ||Vs^-||^2 \\
&\geq ||\nabla^+ s^+||^2 + (\mathcal{R}^+ s^+, s^+) - \frac{\delta}{2}||Vs||^2 - 3(V^2 s^+, s^+) + ||Vs^-||^2 \\
&\geq \delta(V^2 s^+, s^+) - \frac{\delta}{2}||Vs||^2 + ||Vs^-||^2 \geq \frac{\delta}{2}||Vs||^2 = (\frac{\delta V^2}{2} s, s) \geq c^2||s||^2.
\end{aligned}$$

The Application follows now by (SD_3). □

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