REMARKS ON AN INEQUALITY 
of Rogers and Shephard

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Abstract. A classical inequality of Rogers and Shephard states that if \( K \) is a centered convex body of volume 1 in \( \mathbb{R}^n \), then
\[
1 \leq g(K,k;F) := \left( \text{vol}_k(P_F(K)) \text{vol}_{n-k}(K \cap F^\perp) \right)^{1/k} \leq \left( \frac{n}{k} \right)^{1/k} \leq \frac{cn}{k}
\]
for every \( F \in G_{n,k} \), where \( c > 0 \) is an absolute constant. We show that if \( K \) is origin symmetric and isotropic, then, for every \( 1 \leq k \leq n-1 \), a random \( F \in G_{n,k} \) satisfies
\[
c_1 L_K^{-1} \sqrt{n/k} \leq g(K,k;F) \leq c_2 \sqrt{n/k} (\log n)^2 L_K
\]
with probability greater than \( 1 - e^{-k} \), where \( L_K \) is the isotropic constant of \( K \) and \( c_1, c_2 > 0 \) are absolute constants.

1. Introduction

Let \( K \) be a convex body of volume 1 in \( \mathbb{R}^n \) with \( 0 \in \text{int}(K) \). For every \( 1 \leq k \leq n-1 \) and any \( F \in G_{n,k} \) we define
\[
g(K,k;F) := \left( \text{vol}_k(P_F(K)) \text{vol}_{n-k}(K \cap F^\perp) \right)^{1/k},
\]
where \( F^\perp \) denotes the orthogonal subspace of \( F \) in \( \mathbb{R}^n \). A classical inequality of Rogers and Shephard [13] (see also Chakerian [5]) states that if \( K \) is origin symmetric, then
\[
1 \leq g(K,k;F) \leq \left( \frac{n}{k} \right)^{1/k} \leq \frac{cn}{k},
\]
where \( c_0 > 0 \) is an absolute constant. The right-hand side inequality holds true under the more general assumption that \( 0 \in \text{int}(K) \). On the other hand, Spingarn [13] showed that the lower bound remains valid if we assume that \( K \) is centered, i.e., that the barycenter of \( K \) is at the origin.

Both estimates are sharp: let \( \{e_1, \ldots, e_n\} \) be an orthonormal basis of \( \mathbb{R}^n \) and set \( F = \text{span}\{e_1, \ldots, e_k\} \). Consider a convex body \( A \subset F \) and a convex body \( B \subset F^\perp \) with \( 0 \in \text{int}(A) \cap \text{int}(B) \). One can check that if \( K = A \times B = \{a+b : a \in A, b \in B\} \), then \( P_F(K) = A, K \cap F^\perp = B \) and \( \text{vol}_n(K) = \text{vol}_k(A) \text{vol}_{n-k}(B) \). On the other
hand, if we consider the convex body

\[ K' = \text{conv}(A \cup B) = \{(1-t)a + tb : a \in A, b \in B, 0 \leq t \leq 1\}, \]

then \( P_{F}(K') = A, K' \cap F^\perp = B \) and \( \text{vol}_n(K') = \binom{n}{k} \text{vol}_k(A) \text{vol}_{n-k}(B) \).

Our starting point is the observation that the behavior of \( g(\mathcal{E}, k; F) \) lies “in the middle” when \( \mathcal{E} \) is an ellipsoid.

**Proposition 1.1.** For every ellipsoid \( \mathcal{E} \) in \( \mathbb{R}^n \) and for all \( 1 \leq k \leq n - 1 \) and \( F \in G_{n,k} \) the product \( \text{vol}_k(P_{F}(\mathcal{E})) \text{vol}_{n-k}(\mathcal{E} \cap F^\perp) \) is independent of the subspace \( F \). More precisely, we have

\[
(1.3) \quad \text{vol}_k(P_{F}(\mathcal{E})) \text{vol}_{n-k}(\mathcal{E} \cap F^\perp) = \frac{\text{vol}_k(B_2^k) \text{vol}_{n-k}(B_2^{n-k})}{\text{vol}_n(B_2^n)} \text{vol}_n(\mathcal{E}).
\]

Therefore,

\[
(1.4) \quad \left(\frac{c_1 n}{k}\right)^{k/2} \text{vol}_n(\mathcal{E}) \leq \text{vol}_k(P_{F}(\mathcal{E})) \text{vol}_{n-k}(\mathcal{E} \cap F^\perp) \leq \left(\frac{c_2 n}{k}\right)^{k/2} \text{vol}_n(\mathcal{E}),
\]

where \( c_1, c_2 > 0 \) are absolute constants.

For the reader’s convenience we include a proof of this observation in Section 3. Assuming that \( \text{vol}_n(\mathcal{E}) = 1 \), from Proposition 1.1 we see that

\[
(1.5) \quad g(\mathcal{E}, k; F) \simeq \sqrt{n/k}
\]

for all \( 1 \leq k \leq n - 1 \) and \( F \in G_{n,k} \). The question that we discuss in this note is if this is the typical (with respect to \( F \in G_{n,k} \)) behavior of \( g(K, k; F) \) for any symmetric (or, more generally, centered) convex body \( K \) of volume 1 in \( \mathbb{R}^n \). Our main result provides an (almost sharp) affirmative answer if we assume that \( K \) is in isotropic position.

**Theorem 1.2.** Let \( K \) be an origin symmetric isotropic convex body in \( \mathbb{R}^n \). For every \( 1 \leq k \leq n - 1 \) a random \( F \in G_{n,k} \) satisfies

\[
(1.6) \quad c_1 L_K^{-1} \sqrt{n/k} \leq g(K, k; F) \leq c_2 \sqrt{n/k} (\log n)^2 L_K
\]

with probability greater than \( 1 - e^{-k} \), where \( c_1, c_2 > 0 \) are absolute constants.

Our approach is presented in Section 4 and leads to some general lower and upper bounds that might be useful for other classical positions of \( K \), such as the minimal surface area position or minimal mean width position or John position. In Section 5 we use the additional information that one has when \( K \) is isotropic, and obtain the bounds of Theorem 1.2. The left-hand side inequality in (1.6) remains valid for any isotropic convex body \( K \) in \( \mathbb{R}^n \). For the right-hand side inequality we employ a recent result of E. Milman on the mean width of origin symmetric isotropic convex bodies (see [5]); this forces the assumption of symmetry in Theorem 1.2.

Background information is provided in Section 2 and in the beginning of Section 5.

2. Notation and background information

We work in \( \mathbb{R}^n \), which is equipped with a Euclidean structure \( \langle \cdot, \cdot \rangle \). We denote by \( \| \cdot \|_2 \) the corresponding Euclidean norm, and write \( B_2^n \) for the Euclidean unit ball, and \( S^{n-1} \) for the unit sphere. The volume of an \( s \)-dimensional set \( A \) is denoted by \( \text{vol}_s(A) \). We write \( \omega_n \) for the volume of \( B_2^n \) and \( \sigma_n \) for the rotationally invariant probability measure on \( S^{n-1} \). The Grassmann manifold \( G_{n,k} \) of \( k \)-dimensional
The letters $c, c', c_1, c_2$, etc. denote absolute positive constants whose value may change from line to line. Whenever we write $a \simeq b$, we mean that there exist absolute constants $c_1, c_2 > 0$ such that $c_1 a \leq b \leq c_2 a$. Similarly, if $K, L \subseteq \mathbb{R}^n$ we will write $K \simeq L$ if there exist absolute constants $c_1, c_2 > 0$ such that $c_1 K \subseteq L \subseteq c_2 K$. We also write $\mathcal{A}$ for the homothetic image of volume 1 of a convex body $A \subseteq \mathbb{R}^n$, i.e., $\mathcal{A} := \text{vol}_n(A)^{-1/n} A$.

A convex body is a compact convex subset $K$ of $\mathbb{R}^n$ with non-empty interior. We say that $K$ is origin symmetric if $-x \in K$ whenever $x \in K$. We say that $K$ is centered if it has barycenter at the origin, i.e., $\int_K \langle x, \theta \rangle dx = 0$ for every $\theta \in S^{n-1}$.

The support function $h_K : \mathbb{R}^n \to \mathbb{R}$ of $K$ is defined by $h_K(x) = \max \{ \langle x, y \rangle : y \in K \}$. The radius of $K$ is defined as $R(K) = \max \{ \|x\|_2 : x \in K \}$ and, if the origin is an interior point of $K$, the polar body $K^o$ of $K$ is

$$K^o := \{ y \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } x \in K \}. \tag{2.1}$$

We will use the fact that

$$c^n \text{vol}_n(B_2^n)^2 \leq \text{vol}_n(K) \text{vol}_n(K^o) \leq \text{vol}_n(B_2^n)^2 \tag{2.2}$$

for every centered convex body $K$ in $\mathbb{R}^n$. The right-hand side inequality is the Blaschke-Santaló inequality, while the left-hand side inequality is due to Bourgain and V. Milman [3] and holds true if we just assume that $0 \in \text{int}(K)$.

For each $p > -n$, $p \neq 0$, we set

$$I_p(K) := \left( \int_K \|x\|_2^p dx \right)^{1/p} \tag{2.3}$$

and for each $-\infty < p < \infty$, $p \neq 0$, we define the $p$-mean width of $K$ by

$$w_p(K) := \left( \int_{S^{n-1}} h_K^p(\theta) d\sigma_n(\theta) \right)^{1/p}. \tag{2.4}$$

From Hölder’s inequality, both are increasing functions of $p$. The mean width of $K$ is the quantity $w(K) = w_1(K)$. Note that

$$w_{-n}(K) = \left( \frac{\text{vol}_n(B_2^n)}{\text{vol}_n(K^o)} \right)^{\frac{1}{n}}. \tag{2.5}$$

This is immediate if we express $\text{vol}_n(K^o)$ in polar coordinates. If $K$ is an origin symmetric convex body in $\mathbb{R}^n$ and $\| \cdot \|_K$ is the norm induced to $\mathbb{R}^n$ by $K$, we set

$$M(K) = \int_{S^{n-1}} \|x\|_K d\sigma_n(x)$$

and write $b(K)$ for the smallest positive constant $b$ with the property $\|x\|_K \leq b\|x\|_2$ for all $x \in \mathbb{R}^n$. From V. Milman’s proof of Dvoretzky’s theorem (see [10]) we know that if $k \leq cn(M(K)/b(K))^2$, then for most $F \in G_{n,k}$ we have $K \cap F \simeq \frac{1}{M(K)} B_F$.

For every convex body $K$ in $\mathbb{R}^n$ and for every $1 \leq k \leq n-1$ we define the normalized $k$-th quermassintegral of $K$ by

$$Q_k(K) = \left( \frac{1}{\omega_k} \int_{G_{n,k}} \text{vol}_k(P_F(K)) d\nu_{n,k}(F) \right)^{1/k} \tag{2.6}.$$
Note that $Q_1(K) = w(K)$. From the Aleksandrov-Fenchel inequality (see [14]) it follows that $Q_k(K)$ is a decreasing function of $k$. In particular,

$$\left( \int_{G_{n,k}} \operatorname{vol}_k(P_F(K)) \, d\nu_{n,k}(F) \right)^{1/k} \leq \frac{c_1 w(K)}{\sqrt{k}},$$

where $c_1 > 0$ is an absolute constant. We refer to the books [14] and [10] for basic facts from the Brunn-Minkowski theory and the asymptotic theory of finite dimensional normed spaces.

The next two functionals will play an essential role in our argument.

(i) $p$-mean projection function. For every $1 \leq k \leq n - 1$ and for every $p \neq 0$ we define the $p$-mean projection function $W_{[k,p]}(K)$ by

$$W_{[k,p]}(K) := \left( \int_{G_{n,k}} \operatorname{vol}_k(P_F(K))^p \, d\nu_{n,k}(F) \right)^{\frac{1}{kp}}.$$

We also set $W_{[n]}(K) := \operatorname{vol}_n(K)^{1/n}$.

(ii) $p$-mean section function. For every $1 \leq k \leq n - 1$ and for every $p \neq 0$ we define the $p$-mean section function $\tilde{W}_{[k,p]}(K)$ by

$$\tilde{W}_{[k,p]}(K) = \left( \int_{G_{n,k}} \operatorname{vol}_{n-k}(K \cap F^{-\perp})^p \, d\nu_{n,k}(F) \right)^{\frac{1}{kp}}.$$

The normalized dual $k$-th quermassintegral of $K$ is the quantity

$$\tilde{W}_{[k]}(K) := \tilde{W}_{[k,1]}(K).$$

3. Ellipsoids

We start with the proof of Proposition 1.1. We will use the classical fact that Steiner symmetrization transforms an ellipsoid to an ellipsoid (see for example [2]). Here we state it as a lemma and include its proof for the sake of completeness.

Lemma 3.1. For every $u \in S^{n-1}$ and for every ellipsoid $E$ the Steiner symmetral $S_u(E)$ of $E$ with respect to $u$ is an ellipsoid.

Proof. Assume without loss of generality that the ellipsoid is centered at the origin. Consider a positive definite map $T : \mathbb{R}^n \to \mathbb{R}^n$ so that

$$\mathcal{E} = \{ x \in \mathbb{R}^n : \langle Tx, x \rangle \leq 1 \}.$$

By the definition of Steiner symmetrization, a point $y \in \mathbb{R}^n$ belongs to $S_u(\mathcal{E})$ if the line $L = \{ y + \lambda u : \lambda \in \mathbb{R} \}$ intersects $\mathcal{E}$ and

$$|\langle y, u \rangle| \leq \frac{1}{2} \text{length}(\mathcal{E} \cap L).$$

The assumption that $L$ intersects $\mathcal{E}$ means that there exists $\lambda \in \mathbb{R}$ so that $\langle T(y + \lambda u), (y + \lambda u) \rangle \leq 1$. The left-hand side is a quadratic function of $\lambda$, so its discriminant is non-negative, that is,

$$\langle Ty, u \rangle^2 + \langle Tu, u \rangle - \langle Tu, u \rangle \langle Ty, y \rangle \geq 0.$$
In this case the length in (3.1) equals 
\[ 2\sqrt{\langle Ty, u \rangle^2 - \langle Tu, u \rangle(\langle Ty, y \rangle - 1)} / \langle Tu, u \rangle. \]

Substituting in (3.1) we get that 
\[ S_u(\mathcal{E}) = \left\{ y \in \mathbb{R}^n : \langle Tu, u \rangle^2 \langle y, u \rangle^2 \leq \langle Ty, u \rangle^2 - \langle Tu, u \rangle(\langle Ty, y \rangle - 1) \right\}. \]

This set is clearly an ellipsoid (it is defined by a quadratic form).

Note. In fact, it is known that Lemma 3.1 characterizes ellipsoids in the following sense: if \( K \) is a convex body with the property that all its Steiner symmetrals \( S_u(K) \) are affine images of \( K \), then \( K \) is an ellipsoid (see e.g. [7]).

**Proof of Proposition 1.1.** Assume without loss of generality that \( E \) is centered at the origin. We first prove (1.3). We distinguish two cases.

**Case 1:** \( F \) is generated by the unit vectors of \( k \) semiaxes of \( E \). In this case if \( \lambda_1, \ldots, \lambda_n \) are the positive lengths of the ellipsoid’s semiaxes, then obviously 
\[ \text{vol}_k(P_F(\mathcal{E})) \text{vol}_{n-k}(\mathcal{E} \cap F^\perp) = \left( \prod_{j=1}^{n} \lambda_j \right) \text{vol}_k(B_2^k) \text{vol}_{n-k}(B_2^{n-k}) = \frac{\text{vol}_k(B_2^k)}{\text{vol}_n(B_2^n)} \text{vol}_n(\mathcal{E}). \]

**Case 2:** \( F \) is any element of \( G_{n,k} \). Let \( u_1, \ldots, u_k \) be any orthonormal basis of \( F \). We write \( \mathcal{E}' = S_{u_1} \ldots (S_{u_k} \mathcal{E}) \ldots \) for the ellipsoid obtained by successive Steiner symmetrizations of \( \mathcal{E} \) in the directions \( u_1, \ldots, u_k \). By the properties of Steiner symmetrization we have that 
\[ \text{vol}_k(P_F(\mathcal{E})) = \text{vol}_k(P_F(\mathcal{E}')) \quad \text{and} \quad \text{vol}_{n-k}(\mathcal{E} \cap F^\perp) = \text{vol}_{n-k}(\mathcal{E}' \cap F^\perp). \]

From Lemma 3.1 it follows that \( \mathcal{E}' \) is an ellipsoid which in addition has the same volume as \( \mathcal{E} \). Moreover, observe that Case 1 applies now to the ellipsoid \( \mathcal{E}' \) and the subspace \( F \). Thus, we get 
\[ \text{vol}_k(P_F(\mathcal{E})) \text{vol}_{n-k}(\mathcal{E} \cap F^\perp) = \text{vol}_k(P_F(\mathcal{E}')) \text{vol}_{n-k}(\mathcal{E}' \cap F^\perp) = \frac{\text{vol}_k(B_2^k)}{\text{vol}_n(B_2^n)} \text{vol}_n(\mathcal{E}'). \]

completing the proof of (1.3).

Since \( \text{vol}_n(B_2^n) = \pi^{n/2} / \Gamma(1 + n/2) \) it is elementary to check that (1.4) holds true as well.

**4. General bounds**

Let \( K \) be a centered convex body of volume 1 in \( \mathbb{R}^n \). In order to obtain a lower bound for \( g(K, k; F) \) we will estimate the expectation \( \mathbb{E}_{\nu_{n,k}} \left[ (g(K, k; F))^{-a} \right] \) for
some $a > 0$. For any pair $(p, q)$ of conjugate exponents, using Hölder’s inequality we write

\begin{equation}
\int_{G_{n,k}} \frac{1}{\text{vol}_k(P_F(K))} \text{vol}_{n-k}(K \cap F^\perp) \, d\nu_{n,k}(F) \leq \left( \int_{G_{n,k}} \frac{1}{\text{vol}_k(P_F(K))^p} \, d\nu_{n,k}(F) \right)^{1/p} \left( \int_{G_{n,k}} \frac{1}{\text{vol}_{n-k}(K \cap F^\perp)^q} \, d\nu_{n,k}(F) \right)^{1/q}.
\end{equation}

For the first integral in the right-hand side of (4.1) one may use the next lemma (from [6]) which relates it to the mixed widths of $K$.

\textbf{Lemma 4.1.} Let $K$ be a centered convex body of volume 1 in $\mathbb{R}^n$. Then, for every $1 \leq k \leq n-1$ and $p \geq 1$,

\begin{equation}
W_{[k,-p]}(K) = \left( \int_{G_{n,k}} \text{vol}_k(P_F(K))^{-p} \, d\nu_{n,k}(F) \right)^{-\frac{1}{kp}} \geq c_1 \frac{w_{-kp}(K)}{\sqrt{k}},
\end{equation}

where $c_1 > 0$ is an absolute constant.

\textbf{Proof.} Using Hölder’s inequality, the Blaschke-Santaló and the reverse Santaló inequality, for every $p \geq 1$ we can write

\begin{align*}
\left( \int_{G_{n,k}} \text{vol}_k(P_F(K))^{-p} \, d\nu_{n,k}(F) \right)^{\frac{1}{kp}} &\approx \left( \int_{G_{n,k}} \frac{\text{vol}_k((P_F(K))^p)}{\omega_k^{2p}} \, d\nu_{n,k}(F) \right)^{\frac{1}{kp}} \\
&\approx \sqrt{k} \left( \int_{G_{n,k}} \left( \int_{S_F} \frac{1}{h_k^{P_F(K)}(\theta)} d\sigma_F(\theta) \right)^p \, d\nu_{n,k}(F) \right)^{\frac{1}{kp}} \\
&\approx \sqrt{k} \left( \int_{G_{n,k}} \left( \int_{S_F} \frac{1}{h_k^{P_F(K)}(\theta)} d\sigma_F(\theta) \right)^p \, d\nu_{n,k}(F) \right)^{\frac{1}{kp}} \\
&\leq c\sqrt{k} \left( \int_{G_{n,k}} \int_{S_F} \frac{1}{h_k^{P_F(K)}(\theta)} d\sigma_F(\theta) \, d\nu_{n,k}(F) \right)^{\frac{1}{kp}} \\
&= c\sqrt{k} \left( \int_{S^{n-1}} \frac{1}{h_k^{P_F(K)}(\theta)} d\sigma(\theta) \right)^{\frac{1}{kp}} \\
&= c\sqrt{k} w_{-1-kp}(K).
\end{align*}

The lemma follows. \hfill \Box

We set $p := n/k > 1$. Then, from Lemma 4.1, (2.5) and (2.2) we get

\begin{equation}
W_{[k,-n/k]}(K) \geq \frac{w_{-n}(K)}{c_1 \sqrt{k}} \approx \frac{1}{c_1 \sqrt{k}} \left( \frac{\text{vol}_n(B_2^\delta)}{\text{vol}_n(K^0)} \right)^{1/n} \approx \sqrt{n/k}.
\end{equation}
This gives:

**Lemma 4.2.** Let $K$ be a centered convex body of volume 1 in $\mathbb{R}^n$. Then, for every $1 \leq k \leq n - 1$,

\[
W_{[k,n/k]}^{-1}(K) = \left( \int_{G_{n,k}} \frac{1}{\text{vol}_k(P_F(K))} \text{vol}_{n/k}(F) \, d\nu_{n,k}(F) \right)^{1/n} \leq c_2 \sqrt{k/n}
\]

where $c_2 > 0$ is an absolute constant.

Taking into account (4.1) we get the next general estimate.

**Proposition 4.3.** Let $K$ be a centered convex body of volume 1 in $\mathbb{R}^n$. For any $1 \leq k \leq n - 1$ we have

\[
\int_{G_{n,k}} \frac{1}{\text{vol}_k(P_F(K))} \text{vol}_{n-k}(K \cap F^\perp) \, d\nu_{n,k}(F) \leq \left( c_1 \sqrt{k/n} \right)^k \left( \int_{G_{n,k}} \frac{1}{\text{vol}_{n-k}(K \cap F^\perp) \pi_n} \right)^{n-k/n},
\]

where $c_1 > 0$ is an absolute constant.

We turn to the upper bound. The next proposition shows that the normalized dual quermassintegrals $\tilde{W}_{[k]}(K)$ are strongly related to the quantities $I_p(K)$.

**Lemma 4.4.** Let $K$ be a convex body of volume 1 in $\mathbb{R}^n$ and let $1 \leq k \leq n - 1$. Then,

\[
\tilde{W}_{[k]}(K) I_{-k}(K) = \left( \frac{(n-k)\omega_{n-k}}{n\omega_n} \right)^{1/k} = \tilde{W}_{[k]}(\mathbb{B}_2^n) I_{-k}(\mathbb{B}_2^n).
\]

Direct computation shows that $\left( \frac{(n-k)\omega_{n-k}}{n\omega_n} \right)^{1/k} \approx \sqrt{n}$.

**Proof.** We integrate in polar coordinates:

\[
I_{-k}(K) = \frac{n\omega_n}{n-k} \int_{S^{n-1}} \frac{1}{\|x\|^{n-k}} d\sigma(x)
\]

\[
= \frac{n\omega_n}{(n-k)\omega_{n-k}} \int_{G_{n,n-k}} \omega_{n-k} \int_{S_F} \frac{1}{\|\theta\|^{n-k}} d\sigma(\theta) d\nu_{n,n-k}(F)
\]

\[
= \frac{n\omega_n}{(n-k)\omega_{n-k}} \int_{G_{n,n-k}} \text{vol}_{n-k}(K \cap F) d\nu_{n,n-k}(F)
\]

\[
= \frac{n\omega_n}{(n-k)\omega_{n-k}} \int_{G_{n,n-k}} \text{vol}_{n-k}(K \cap F^\perp) d\nu_{n,k}(F),
\]

and the result follows from the definition of $\tilde{W}_{[k]}(K)$.

It was proved in [12] that if $K$ is a centered convex body of volume 1 in $\mathbb{R}^n$, then for any $p > -n$ we have

\[
I_p(K) \geq I_p(\mathbb{B}_2^n).
\]
One can also check that \( \tilde{W}_{[k]}(B^n_2) \simeq 1 \) for all \( 1 \leq k \leq n - 1 \). Then, Lemma 4.4 immediately gives:

**Lemma 4.5.** Let \( K \) be a centered convex body of volume 1 in \( \mathbb{R}^n \). Then, for every \( 1 \leq k \leq n - 1 \),

\[
\tilde{W}_{[k]}(K) \leq \tilde{W}_{[k]}(B^n_2) \simeq 1.
\]

Now we write

\[
\int_{G_{n,k}} \left( \text{vol}_k(P_F(K)) \text{vol}_{n-k}(K \cap F^\perp) \right)^{1/2} d\nu_{n,k}(F)
\]

\[
\leq \left( \int_{G_{n,k}} \text{vol}_k(P_F(K)) d\nu_{n,k}(F) \right)^{1/2} \left( \int_{G_{n,k}} \text{vol}_{n-k}(K \cap F^\perp) d\nu_{n,k}(F) \right)^{1/2},
\]

and taking into account Lemma 4.5 we get the next general estimate.

**Proposition 4.6.** Let \( K \) be a centered convex body of volume 1 in \( \mathbb{R}^n \). For any \( 1 \leq k \leq n - 1 \) we have

\[
\int_{G_{n,k}} \left( \text{vol}_k(P_F(K)) \text{vol}_{n-k}(K \cap F^\perp) \right)^{1/2} d\nu_{n,k}(F) \leq c_2 \left( \int_{G_{n,k}} \text{vol}_k(P_F(K)) d\nu_{n,k}(F) \right)^{1/2},
\]

where \( c_2 > 0 \) is an absolute constant.

Taking into account (2.7) we see that

\[
\int_{G_{n,k}} \left( \text{vol}_k(P_F(K)) \text{vol}_{n-k}(K \cap F^\perp) \right)^{1/2} d\nu_{n,k}(F) \leq \left( \frac{c_3 w(K)}{\sqrt{k}} \right)^{k/2},
\]

where \( c_3 > 0 \) is an absolute constant. Then, Markov’s inequality implies the following.

**Proposition 4.7.** Let \( K \) be a centered convex body of volume 1 in \( \mathbb{R}^n \). For any \( 1 \leq k \leq n - 1 \) we have that a random \( F \in G_{n,k} \) satisfies

\[
g(K, k; F) = \left( \text{vol}_k(P_F(K)) \text{vol}_{n-k}(K \cap F^\perp) \right)^{1/k} \leq \frac{c_4 w(K)}{\sqrt{k}}
\]

with probability greater than \( 1 - e^{-k} \), where \( c_4 > 0 \) is an absolute constant.

5. THE ISOPTROPIC CASE

Recall that a convex body \( K \) in \( \mathbb{R}^n \) is called isotropic if it has volume 1, it is centered, i.e., its barycenter is at the origin, and its inertia matrix is a multiple of the identity: there exists a constant \( L_K > 0 \) such that

\[
\int_K \langle x, \theta \rangle^2 dx = L_K^2
\]

for every \( \theta \) in the Euclidean unit sphere \( S^{n-1} \). More generally, a log-concave probability measure \( \mu \) on \( \mathbb{R}^n \) is called isotropic if its barycenter is at the origin and its inertia matrix is the identity; in this case, the isotropic constant of \( \mu \) is defined as

\[
L_{\mu} := \sup_{x \in \mathbb{R}^n} \left( f_\mu(x) \right)^{1/n},
\]
where \( f_\mu \) is the density of \( \mu \) with respect to the Lebesgue measure. Note that a centered convex body \( K \) of volume 1 in \( \mathbb{R}^n \) is isotropic if and only if the log-concave probability measure \( \mu_K \) with density \( x \mapsto L_{K}^{n}1_{K/L_{K}}(x) \) is isotropic. The reader may find a detailed and updated exposition of the theory of isotropic log-concave measures in the book [4].

Let \( \mu \) be a probability measure on \( \mathbb{R}^n \) with density \( f_\mu \) with respect to the Lebesgue measure. For every \( 1 \leq k \leq n-1 \) and every \( E \in G_{n,k} \), the marginal of \( \mu \) with respect to \( E \) is the probability measure with density

\[
(5.3) \quad f_{\pi_E \mu}(x) = \int_{x+E^\perp} f_\mu(y) dy.
\]

It is easily checked that if \( \mu \) is centered, isotropic or log-concave, then \( \pi_E \mu \) is also centered, isotropic or log-concave, respectively. For every log-concave probability measure \( \mu \) on \( \mathbb{R}^n \) and any \( p > 0 \) we define the set \( K_p(\mu) \) as follows:

\[
K_p(\mu) = \left\{ x \in \mathbb{R}^n : \int_0^\infty f_\mu(rx) r^{p-1} dr \geq \frac{f_\mu(0)}{p} \right\}.
\]

The bodies \( K_p(\mu) \) were introduced by K. Ball [1] who showed that they are convex. The next proposition is a generalization of a result of Ball from the same work (see also [9], and [4] for the precise statement below); it gives a very useful expression for the volume of central sections of an isotropic convex body.

**Proposition 5.1.** Let \( K \) be an isotropic convex body in \( \mathbb{R}^n \). We denote by \( \mu_K \) the isotropic log-concave measure with density \( L_{K}^{n}1_{K/L_{K}} \). Then, for every \( 1 \leq k \leq n-1 \) and \( F \in G_{n,k} \), the body \( \overline{K_{k+1}(\pi_F(\mu_K))} \) satisfies

\[
(5.4) \quad \text{vol}_{n-k}(K \cap F^\perp)^{1/k} \simeq \frac{L_{K}}{L_{K+1}(\pi_F(\mu_K))}.
\]

Assume that \( K \) is an isotropic convex body in \( \mathbb{R}^n \). From Proposition 5.1 we know that, for every \( 1 \leq k \leq n-1 \) and \( F \in G_{n,k} \),

\[
(5.5) \quad \text{vol}_{n-k}(K \cap F^\perp)^{-1/k} \simeq \frac{L_{K}}{L_{K+1}(\pi_F(\mu_K))} \leq c_2 L_K,
\]

because \( L_C \geq c \) for every convex body \( C \), where \( c > 0 \) is an absolute constant (see for example Proposition 2.3.12 in [4]). Therefore, Proposition 4.3 gives

\[
\int_{G_{n,k}} \frac{1}{\text{vol}_k(P_F(K)) \text{vol}_{n-k}(K \cap F^\perp)} d\nu_{n,k}(F) \leq \left(c_1 \sqrt{n/k}\right)^k (c_2 L_K)^k \leq (c_3 \sqrt{k/nL_{K}})^k.
\]

From Markov’s inequality we get:

**Proposition 5.2.** Let \( K \) be an isotropic convex body in \( \mathbb{R}^n \). For every \( 1 \leq k \leq n-1 \), a random \( F \in G_{n,k} \) satisfies

\[
(5.6) \quad g(K, k; F) := \left( \text{vol}_k(P_F(K)) \text{vol}_{n-k}(K \cap F^\perp) \right)^{1/k} \geq \frac{c_4 \sqrt{n/k}}{L_K}
\]

with probability greater than \( 1 - e^{-k} \), where \( c_4 > 0 \) is an absolute constant.
For the upper bound we use (2.7) and a recent result of E. Milman [8]: if \( K \) is isotropic, and if we make the additional assumption that \( K \) is origin symmetric, then
\[
w(K) \leq c_5 \sqrt{n} (\log n)^2 L_K.
\]
Thus, directly applying Proposition 4.7 we get:

**Proposition 5.3.** Let \( K \) be an origin symmetric isotropic convex body in \( \mathbb{R}^n \). For every \( 1 \leq k \leq n - 1 \) a random \( F \in G_{n,k} \) satisfies
\[
g(K, k; F) := \left( \frac{\text{vol}_k(P_F(K))}{\text{vol}_n - k(K \cap F^\perp)} \right)^{\frac{1}{2}} \leq c_6 \sqrt{n/k} (\log n)^2 L_K
\]
with probability greater than \( 1 - e^{-k} \).

Combining Proposition 5.2 and Proposition 5.3 we obtain Theorem 1.2.

**Remark 5.4.** (i) It is known that for every isotropic convex body \( K \) in \( \mathbb{R}^n \) we can find an origin-symmetric isotropic convex body \( T \) with the property that \( L_T \simeq L_K \) (see [4, Proposition 2.5.10]): if we define a function \( f \) supported on \( K - K \) by
\[
f(x) = (1_K * 1_{-K})(x) = \int_{\mathbb{R}^n} 1_K(y) 1_{-K}(x - y) \, dy = \text{vol}_n(K \cap (x + K)),
\]
then \( f \) is an even isotropic log-concave density and one can check that \( L_f = \sqrt{2} L_K \).

It follows from the convex body \( T = K_{n+2}(f) \) has the desired properties. From Proposition 4.6 we see that the upper bound in Theorem 1.2 remains valid for a not necessarily symmetric isotropic convex body \( K \) and some \( 1 \leq k \leq n - 1 \), provided that
\[
\int_{G_{n,k}} \text{vol}_k(P_F(K))d\nu_{n,k}(F) \leq C_k \int_{G_{n,k}} \text{vol}_k(P_F(T))d\nu_{n,k}(F).
\]

(ii) The logarithmic terms in (5.7) cannot be completely eliminated as long as the proof passes through estimates of the mean width of \( K \). This is evident from the case of \( K = \overline{B}_1^n \), where \( w(\overline{B}_1^n) \simeq \sqrt{n \log(1 + n)} \). However, some of these terms may not be needed. For example, if the body is in the \( \ell \)-position (see [4, Section 1.11]), then the reverse Urysohn inequality \( w(K) \leq c\sqrt{n} \log n \) and Proposition 4.7 imply that \( g(K, k; F) \leq c_6 \sqrt{n/k} \log n \) for a random \( F \in G_{n,k} \).

**References**


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