PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 144, Number 2, February 2016, Pages 733–738
http://dx.doi.org/10.1090/proc12748
Article electronically published on June 30, 2015

A FUNCTION WHOSE GRAPH HAS POSITIVE DOUBLING MEASURE

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(Communicated by Jeremy Tyson)

Abstract. We show that a doubling measure on the plane can give positive measure to the graph of a continuous function. This answers a question by Wang, Wen and Wen posed in a 2013 paper. Moreover, we show that the doubling constant of the measure can be chosen to be arbitrarily close to the doubling constant of the Lebesgue measure.

1. Introduction

A Borel regular measure μ on $\mathbb{R}^d$ is called doubling if there exists a constant $C < \infty$ such that

$$0 < \mu(Q_1) \leq C\mu(Q_2) < \infty,$$

for any pair of adjacent cubes Q_1 and Q_2 with the same side length. A constant C for which this holds is called a doubling constant of μ. In this note we are in dimension two, and so the cubes are called squares. They are understood to be product sets $[x, x+s] \times [y, y+s]$ without rotation, for side length $s > 0$. Squares are called adjacent if they intersect. A set $E \subset \mathbb{R}^d$ is called *thin* if $\mu(E) = 0$ for all doubling measures of $\mathbb{R}^d$. In this paper we give a negative answer to a question posed by Wang, Wen and Wen in [3, Problem 1]:

> Is the $\operatorname{graph}(f) := \{(x, f(x)) : x \in [0,1]\}$ of a continuous function $f : [0,1] \to [0,1]$ thin?

The answer to the aforementioned question is contained in the following

Theorem 1.1. *For any $\epsilon > 0$ there exist a doubling measure μ on $[0,1]^2$ with doubling constant less than $1+\epsilon$ and a continuous function $f\colon [0,1] \to [0,1]$ such that*

$$\mu(\operatorname{graph}(f)) > (1-\epsilon)\mu([0,1]^2). \tag{1.1}$$

By a density point argument one easily sees that upper-porous sets are thin. Thus the graph of a function $f\colon \mathbb{R}^d \to \mathbb{R}$ satisfying

$$\operatorname{lip}[f](x) = \liminf_{r\to 0} \sup_{y\in B(x,r)} \frac{|f(x) - f(y)|}{r} < \infty \tag{1.2}$$

at every point $x \in \mathbb{R}^d$ is thin. In particular, the graph of any Lipschitz function is thin. Interestingly, being a graph is to some extent necessary for this property: Garnett, Killip and Schul showed in [1] that there exist rectifiable curves in $\mathbb{R}^d$ that

Received by the editors January 28, 2015 and, in revised form, February 3, 2015.
2010 *Mathematics Subject Classification.* Primary 28A12; Secondary 30L10.
Key words and phrases. Doubling measure, thin set, fat set.

are not thin. Here, a rectifiable curve is defined as the image of a continuous curve with finite length.

It would be interesting to further investigate which doubling measures give zero measure to which graphs. For instance, one could study the connection between the doubling constant and the modulus of continuity in this problem. To our knowledge not much is known on such questions beyond the obviously thin graphs of functions with finite lip as defined in (1.2), and the example provided in this note. For instance the following natural question is still open:

Question 1.2. Is the graph of a Hölder continuous function $f : [0, 1] \to [0, 1]$ thin?

2. Proof of the theorem

Let us now prove Theorem 1.1. By Lusin's theorem, we can reduce the proof to the case of measurable functions. Indeed, let μ be a doubling measure, and let $f\colon [0, 1] \to [0, 1]$ be a measurable function such that

$$\mu(\mathrm{graph}(f)) = (1 - \epsilon_1)\mu([0, 1]^2) > (1 - \epsilon)\mu([0, 1]^2).$$

Since the projection $\pi^x_\sharp\mu$ of the measure μ to the real line is a Borel regular measure, by Lusin's theorem for any $\epsilon_0 > 0$ we have a continuous function $\hat{f}\colon [0, 1] \to [0, 1]$ such that

$$\pi^x_\sharp\mu(\Big\{x : f(x) \neq \hat{f}(x)\Big\}) < \epsilon_0\mu([0, 1]^2).$$

This implies that

$$\begin{aligned}\mu(\mathrm{graph}(\hat{f})) &\geq \mu(\mathrm{graph}(f)) - \pi^x_\sharp\mu(\Big\{x : f(x) \neq \hat{f}(x)\Big\})\\ &> (1 - \epsilon_1)\mu([0, 1]^2) - \epsilon_0\mu([0, 1]^2) > (1 - \epsilon)\mu([0, 1]^2),\end{aligned}$$

when $\epsilon_1 + \epsilon_0 \leq \epsilon$. Thus, to prove Theorem 1.1 it is enough to construct a doubling measure μ on $[0, 1]^2$ with doubling constant less than $1+\epsilon$ and a measurable function $f\colon [0, 1] \to [0, 1]$ such that

$$\mu(\mathrm{graph}(f)) > (1 - \epsilon)\mu([0, 1]^2). \tag{2.1}$$

Remark 2.1. Notice that instead of using Lusin's theorem one could also explicitly construct the continuous function $\hat{f}\colon [0, 1] \to [0, 1]$. This can be done for example by selecting differently the approximating graph at the leftmost and the rightmost construction subrectangles of every construction rectangle (see Section 2.2 for the construction of the graph). Since such an explicit continuous function would not have a nice modulus of continuity, we prefer to construct just a measurable function.

2.1. Constructing the measure.

We construct the desired measure using a 4-adic distribution of mass. The weights used in the distribution will change during the iteration process.

Take $0 < p < 1$ and define $q = 2 - p$. These will be fixed throughout the construction, and the basic distribution of mass will be done according to these two numbers. The idea of the distribution of mass and the respective approximation of the graph after three iteration steps is shown in Figure 1. This same pattern would then be repeated in every subrectangle with increased number of steps in the 4-adic division. It is relatively easy to see that while this increasing of steps would result in positive mass for the graph, it would also blow up the doubling constant. To ensure that the measure stays doubling, we have to proceed with a little more care, the main idea being the same.

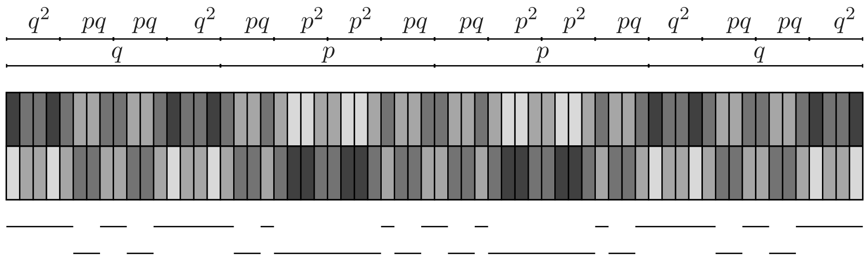

FIGURE 1. Illustration of the completed second step. Measure inside rectangles is spread using weights p and q in a 4-adic division. This construction with its dual construction (having p and q swapped) placed above each other gives the basic idea of the measure. In order to obtain a nicely doubling measure, the measure has to be corrected near the line where the two constructions meet. In constructing the graph, we drop from above and below all the rectangles that have less than avarage measure; as well as the areas where the measure had to be redefined. An approximating graph is also illustrated.

Measure μ_k after each iteration step will be a weighted Lebesgue measure

$$d\mu_k(x,y) = \prod_{i=1}^{k} w_i(x,y)d\mathcal{L}(x,y),$$

where weights w_i are constant in 4-adic squares of level i. The weak* limit μ of the measures μ_k as $k \to \infty$ will then have the desired properties. Let us now define the weights w_i.

We define the weights w_i using a two step construction. For this purpose we take two sequences $(m_j)_{j=1}^{\infty}, (n_j)_{j=1}^{\infty}$ of positive odd integers larger than 2 that shall be later specified. Let us call the unit square $[0,1]^2$ the *level* 0 *construction rectangle.* For brevity we denote $M_k = \sum_{j=1}^{k}(m_j + n_j)$. We will define the weights in 4-adic scales by repeatedly first uniformly distributing in m_j scales and then doing the actual redistribution of mass in the following n_j scales.

Let us now give the precise definition for the weights. Suppose we are given the level k construction rectangles $[l,r] \times [b,t]$. The weights w_i for steps $M_k < i \leq M_{k+1}$ and level $k+1$ construction rectangles inside the rectangles $[l,r] \times [b,t]$ are defined as follows.

First step: *Uniform distribution.* We define $w_i = 1$ for all steps $M_k < i \leq M_k + m_{k+1}$. The purpose for doing this is to have the side length $4^{-M_k - m_{k+1}}$ of the squares compared to the height $t-b$ of the construction rectangle to be sufficiently small. As we will see in the second step and in the final computations, the m_k will determine the size of the set where we move from the weights of the type p^n to the weights of the type q^n.

Second step: *Non-uniform distribution.* The basic idea of the construction was shown in Figure 1, but, in order to keep the measure nicely doubling, we only gradually let the weights change from p^n to q^n.

In order to obtain this doubling transition at the top and bottom of the construction rectangles, we define on the upper halves of the construction rectangles of level k living in $[0,1]\times[b,t]$ the weights for $M_k + m_{k+1} < i \le M_{k+1}$ as

$$w_i(x,y) = \begin{cases} q, & \text{if } x \in A_i \text{ and } y \in B_i, \\ p, & \text{if } x \notin A_i \text{ and } y \in B_i, \\ 1, & \text{otherwise,} \end{cases}$$

where

$$A_i := \bigcup_{j \in \{0,1,\ldots,4^{(i-1)}-1\}} \left[\frac{j}{4^{(i-1)}} + \frac{1}{4^i}, \frac{j}{4^{(i-1)}} + \frac{3}{4^i}\right]$$

and

$$B_i := \left[\frac{t+b}{2} + \sum_{j=M_k+m_{k+1}+1}^{i-1} \frac{1}{4^j}, t - \sum_{j=M_k+m_{k+1}+1}^{i-1} \frac{1}{4^j}\right].$$

On the lower halves of the construction rectangles of level k living in $[0,1]\times[b,t]$, we swap the roles of p and q in defining the weights w_i for $M_k + m_{k+1} < i \le M_{k+1}$. In other words, we define

$$w_i(x,y) = \begin{cases} p, & \text{if } x \in A_i \text{ and } \frac{t-b}{2} + y \in B_i, \\ q, & \text{if } x \notin A_i \text{ and } \frac{t-b}{2} + y \in B_i, \\ 1, & \text{otherwise,} \end{cases}$$

where A_i and B_i are as above.

After this we define the *construction rectangles of level* $k+1$ inside $[0,1]\times[b,t]$ to be the $4^{-M_{k+1}}$-wide rectangles

$$\left[\frac{j}{4^{M_{k+1}}}, \frac{j+1}{4^{M_{k+1}}}\right] \times \left[b + \frac{1}{4^{M_k+m_{k+1}}}, \frac{t+b}{2} - \frac{1}{4^{M_k+m_{k+1}}}\right]$$

and

$$\left[\frac{j}{4^{M_{k+1}}}, \frac{j+1}{4^{M_{k+1}}}\right] \times \left[\frac{t+b}{2} + \frac{1}{4^{M_k+m_{k+1}}}, t - \frac{1}{4^{M_k+m_{k+1}}}\right]$$

for $j \in \{0,1,\ldots,4^{M_{k+1}}-1\}$.

We call the complement of the union of all the construction rectangles of level $k+1$ inside the level k construction rectangles the *leftover part of level* $k+1$. On the leftover part of level $k+1$, we define the weight $w_i = 1$ for $i > M_{k+1}$.

2.2. **Approximating the graph.** The next thing is to construct the approximative graph of the desired function. We define the approximation of the graph always after the second step is completed. That is, the kth approximative graph is defined at the scale 4^{-M_k}.

Let us denote the first approximative graph by R_1. This will simply be the set of level 1 construction rectangles where we have more weights q than weights p at the level $M_1 = m_1 + n_1$ (see Figure 2). In other words, we choose upper or lower construction rectangles depending which of these has larger mass, i.e.,

$$R_1 = \Big\{(x,y) : w_i(x,y) \in \{p,q\}\ \forall i = m_1+1,\ldots,m_1+n_1 \\ \text{and} \prod_{i=m_1+1}^{m_1+n_1} w_i(x,y) > (pq)^{n_1/2}\Big\}.$$

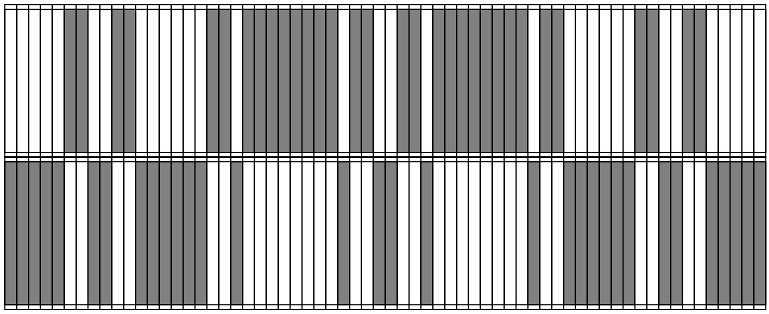

FIGURE 2. Approximation of the graph is simply the better choice of the two options; at each point x we choose the construction rectangle (either upper or lower) that has more mass.

We can estimate the total measure of the construction rectangles of level 1 that are not chosen by

$$2^{-n_1} \sum_{i=0}^{(n_1-1)/2} \binom{n_1}{i} q^i p^{n_1-i} \le \frac{1}{2}(pq)^{\frac{n_1-1}{2}}.$$

Notice that $pq = 1-(1-p)^2 < 1$. The measure of the leftover part of level 1 is at most $2 \cdot 4^{-m_1}$. Similarly, for $k > 1$, let us denote by R_k the union of construction rectangles inside R_{k-1} that have more weights q than weights p in the range $M_{k-1} + m_k < i \le M_k$. The leftover part of level k is included in 2^{k+1} strips of width 1 and of height at most $2^{-k}4^{-m_k}$. Therefore, the above estimates for the measures of the leftover part and the construction rectangles that are not chosen hold with n_1 and m_1 replaced by n_k and m_k, respectively. Now we define the kth approximation of the graph S_k and the actual graph S as

$$S_k = \bigcap_{j=1}^{k} R_j \qquad \text{and} \qquad S = \bigcap_{j=1}^{\infty} R_j.$$

It is clear that we get a graph of a function in the limit: when looking at any point $x \in [0,1]$ on the line $\{x\} \times [0,1]$, the sequence of the approximations, $S_j \cap \{x\} \times [0,1]$, is just a sequence of nested closed intervals whose length tends to zero.

When we choose the sequences (n_i) and (m_i) appropriately, we have

$$\mu(S) \ge 1 - \sum_{k=1}^{\infty} \frac{1}{2}(pq)^{\frac{n_k-1}{2}} - \sum_{k=1}^{\infty} \frac{1}{4^{m_k-1}} > 1-\epsilon,$$

so that we have (2.1). For example, given $0 < p < 1$, $q = 2-p$ and $\epsilon > 0$, we can take any odd integers

$$n_1 > \frac{\log((1-pq)\epsilon)}{\log(pq)} \qquad \text{and} \qquad m_1 > -\frac{\log(2\epsilon)}{\log(4)} + 2,$$

and define $n_k = n_{k-1} + 2$, $m_k = m_{k+1} + 2$ for all $k > 1$.

By the construction, it is clear that the function is Borel: any pre-image of an open interval is simply a countable union of intervals.

2.3. **The measure is doubling.** The only thing left to check is that we actually have a doubling measure as claimed. For this purpose we rephrase [2, Lemma 2]. After this it is easy to check that the construction at hand satisfies the assumptions of the lemma.

Lemma 2.2 ([2, Lemma 2])**.** *Let $\mathcal{D}_k$ be a collection of N-adic squares of side length N^{-k} in $[0,1]^2$. Suppose that $\epsilon > 0$, and $\{\mu_k\}_{k=1}^{\infty}$ is a sequence of probability measures on $[0,1]^2$ satisfying:*

(1) *$\forall k$, $\mu_k|_Q = C_Q\mathcal{L}$ for all $Q \in \mathcal{D}_k$ where $\mathcal{L}$ is Lebesgue measure and $C_Q > 0$ is a constant weight,*
(2) *$\mu_k(Q) = \mu_{k+1}(Q)$ for all $Q \in \mathcal{D}_k$,*
(3) *$\mu_k(Q) \leq \epsilon\mu_k(G)$ for $Q, G \in \mathcal{D}_k$ adjacent.*

Then the sequence $\{\mu_k\}$ converges in a weak* sense to a C-doubling measure μ, where $C = C(\epsilon) \to 1$ when $\epsilon \to 1$.

It is easy to see that our construction satisfies the assumptions of Lemma 2.2 with $N = 4$. Indeed, the first two requirements are clearly satisfied. The third one simply follows from the main idea of this type of doubling mass distribution: on adjacent squares $Q, G \in \mathcal{D}_k$ the weights $w_j(x,y)$, $j \leq k$, can only differ on one index j, from which it follows that $\mu_k(Q) \leq \frac{q}{p}\mu_k(G)$.

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