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THE UPPER AND LOWER BOUNDS ON NON-REAL EIGENVALUES OF INDEFINITE STURM-LIOUVILLE PROBLEMS

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Abstract. The present paper gives a priori upper and lower bounds on non-real eigenvalues of regular indefinite Sturm-Liouville problems only under the integrability conditions. More generally, a lower bound on non-real eigenvalues of the self-adjoint operator in Krein space is obtained.

1. Introduction

Consider the regular indefinite Sturm-Liouville eigenvalue problem

$$-(p(x)y'(x))' + q(x)y(x) = \lambda w(x)y(x), \;\; x \in (a,b), \tag{1.1}$$

$$y(a)\cos\theta_1 - py'(a)\sin\theta_1 = 0, \; y(b)\cos\theta_2 - py'(b)\sin\theta_2 = 0, \tag{1.2}$$

where p, q, w are real-valued functions satisfying the standard conditions

$$p(x) > 0, \; w(x) \neq 0 \;\text{ a.e. on } [a,b], \; \frac{1}{p}, \; q, \; w \in L^1[a,b], \tag{1.3}$$

λ is the spectral parameter, θ_1, $\theta_2 \in [0,\pi)$ and the weight function $w(x)$ changes sign on $[a,b]$, namely, both sets $\{x \in [a,b] : w(x) > 0\}$ and $\{x \in [a,b] : w(x) < 0\}$ have a positive Lebesgue measure. In this case, problem (1.1), (1.2) is called *right-indefinite*.

Let $L^2_{|w|}[a,b]$ be the Hilbert space of all weighted square-integrable functions on $[a,b]$ equipped with the inner product $(f,g)_{|w|} := \int_a^b |w| f\overline{g}$ and the norm $\|f\|^2_{|w|} := \int_a^b |w||f|^2$. Set

$$\tau y := -(py')' + qy.$$

The operator $S := \frac{1}{|w|}\tau$ associated to the corresponding *right-definite* eigenvalue problem

$$-(p(x)y')' + q(x)y = \lambda |w(x)| y, \;\; x \in (a,b), \tag{1.4}$$

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subject to (1.2) is self-adjoint in the Hilbert space $(L^2_{|w|}[a,b], (\cdot,\cdot)_{|w|})$ and has a discrete spectrum bounded below without finite accumulation.

On the other hand, the operator $T := \frac{1}{w}\tau$ associated to the indefinite problem (1.1) and (1.2) is self-adjoint in the Krein space $(L^2_{|w|}[a,b], [\cdot,\cdot]_w)$ with respect to the indefinite inner product (cf. [7])

$$[f,g]_w := \int_a^b \overline{g(x)} w(x) f(x) \mathrm{d}x, \quad f,\, g \in L^2_{|w|}[a,b],$$

and has a discrete spectrum containing a set of real eigenvalues unbounded below and above ([5, Theorem 2.2] and [12, Theorem 5.1]) and a possible set of non-real eigenvalues only if S has negative eigenvalues (cf. [7, 11] or [10, Theorem 3.3]). Moreover, the indefiniteness of the weight function $w(x)$ gives rise to the existence of "ghost states". Let λ be an eigenvalue and ϕ a corresponding eigenfunction. If $\lambda \int_0^1 w|\phi|^2 = 0$, we call ϕ a *neutral or ghost state*. It is well known that a complex eigenvalue corresponds to a neutral state. If λ is real and $\lambda \int_0^1 w|\phi|^2 < 0$, then λ is called either a *positive eigenvalue of negative type* or a *negative eigenvalue of positive type*, and ϕ is also called a *non-degenerate real ghost state* [12].

The indefinite nature of the problem had not been discussed until the work of Haupt, Richardson, and others (cf. [9, 14, 15]). For a review of early works on indefinite problems, see [12]. The peculiar properties of the spectrum of the indefinite operator T has attracted much attention (cf. [1, 4, 6, 8, 16]).

In 1986, Mingarelli made a summary of regular indefinite Sturm-Liouville problems and raised an open problem (cf. [12]): can one find an a priori bound on the modulus, real and imaginary parts of the "largest" non-real eigenvalue which might appear? In 2003, Kong, Müller, Wu and Zettl [10, Remark 4.4, Example 4.5] posed a similar question of how to estimate the magnitude of the non-real eigenvalues or their imaginary parts of T in terms of the negative eigenvalues of S and gave an example to show that such an estimate does not exist (see also [19, Remark 11.4.1]).

Recently, in 2013 and 2014, upper bounds on non-real eigenvalues of T have been obtained explicitly in terms of coefficients in [13, 17] under the condition that the weight function w can change signs only once or is absolutely continuous. In 2014, the condition that the weight function can change signs any time was weakened in [2]. In 2013, Behrndt, Philipp and Trunk obtained a bound on non-real eigenvalues in a special singular case (see [3]). In 2015, the upper bounds on non-real eigenvalues of a p-Laplacian with indefinite weight have been given in [18].

In this paper, we will give both upper and lower bounds on the eigenvalues to neutral states and the positive (negative) eigenvalues of negative (positive, resp.) type of T without any additional restrictions to the standard conditions (1.3). The results obtained are complete answers to the questions raised by Mingarelli in 1986. Besides, our results and examples will show that it is basically the oscillation of $w(x)$ that drives away the non-real eigenvalues from the real line.

The arrangement of the present paper is as follows. The main results, Theorems 2.1, 2.2 and 2.4, are stated in Section 2 and their proofs are given in Section 3. Two examples are given in Section 4 to illustrate the results.

2. Main results

To simplify our statements, we need to fix some notation. Denote by $\|\cdot\|_1$ the $L^1[a,b]$-norm and by $\|\cdot\|_\infty$ the $L^\infty[a,b]$-norm. The positive and negative parts of a function q will be denoted by $q_\pm = \max\{\pm q, 0\}$. Set

$$N = N(\theta_1, \theta_2, q) = |\cot^* \theta_1| + |\cot^* \theta_2| + \|q_-\|_1,$$
$$\delta = \min\left\{\frac{1}{4N}, \frac{B}{2}\right\}, \quad m = 1 + \max\{\lfloor 8NB \rfloor, 4\}, \tag{2.1}$$

where $B := \int_a^b 1/p$, $\lfloor\cdot\rfloor$ is the integer floor function and

$$\cot^* \theta = \begin{cases} \cot\theta, & \theta \in (0, \pi), \\ 0, & \theta = 0. \end{cases}$$

Let $x_k = \frac{k}{m}B$, $k = 0, 1, 2, \dots, m$. Then each interval $I_k := [x_{k-1}, x_k]$ has an equal length of $\frac{B}{m} < \frac{\delta}{2}$, $k = 1, \dots, m$.

Set

$$s = \int_a^x \frac{1}{p} =: f(x),\ x \in [a, b], \text{ and } \widetilde{w}(s) := p(x)w(x),\ s \in [0, B],$$

with $x = f^{-1}(s)$. It follows from $\int_0^B |\widetilde{w}| = \int_a^b w$ and $w \in L^1[a, b]$ that $\widetilde{w} \in L^1[0, B]$ and

$$\frac{d}{ds}\left(\int_0^s \widetilde{w}\right) = \widetilde{w}(s),\ \frac{d}{ds}\left(\int_0^s |\widetilde{w}|\right) = |\widetilde{w}(s)|, \text{ a.e. } s \in (0, B). \tag{2.2}$$

Therefore, for each $I_k = [x_{k-1}, x_k]$, $x_k = \frac{k}{m}B$, $k = 1, \dots, m$, there exists ξ_k such that $\widetilde{w}(\xi_k) \neq 0$ and the two equalities in (2.2) hold simultaneously for $s = \xi_k$. Since

$$\lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_{\xi_k}^{\xi_k + \Delta x} \widetilde{w} = \widetilde{w}(\xi_k),\ \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_{\xi_k}^{\xi_k + \Delta x} |\widetilde{w}| = |\widetilde{w}(\xi_k)|,$$

one sees that

$$\lim_{\Delta x \to 0} \frac{\int_{\xi_k}^{\xi_k + \Delta x} \widetilde{w}}{\int_{\xi_k}^{\xi_k + \Delta x} |\widetilde{w}|} = \operatorname{sgn}(\widetilde{w}(\xi_k)).$$

If $\operatorname{sgn}(\widetilde{w}(\xi_k)) = 1$, we can then find an $\eta_k > 0$ small enough such that for every $x' \in (\xi_k - \eta_k, \xi_k)$ and $x'' \in (\xi_k, \xi_k + \eta_k)$,

$$\int_{x'}^{\xi_k} \widetilde{w} > \frac{9}{10} \int_{x'}^{\xi_k} |\widetilde{w}| \quad \text{and} \quad \int_{\xi_k}^{x''} \widetilde{w} > \frac{9}{10} \int_{\xi_k}^{x''} |\widetilde{w}|$$

and hence,

$$\int_{x'}^{x''} \widetilde{w} > \frac{9}{10} \int_{x'}^{x''} |\widetilde{w}|.$$

Similarly, if $\operatorname{sgn}(\widetilde{w}(t_k)) = -1$, we can get

$$\int_{x'}^{x''} \widetilde{w} < -\frac{9}{10} \int_{x'}^{x''} |\widetilde{w}|.$$

In a word, for every interval I_k there exist a $\xi_k \in F \cap I_k$ and an $\eta_k > 0$ such that $[\xi_k - \eta_k, \xi_k + \eta_k] \subset I_k$ and for every interval $I \subset [\xi_k - \eta_k, \xi_k + \eta_k]$ containing ξ_k we have

$$\left|\int_I \widetilde{w}\right| \geq \frac{9}{10} \int_I |\widetilde{w}|. \tag{2.3}$$

Define

$$\eta = \min\{\eta_k : k = 1, \cdots, m\} \tag{2.4}$$

and set

$$w_t := \min\Big\{ \int_s^{s+t} p(f^{-1}(\cdot))\big|w(f^{-1}(\cdot))\big| : \ s \in [0, T-t]\Big\}, \ \ t \in [0, B]. \tag{2.5}$$

Obviously, $w_t > 0$ for $t > 0$. We are now in a position to state the first result of upper bounds on the eigenvalues corresponding to neutral states, particularly, the non-real eigenvalues, and non-zero real eigenvalues of sign-opposite type.

Theorem 2.1. *If λ corresponds to an eigenfunction y of problem* (1.1), (1.2) *with $\lambda \int_a^b w|y|^2 \leq 0$, then the eigenvalue λ satisfies*

$$|\operatorname{Re}\lambda| \leq \frac{1}{3\eta w_\eta}(9 + 16\eta\|q\|_1), \quad |\operatorname{Im}\lambda| \leq \frac{8}{3\eta w_\eta}. \tag{2.6}$$

It is easily seen that the constants η and w_η involved in (2.6) are determined by θ_1, θ_2, p, q and w. Clearly, Theorem 2.1 generalizes the results in [13, Theorems 1.2-1.3] and [17, Theorems 1.2-1.3], since no additional restrictions are imposed on w. Thus, Theorem 2.1 completely answers the problem raised by Mingarelli in [12, p.120] and Kong, Müller, Wu and Zettl in [10].

Next, we will find lower bounds on non-real eigenvalues and real eigenvalues of sign-opposite type of a self-adjoint operator in the Krein space in terms of the eigenvalues of the corresponding self-adjoint operator in the Hilbert space.

First, let us introduce some symbols. Let S be a self-adjoint operator in the Hilbert space H with the inner product $(\cdot,\cdot)$ and the norm $\|\cdot\|^2 = (\cdot,\cdot)$. Denote the resolvent set of S by $\rho(S)$ and the spectrum of S by $\sigma(S)$. Let J be a fundamental symmetry in H satisfying $J^2 = I$ and $T := JS$. Then T is a self-adjoint operator in the Krein space $K := (H, [\cdot,\cdot])$, i.e., for any two $y_1,\ y_2 \in H$, $[Ty_1, y_2] = [y_1, Ty_2]$, where the inner product of K is $[\cdot,\cdot] := (J\cdot,\cdot)$.

Now, we state the lower bound result on T.

Theorem 2.2. *Let T and S be defined as above. Suppose that $0 \in \rho(S)$ and S^{-1} is compact. Let*

$$\mu^+ := \min\sigma(S) \cap (0,\infty),\ \mu^- := \min\{|\lambda| : \lambda \in \sigma(S) \cap (-\infty, 0)\},$$

where $\min\emptyset := \infty$. Then, for each eigenvalue λ of T we have

$$|\lambda| \geq \min\{\mu^+, \mu^-\}. \tag{2.7}$$

Moreover, if λ corresponds to an eigenvector ϕ of T with $[\phi,\phi] = 0$, then the following, in general stronger, estimate holds:

$$|\lambda|^2 \geq \mu^+\mu^-. \tag{2.8}$$

In the following result, the equality in (2.8) holds.

Corollary 2.3. *Suppose T is a 2×2 matrix, $J = \operatorname{diag}\{1,-1\}$ and $S = JT$ is Hermitian and has two real eigenvalues λ_1 and λ_2 satisfying $\lambda_1 < 0 < \lambda_2$. Then for any non-real eigenvalue λ of T, it holds that $|\lambda|^2 = -\lambda_1\lambda_2$.*

We now apply Theorem 2.2 to the problem (1.1), (1.2). To do this, we define a number ω which reflects the influence of the oscillation of the weight function as

$$\omega := \frac{1}{\|w\|_1}\max\left\{\left|\int_{t_1}^{t_2} p(f^{-1}(\cdot))w(f^{-1}(\cdot))\right| : [t_1, t_2] \subset [0, B]\right\}. \tag{2.9}$$

We see that $\omega > 0$ and that, roughly speaking, the interval where ω is attained is where $p(f^{-1}(\cdot))w(f^{-1}(\cdot))$ oscillates "most slowly".

Let λ_k be the k-th eigenvalue of the right-definite problem (1.4),(1.2). Set

$$N_k = N + |\lambda_k|\|w\|_1, \quad \delta_k := \min\left\{\frac{B}{4N_k}, \frac{B}{2}\right\}, \quad k \geq 1. \tag{2.10}$$

With this notation, we will prove the following result.

Theorem 2.4. *Suppose that $\lambda_{h-1} = \mu^- < 0 < \mu^+ = \lambda_h$, $h \geq 2$. If λ corresponds to an eigenfunction y of problem* (1.1), (1.2) *with $\lambda \int_a^b w|y|^2 \leq 0$, then the eigenvalue λ satisfies*

$$|\lambda|^2 \geq \frac{\lambda_h \lambda_{h-1}^2 w_\delta^2}{16(\lambda_h - \lambda_{h-1})\omega^2 \|w\|_1} \left(\sum_{k=1}^{h-1} \frac{(1 + \sqrt{BN} + \sqrt{BN_k})^2}{w_{\delta_k}} \right)^{-1}. \tag{2.11}$$

We will give an example (see (4.1) below) to show that the estimate of the lower bound in (2.11) is more accurate than the one given in (2.8) if ω is sufficiently small.

3. The proofs

The proofs of Theorems 2.1, 2.2 and 2.4 and Corollary 2.3 are given in this section. For the sake of simplicity, we prove Theorems 2.1 and 2.4 in the special case of the problem (1.1), (1.2) on the interval $(0, B)$ where $p = 1$ and then use the Sturm transformation to deal with the general case.

Let λ and ϕ be a pair of eigenvalue and eigenfunction of the problem (1.1), (1.2) on $[0, B]$ with $p \equiv 1$, i.e., the problem

$$-\phi''(s) + q(s)\phi(s) = \lambda q(s)\phi(s), \ s \in (0, B), \tag{3.1}$$

with the boundary condition

$$\phi(0)\cos\theta_1 - \phi'(0)\sin\theta_1 = 0, \phi(B)\cos\theta_2 - \phi'(B)\sin\theta_2 = 0. \tag{3.2}$$

Lemma 3.1. *If λ corresponds to an eigenfunction ϕ of problem* (3.1), (3.2) *with $\lambda \int_0^B w|\phi|^2 \leq 0$ and $\|\phi\|_\infty = 1$, then*

$$\int_0^B |\phi'|^2 \leq N = |\cot^* \theta_1| + |\cot^* \theta_2| + \|q_-\|_1. \tag{3.3}$$

Proof. Multiplying both sides of equation (3.1) by $\overline{\phi}$ and integrating by parts on $[0, B]$, we have from the boundary condition (3.2) that

$$|\phi(0)|^2 \cot^* \theta_1 - |\phi(B)|^2 \cot^* \theta_2 + \int_0^B |\phi'|^2 + \int_0^B q|\phi|^2 = \lambda \int_0^B w|\phi|^2. \tag{3.4}$$

Then, (3.3) follows from (3.4), $\lambda \int_0^B w|\phi|^2 \leq 0$ and $\|\phi\|_\infty = 1$. □

Lemma 3.2. *If λ corresponds to an eigenfunction ϕ of problem* (3.1), (3.2) *with $\lambda \int_0^B w|\phi|^2 \leq 0$ and $\|\phi\|_\infty = 1$, then there exists an interval $I_\phi \subset [0, B]$, with $\delta = \min\left\{\frac{1}{4N}, \frac{B}{2}\right\}$ in length, such that $|\phi(\cdot)| \geq \frac{1}{2}$ on I_ϕ.*

Proof. For any interval $I \subset [0, B]$ of length δ, by the Cauchy-Schwarz inequality and (3.3), we have

$$\left(\int_I |\phi'| \right)^2 \leq \int_I 1 \int_0^B |\phi'|^2 \leq \frac{1}{4}. \tag{3.5}$$

Let $x_0 \in [0, B]$ be such that $|\phi(x_0)| = \|\phi\|_\infty = 1$. Then for $x \in [0, B]$ and $|x - x_0| \le \delta$,

$$||\phi(x)| - 1| \le |\phi(x) - \phi(x_0)| = \left| \int_{x_0}^{x} \phi' \right| \le \frac{1}{2}$$

by (3.5), and hence $|\phi(x)| \ge 1/2$ on $I_\phi = [x_0 - \delta, x_0]$ or $[x_0, x_0 + \delta]$. □

Using these lemmas, we give the proof of Theorem 2.1.

Proof of Theorem 2.1. First we will prove the theorem in the case $p \equiv 1$ on $(0, B)$. Suppose that λ corresponds to an eigenfunction ϕ of problem (3.1) with $\|\phi\|_\infty = 1$ and $\lambda \int_0^B w|\phi|^2 \le 0$.

Let us split $[0, B]$ into m subintervals $I_k = [x_{k-1}, x_k]$, $k = 1, \ldots, m$, where $x_k = \frac{k}{m}B$, $k = 0, 1, \ldots, m$, and $x_k - x_{k-1} = \frac{B}{m} < \frac{\delta}{2}$; see definitions of m and δ in (2.1). Since the interval I_ϕ in Lemma 3.2 has length δ, there must be a subinterval $I_k \subset I_\phi$, and hence, we can select $\xi_k \in I_k \subset I_\phi$ such that $w(\xi_k) \ne 0$ and the two equalities in (2.2) hold simultaneously at $s = \xi_k$. Set $a_k = \xi_k - \eta$, $b_k = \xi_k + \eta$, where η is defined in (2.4). Then $[a_k, b_k] \subset [\xi_k - \eta_k, \xi_k + \eta_k]$ and (2.3) holds for every interval $I \subset [a_k, b_k]$ containing ξ_k, $k = 1, \cdots, m$.

Then $2\xi_k = a_k + b_k$, $b_k - a_k = 2\eta$ and from (2.3) we have

$$\left| \operatorname{sgn} w(\xi) \int_x^{x+\eta} w \right| \ge \frac{9}{10} \int_x^{x+\eta} |w|, \quad x \in [a_k, \xi_k].$$

We now assume $\operatorname{sgn} w(\xi) = 1$. The case when $\operatorname{sgn} w(\xi) = -1$ can be dealt with in a similar way. Since $w = w_+ - w_-$ and $|w| = w_+ + w_-$, with some manipulation we get for any $x \in [a_k, \xi_k]$,

$$\int_x^{x+\eta} w_+ \ge 19 \int_x^{x+\eta} w_-, \qquad \int_x^{x+\eta} w_+ \ge \frac{19}{20} \int_x^{x+\eta} |w|. \tag{3.6}$$

From $1 \ge |\phi| \ge \frac{1}{2}$ on $[a_k, b_k]$, (3.6) and the definition of w_t in (2.5), we estimate

$$\begin{aligned} \left| \int_x^{x+\eta} w|\phi|^2 \right| &= \left| \int_x^{x+\eta} (w_+ - w_-)|\phi|^2 \right| \ge \int_x^{x+\eta} \frac{w_+}{4} - \int_x^{x+\eta} w_- \\ &\ge \left(\frac{1}{4} - \frac{1}{19} \right) \frac{19}{20} \int_x^{x+\eta} |w| = \frac{3}{16} \int_x^{x+\eta} |w| \ge \frac{3}{16} w_\eta. \end{aligned} \tag{3.7}$$

By integration, from (3.1), we have

$$\phi' \overline{\phi} \,|_x^{x+\eta} + \int_x^{x+\eta} |\phi'|^2 + \int_x^{x+\eta} q|\phi|^2 = \lambda \int_x^{x+\eta} w|\phi|^2. \tag{3.8}$$

Taking norms and considering $\|\phi\|_\infty = 1$, (3.3) and (3.7) yield

$$\frac{3}{16}|\lambda| w_\eta \le |\phi'(x + \eta)| + |\phi'(x)| + N + \|q\|_1, \tag{3.9}$$

$$\frac{3}{16}|\mathrm{Im}\lambda| w_\eta \le |\phi'(x + \eta)| + |\phi'(x)|, \quad x \in [a_k, \xi_k]. \tag{3.10}$$

Integrating with respect to x over $[a_k, \xi_k]$ and using the Cauchy-Schwarz inequality and $\eta \le \delta/4$, we obtain that, for $t = 0$ or $t = \eta$,

$$\int_a^\xi |\phi'(x + t)| \mathrm{d}x \le \left(\eta \int_a^\xi |\phi'(x + t)|^2 \mathrm{d}x \right)^{\frac{1}{2}} \le \sqrt{\delta N/4} \le \frac{1}{4}. \tag{3.11}$$

Then, from (3.9) and (3.10), we have

$$\begin{aligned} &\frac{3}{16}|\lambda|\eta w_\eta \le \frac{1}{4} + \frac{1}{4} + \eta N + \eta\|q\|_1 \le \frac{9}{16} + \eta\|q\|_1, \\ &\frac{3}{16}|\operatorname{Im}\lambda|\eta w_\eta \le \frac{1}{2}, \end{aligned} \tag{3.12}$$

which imply the inequalities in (2.6).

For the general case, since $p > 0$ and $1/p \in L^1[a,b]$, the Sturm transformation

$$s = \int_a^x \frac{1}{p} =: f(x), \;\; \widetilde{y}(s) = y(f^{-1}(s)), \;\; B = \int_a^b \frac{1}{p}$$

reduces the problem (1.1) and (1.2) to the problem

$$-\widetilde{y}''(s) + \widetilde{q}(s)\widetilde{y}(s) = \lambda\widetilde{w}(s)\widetilde{y}(s), \;\; s \in (0,B), \tag{1.1$'$}$$

with the boundary conditions

$$\widetilde{y}(0)\cos\theta_1 - \widetilde{y}'(0)\sin\theta_1 = 0, \;\; \widetilde{y}(B)\cos\theta_2 - \widetilde{y}'(B)\sin\theta_2 = 0, \tag{1.2$'$}$$

where

$$\widetilde{q}(s) = p(x)q(x), \;\; \widetilde{w}(s) = p(x)w(x), \;\; \widetilde{y}(s) = y(x), \tag{3.13}$$

with $x = (f^{-1}(s))$. Clearly, $\widetilde{q}$ and $\widetilde{w}$ satisfy (1.3) with $[a,b]$ replaced by $[0,B]$. Besides,

$$\int_a^b w|y|^2 = \int_0^B \widetilde{w}|\widetilde{y}|^2 \;\text{ and }\; \int_a^b q = \int_0^B \widetilde{q} \tag{3.14}$$

and the eigenvalues of (1.1), (1.2) are just the same as the ones of (1.1$'$), (1.2$'$). Using this transformation and the above proof in the special case, we obtain an upper bound (2.6) for the general case. The proof of Theorem 2.1 is finished. □

Next we prove Theorem 2.2.

Proof of Theorem 2.2. Let λ_k be the k-th eigenvalue of S and let a corresponding eigenfunction ϕ_k satisfying $\{\phi_k,\, k \ge 1\}$ be the orthonormal system. Let ϕ be an eigenvector of T and λ the corresponding eigenvalue such that $\|\phi\|^2 = 1$.

Since S is self-adjoint and S^{-1} is compact, expanding ϕ via the orthonormal system $\{\phi_k\}$, i.e., $\phi = \sum_{k=1}^\infty d_k\phi_k,\; d_k = (\phi,\phi_k),\; k \ge 1$, we have

$$\sum_{k=1}^\infty |d_k|^2 = 1, \tag{3.15}$$

$$\sum_{k=1}^\infty |d_k|^2\lambda_k = (S\phi,\phi) = [T\phi,\phi] = \lambda[\phi,\phi], \tag{3.16}$$

$$\sum_{k=1}^\infty |d_k|^2\lambda_k^2 = (S\phi,S\phi) = (T\phi,T\phi) = |\lambda|^2. \tag{3.17}$$

From $|\lambda_k| \ge \min\{\mu^+,\, \mu^-\}$ for $k \ge 1$, by (3.15) and (3.17), we have $|\lambda| \ge \min\{\mu^+,\, \mu^-\}$.

If λ corresponds to an eigenvector ϕ of T satisfying $[\phi,\phi] = 0$, then it follows from (3.15), (3.16) and (3.17) that

$$\sum_{k=1}^\infty |d_k|^2\Big(\lambda_k - \frac{1}{2}(\mu^+ + \mu^-)\Big)^2 = |\lambda|^2 + \frac{1}{4}(\mu^+ + \mu^-)^2. \tag{3.18}$$

Note

$$\left(\lambda_k - \frac{1}{2}(\mu^+ + \mu^-)\right)^2 - \frac{1}{4}(\mu^+ - \mu^-)^2 = (\lambda_k - \mu^-)(\lambda_k - \mu^+) \geq 0. \tag{3.19}$$

From (3.15), (3.19) and (3.18) we get $|\lambda|^2 \geq -\mu^+\mu^-$ and complete the proof of Theorem 2.2. □

Proof of Corollary 2.3. Suppose that the 2×2 matrix T has a non-real eigenvalue λ and corresponding normal eigenvector ϕ and that the Hermitian matrix $S = JT$ has two real eigenvalues λ_1, λ_2, $\lambda_1 < 0 < \lambda_2$, with corresponding normal eigenvectors ϕ_1, ϕ_2, respectively. Then from the above proof, $\phi = d_1\phi_1 + d_2\phi_2$, $|d_1|^2 + |d_2|^2 = 1$ and $|d_1|^2\lambda_1 + |d_2|^2\lambda_2 = 0$. We thus have

$$|\lambda|^2 = |d_1|^2\lambda_1^2 + |d_2|^2\lambda_2^2 = -|d_2|^2\lambda_2\lambda_1 - |d_1|^2\lambda_1\lambda_2 = -\lambda_1\lambda_2.$$

□

To prove Theorem 2.4, we consider the self-adjoint operator $S = \frac{1}{|w|}\tau$ associated to the right-definite problem (1.4), i.e., $\tau(y) = \lambda|w|y$ in the domain

$$\mathcal{D}(S) = \Big\{y \in L^2_{|w|} :\ y, py' \in AC_{\mathrm{loc}}[a,b],\ \frac{1}{|w|}\tau y \in L^2_{|w|},$$
$$y(a)\cos\theta_1 - py'(a)\sin\theta_1 = 0, y(b)\cos\theta_2 - py'(b)\sin\theta_2 = 0\Big\},$$

where $L^2_{|w|} = L^2_{|w|}[a,b]$.

The operator $T = \frac{1}{w}\tau$ associated to the indefinite problem (1.1), (1.2) is self-adjoint in the Krein space $K := (L^2_{|w|}, [\cdot,\cdot]_w)$ with domain $\mathcal{D}(T) = \mathcal{D}(S)$. Let $J = \operatorname{sgn} w$ be the fundamental symmetry of K. Then $S = JT$ and

$$[Ty_1, y_2]_w = (Sy_1, y_2)_{|w|},\ \ y_1,\ y_2 \in \mathcal{D}(T).$$

With these symbols, we prove Theorem 2.4.

Proof of Theorem 2.4. As before, we only prove the theorem in the case $p \equiv 1$ on $[0, B]$.

Let λ_k be the k-th eigenvalue of S and let ϕ_k be the corresponding eigenfunction, such that $\|\phi_k\|_\infty = 1$, $k \geq 1$. Lemma 3.1 gives $\int_0^B |\phi'|^2 \leq N$. We can also show, analogously to the proof of Lemma 3.1, using the notation N_k given in (2.10), that

$$\int_0^B |\phi_k'|^2 \leq N + |\lambda_k|\|w\|_1 = N_k,\ \ k \geq 1. \tag{3.20}$$

Then, as in the proof of Lemma 3.2, we can find intervals I_ϕ of length δ given in (2.1), I_{ϕ_k} of length δ_k given in (2.10), all contained in $[0, B]$, such that $|\phi(x)| > \frac{1}{2}$ on I_ϕ, $|\phi_k(x)| > \frac{1}{2}$ on I_{ϕ_k}, $k \geq 1$. Furthermore, recalling the definition of w_t in (2.5), we obtain

$$\begin{aligned} \|\phi\|^2_{|w|} &= \int_0^B |w||\phi|^2 \geq \int_{I_\phi} |w||\phi|^2 \geq \int_{I_\phi} \frac{|w|}{4} \geq \frac{w_\delta}{4}, \\ \|\phi_k\|_{|w|} &= \int_0^B |w||\phi_k|^2 \geq \int_{I_{\phi_k}} |w||\phi_k|^2 \geq \int_{I_{\phi_k}} \frac{|w|}{4} \geq \frac{w_{\delta_k}}{4},\ k \geq 1. \end{aligned} \tag{3.21}$$

Let $W(x) = \int_0^x w$. Clearly, for any $x \in [0, B]$, $|W(x)| \le \omega \|w\|_1$ by the definition of ω in (2.9). Then,

$$[\phi, \phi_k]_w = W(x)\phi(x)\phi_k(x)\Big|_0^B - \int_0^B W(\phi'\phi_k + \phi\phi_k'), \ k \ge 1.$$

Therefore, from $\|\phi\|_\infty = 1 = \|\phi_k\|_\infty$ and (3.20), by the Cauchy-Schwarz inequality, we have for $k \ge 1$,

$$\begin{aligned} \Big|[\phi, \phi_k]_w\Big| &\le \omega\|w\|_1 \left(1 + \int_0^B |\phi\phi_k'| + \int_0^B |\phi'\phi_k|\right) \\ &\le \omega\|w\|_1 \left(1 + \sqrt{BN} + \sqrt{BN_k}\right). \end{aligned} \tag{3.22}$$

Furthermore, for $k \ge 1$ we have

$$\lambda[\phi, \phi_k]_w = [T\phi, \phi_k]_w = (S\phi, \phi_k)_{|w|} = (\phi, S\phi_k)_{|w|} = \lambda_k(\phi, \phi_k)_{|w|}. \tag{3.23}$$

If we define $d_k = (\phi, \phi_k)_{|w|}/(\|\phi\|_{|w|}\|\phi_k\|_{|w|})$ as before, then, from (3.21), (3.22) and (3.23), for $k \ge 1$,

$$|d_k| \le \frac{4|\lambda|}{\sqrt{w_\delta w_{\delta_k}}|\lambda_k|}\big|[\phi, \phi_k]_w\big| \le \frac{4|\lambda|\,\omega\,\|w\|_1}{\sqrt{w_\delta w_{\delta_k}}|\lambda_k|}\left(1 + \sqrt{BN} + \sqrt{BN_k}\right). \tag{3.24}$$

By (3.16), $-\sum_{k=1}^{h-1} |d_k|^2\lambda_k \ge \sum_{k=h}^{\infty} |d_k|^2\lambda_k$. Hence

$$\sum_{k=1}^{h-1} |d_k|^2(\lambda_h - \lambda_k) \ge \sum_{k=1}^{h-1} |d_k|^2\lambda_h + \sum_{k=h}^{\infty} |d_k|^2\lambda_k \ge \sum_{k=1}^{\infty} |d_k|^2\lambda_h = \lambda_h. \tag{3.25}$$

Finally, from

$$\frac{\lambda_h - \lambda_k}{\lambda_k^2} = \frac{\lambda_h}{\lambda_k^2} - \frac{1}{\lambda_k} \le \frac{\lambda_h - \lambda_{h-1}}{\lambda_{h-1}^2}, \ 1 \le k \le h-1,$$

(3.24) and (3.25) we arrive at (2.11). □

4. Two examples

The positive lower bounds on the non-real eigenvalues obtained in Theorems 2.2 and 2.4 quantitatively show how far away these eigenvalues are from the real axis. From (2.8) of Theorem 2.2, for instance, we see that even when the negative part of $\sigma(S)$ is invariant or, more exactly, when the largest negative eigenvalue λ_{h-1} is invariant, the non-real eigenvalues of T will tend to infinity as the least positive eigenvalue $\lambda_h \to \infty$.

Certainly, the structure of $\sigma(S)$ mainly depends on the potential q. On the other hand, in the case where p, q and $|w|$ do not change, and hence the spectrum of S does not change, but ω, the quantity describing oscillation of w (see (2.9)), tends to zero, the non-real eigenvalues of T still tend to infinity by Theorem 2.4; see Example 4.1.

Example 4.1. Consider the problem

$$-y''(x) - \mu y(x) = \lambda w_n(x)y(x), \;\; x \in (0,1), \;\; y(0) = y(1) = 0, \tag{4.1}$$

where $\pi^2 < \mu < 4\pi^2$ and

$$w_n(x) = \begin{cases} -1, \; x \in \Big(\dfrac{k-1}{n}, \dfrac{2k-1}{2n}\Big), \;\; i = 1, \dots, n, \;\; n > 1, \\ \;\;1, \; \text{otherwise}. \end{cases}$$

For every $n > 1$, the corresponding right-definite problem is

$$-y''(x) - \mu y(x) = \lambda y(x), \quad y(0) = y(1) = 0$$

and has the first two eigenvalues $\lambda_1 = \pi^2 - \mu < 0$ and $\lambda_2 = 4\pi^2 - \mu > 0$ by $\pi^2 < \mu < 4\pi^2$. In this case, $\|q\|_1 = \|q_-\|_1 = \mu$, $N = \mu$, $\|w_n\|_1 = 1$, $(w_n)_\delta = \delta = \frac{1}{4\mu}$, and $(w_n)_{\delta_1} = \delta_1 = 1/(8\mu - 4\pi^2)$. Particularly, $\omega_n = \frac{1}{2n} \to 0$, $n \to \infty$; see (2.9). By Theorem 2.4, the lower bound on non-real eigenvalues λ of (4.1) is given by

$$|\lambda| \geq \frac{n(\mu - \pi^2)}{8\sqrt{3}\pi(1 + \sqrt{\mu} + \sqrt{2\mu - \pi^2})} \sqrt{\frac{4\pi^2 - \mu}{\mu(2\mu - \pi^2)}} = O(n), \tag{4.2}$$

as $n \to \infty$. Besides, $m = 1 + \lfloor 8\mu \rfloor$ and we can select $\eta \geq \min\{\frac{1}{4m}, \frac{1}{4n}\}$; then $w_\eta = \eta$. Obviously, $\min\{\frac{1}{4m}, \frac{1}{4n}\} = \frac{1}{4n}$ for sufficiently large n. Then upper bounds of non-real eigenvalues are obtained by Theorem 2.1,

$$|\operatorname{Re}\lambda| \leq 48n^2 + \frac{64}{3}\mu n = O(n^2) \;\text{ and }\; |\operatorname{Im}\lambda| \leq \frac{128}{3}n^2 = O(n^2), \; n \to \infty. \tag{4.3}$$

We would like to mention here that if we apply Theorem 3.2 of [2] to Example 4.1 we may get better upper bounds which are of order $O(n)$. This is mainly because the conditions of our results are much weaker and we are not able to get better upper bounds for Example 4.1 at present. However, we give another example to which our results are applicable but Theorem 3.2 of [2] is not.

Example 4.2. Consider the problem

$$-y''(x) - \mu y(x) = \lambda w(x)y(x), \;\; x \in (0,1), \;\; y(0) = y(1) = 0, \tag{4.4}$$

where $\pi^2 < \mu < 4\pi^2$ and

$$w(x) = \begin{cases} -1, \; x \in K_\alpha, \\ \;\;1, \; x \in [0,1] \setminus K_\alpha, \end{cases}$$

where K_α is the α-Cantor set, $0 < \alpha < 1$, constructed as follows. Let $X_0 = [0,1]$. Remove the open middle $\frac{\alpha}{3}$-interval of X_0 and denote the remaining by X_1, namely, $X_1 = [0, \frac{1}{2} - \frac{\alpha}{6}] \cup [\frac{1}{2} + \frac{\alpha}{6}, 1]$. Remove the open middle $\frac{\alpha}{3^2}$-intervals of each of the two pieces of X_1 and denote the remaining by X_2. Continue this procedure ad infinitum and let $K_\alpha = \bigcap_{k=1}^{\infty} X_k$.

It is easy to show that K_α is nowhere dense. Hence, Theorem 3.2 of [2] does not apply to this example since it requires the existence of an absolutely continuous function g satisfying $g(x)w(x) > 0$.

In this example, $N = \|q\|_1 = \mu$, $\delta = \frac{1}{4\mu}$, $m = 1 + \lfloor 8\mu \rfloor$, $|w| = 1$, $\|w\|_1 = 1$ and $w_t = t$ for $t > 0$. We also see $\operatorname{mes} K_\alpha = 1 - \alpha$ and for the number ω as in (2.9), we estimate

$$\omega < \max\left\{ \left| \int_{K_\alpha^c} 1 \right|, \left| \int_{K_\alpha} -1 \right| \right\} = \max\{\alpha, 1 - \alpha\}.$$

For every α, the corresponding right-definite problem is

$$-y''(x) - \mu y(x) = \lambda y(x), \quad y(0) = y(1) = 0$$

and, by $\pi^2 < \mu < 4\pi^2$, has the first two eigenvalues $\lambda_1 = \pi^2 - \mu < 0$ and $\lambda_2 = 4\pi^2 - \mu > 0$. Thus, $N_1 = N + |\lambda_1| = 2\mu - \pi^2$ and $\delta_1 = \frac{1}{4N_1} = \frac{1}{8\mu - 4\pi^2}$. The lower bound on non-real eigenvalues λ is given by Theorem 2.4 as

$$|\lambda| \geq \frac{(\mu - \pi^2)}{16\sqrt{3}\pi \max\{\alpha, 1-\alpha\}(1 + \sqrt{\mu} + \sqrt{2\mu - \pi^2})} \sqrt{\frac{4\pi^2 - \mu}{\mu(2\mu - \pi^2)}}. \tag{4.5}$$

For the upper bound on non-real eigenvalues, we divide $[0, 1]$ into m equal subintervals $I_k = [x_{k-1}, x_k]$, $k = 1, \ldots, m$, and try to find a positive number η such that in each of the subintervals there exists an interval of length 2η on which (2.3) holds.

If $\alpha < \frac{1}{40m}$, we may choose $\eta = \frac{1}{2m}$. In fact, since $w = 1 - 2\mathcal{X}(K_\alpha) = 2\mathcal{X}(K_\alpha^c) - 1$, where $K_\alpha^c = [0, 1] \setminus K_\alpha$ and $\mathcal{X}$ is the characteristic function, for any interval I with length η, we have

$$\left| \int_I w \right| = \left| 2 \int_I \mathcal{X}(K_\alpha^c) - \eta \right| > \eta - 2\alpha > \frac{9}{10}\eta = \frac{9}{10} \int_I |w|.$$

Now we assume $\frac{1}{40m} \leq \alpha < 1$. There are 2^k closed intervals in X_k. Each of them has a length $\ell_k = 2^{-k}(1 - \alpha + \alpha(\frac{2}{3})^k)$ and contains an open interval of length $\frac{\alpha}{3^{k+1}}$ removed in the $(k+1)$-th step of the procedure constructing K_α. There is a collection, denoted by Y_k, of $2^k - 1$ open intervals in $[0, 1] \setminus X_k$ and the lengths of these open intervals are larger than $\frac{\alpha}{3^{k+1}}$. If we choose $\eta = \frac{\alpha}{2 \cdot 3^{k+1}}$ and $k = \lfloor \log_2 m \rfloor + 1$, then

$$\ell_k < \frac{1 - \alpha/3}{2^k} < \frac{1 - \alpha/3}{2^{\log_2 m}} = \frac{1}{m} - \frac{\alpha}{3m}.$$

Since $\mu > \pi^2$, we see $m = 1 + \lfloor 8\mu \rfloor \geq 79$ and $k \geq 7$. Therefore,

$$\frac{\alpha}{3m} > \frac{\alpha}{3 \cdot 2^k} > \frac{\alpha}{3^{k+1}} = 2\eta. \tag{4.6}$$

Then, each interval I_k either contains a whole interval in X_k and hence contains an open interval of length $\frac{\alpha}{3^{k+1}} = 2\eta$ belonging to Y_k, or contains an open interval that is an element or at least a part of an element in Y_{k-1} and, by (4.6), has a length larger than 2η. Thus, we have found an interval I of length 2η in each I_k, $k = 1, \ldots, m$, such that $w(x) = 1$ on I and, apparently, (2.3) holds on I.

Since

$$\eta = \frac{\alpha}{2 \cdot 3^{k+1}} = \frac{\alpha}{6 \cdot 3^{\lfloor \log_2 m \rfloor + 1}}$$

and $3^{\log_2 m} = m^{\log_2 3} = m^{1.585}$ with $m = \lfloor 8\mu \rfloor + 1$, we get

$$\frac{\alpha}{18(\lfloor 8\mu \rfloor + 1)^{1.585}} \leq \eta < \frac{\alpha}{6(\lfloor 8\mu \rfloor + 1)^{1.585}}.$$

Finally, by Theorem 2.1, for any non-real eigenvalue λ we have

$$|\operatorname{Re} \lambda| \leq 972(\lfloor 8\mu \rfloor + 1)^{3.17}\alpha^{-2} + 1728\mu(\lfloor 8\mu \rfloor + 1)^{1.585}\alpha^{-1},$$
$$|\operatorname{Im} \lambda| \leq 864(\lfloor 8\mu \rfloor + 1)^{3.17}\alpha^{-2}, \quad \text{if } 0.025(\lfloor 8\mu \rfloor + 1)^{-1} \leq \alpha < 1,$$

and

$$
\begin{aligned}
|\operatorname{Re}\lambda| &\le 12(\lfloor 8\mu\rfloor+1)^2+\frac{32}{3}\mu(\lfloor 8\mu\rfloor+1),\\
|\operatorname{Im}\lambda| &\le \frac{32}{3}(\lfloor 8\mu\rfloor+1)^2, \quad \text{if } 0<\alpha<0.025(\lfloor 8\mu\rfloor+1)^{-1}.
\end{aligned}
$$

References

[1] F.V. Atkinson, D. Jabon, *Indefinite Sturm-Liouville problems*, in "Proceedings of the Focused Research Program on Spectral Theory and Boundary Value Problems", Vol.I, H. Kaper, M.-K. Kwong, and A. Zettl (Eds.), Argonne National Laboratory, Lemont, IL, USA, 1988, pp. 31–45.

[2] Jussi Behrndt, Shaozhu Chen, Friedrich Philipp, and Jiangang Qi, *Estimates on the non-real eigenvalues of regular indefinite Sturm-Liouville problems*, Proc. Roy. Soc. Edinburgh Sect. A **144** (2014), no. 6, 1113–1126, DOI 10.1017/S0308210513001212. MR3283062

[3] Jussi Behrndt, Friedrich Philipp, and Carsten Trunk, *Bounds on the non-real spectrum of differential operators with indefinite weights*, Math. Ann. **357** (2013), no. 1, 185–213, DOI 10.1007/s00208-013-0904-7. MR3084346

[4] Jussi Behrndt, Qutaibeh Katatbeh, and Carsten Trunk, *Non-real eigenvalues of singular indefinite Sturm-Liouville operators*, Proc. Amer. Math. Soc. **137** (2009), no. 11, 3797–3806, DOI 10.1090/S0002-9939-09-09964-X. MR2529889 (2010h:47008)

[5] Paul Binding and Hans Volkmer, *Eigencurves for two-parameter Sturm-Liouville equations*, SIAM Rev. **38** (1996), no. 1, 27–48, DOI 10.1137/1038002. MR1379040 (97f:34015)

[6] R. C. Brown and D. B. Hinton, *Sufficient conditions for weighted inequalities of sum form*, J. Math. Anal. Appl. **112** (1985), no. 2, 563–578, DOI 10.1016/0022-247X(85)90263-X. MR813620 (87d:26017)

[7] Branko Ćurgus and Heinz Langer, *A Kreĭn space approach to symmetric ordinary differential operators with an indefinite weight function*, J. Differential Equations **79** (1989), no. 1, 31–61, DOI 10.1016/0022-0396(89)90112-5. MR997608 (90j:47060)

[8] J. Fleckinger and A. B. Mingarelli, *On the eigenvalues of nondefinite elliptic operators*, Differential equations (Birmingham, Ala., 1983), North-Holland Math. Stud., vol. 92, North-Holland, Amsterdam, 1984, pp. 219–227, DOI 10.1016/S0304-0208(08)73696-X. MR799351 (86j:35072)

[9] Otto Haupt, *Über eine Methode zum Beweise von Oszillationstheoremen* (German), Math. Ann. **76** (1914), no. 1, 67–104, DOI 10.1007/BF01458673. MR1511814

[10] Q. Kong, H. Wu, A. Zettl, and M. Möller, *Indefinite Sturm-Liouville problems*, Proc. Roy. Soc. Edinburgh Sect. A **133** (2003), no. 3, 639–652, DOI 10.1017/S0308210500002584. MR1983691 (2004f:34039)

[11] Angelo B. Mingarelli, *Indefinite Sturm-Liouville problems*, Ordinary and partial differential equations (Dundee, 1982), Lecture Notes in Math., vol. 964, Springer, Berlin-New York, 1982, pp. 519–528. MR693136 (84f:34037)

[12] Angelo B. Mingarelli, *A survey of the regular weighted Sturm-Liouville problem—the non-definite case*, International workshop on applied differential equations (Beijing, 1985), World Sci. Publishing, Singapore, 1986, pp. 109–137. MR901329

[13] Jiangang Qi and Shaozhu Chen, *A priori bounds and existence of non-real eigenvalues of indefinite Sturm-Liouville problems*, J. Spectr. Theory **4** (2014), no. 1, 53–63, DOI 10.4171/JST/61. MR3181385

[14] R. G. D. Richardson, *Theorems of oscillation for two linear differential equations of the second order with two parameters*, Trans. Amer. Math. Soc. **13** (1912), no. 1, 22–34, DOI 10.2307/1988611. MR1500902

[15] R. G. D. Richardson, *Contributions to the study of oscillation properties of the solutions of linear differential equations of the second order*, Amer. J. Math. **40** (1918), no. 3, 283–316, DOI 10.2307/2370485. MR1506360

[16] Lawrence Turyn, *Sturm-Liouville problems with several parameters*, J. Differential Equations **38** (1980), no. 2, 239–259, DOI 10.1016/0022-0396(80)90007-8. MR597803 (82d:34036)

[17] Bing Xie and Jiangang Qi, *Non-real eigenvalues of indefinite Sturm-Liouville problems*, J. Differential Equations **255** (2013), no. 8, 2291–2301, DOI 10.1016/j.jde.2013.06.013. MR3082462

[18] Bing Xie, Jiangang Qi, and Shaozhu Chen, *Non-real eigenvalues of one dimensional p-Laplacian with indefinite weight*, Appl. Math. Lett. **48** (2015), 143–149, DOI 10.1016/j.aml.2015.04.005. MR3348958

[19] Anton Zettl, *Sturm-Liouville theory*, Mathematical Surveys and Monographs, vol. 121, American Mathematical Society, Providence, RI, 2005. MR2170950 (2007a:34005)

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