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DEGREES OF IRREDUCIBLE CHARACTERS OF THE SYMMETRIC GROUP AND EXPONENTIAL GROWTH

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Abstract. We consider sequences of degrees of ordinary irreducible S_n-characters. We assume that the corresponding Young diagrams have rows and columns bounded by some linear function of n with leading coefficient less than one. We show that any such sequence has at least exponential growth and we compute an explicit bound.

1. Introduction

Let $\lambda \vdash n$ be a partition of an integer n and f^λ the degree of the corresponding irreducible character of the symmetric group S_n in characteristic zero.

Given a sequence of partitions $\{\lambda^{(n)}\}_{n\geq 1}$, where $\lambda^{(n)} \vdash n$, an interesting problem is that of studying the asymptotic behavior of the corresponding numerical sequence $\{f^{\lambda^{(n)}}\}_{n\geq 1}$ provided the partitions $\lambda^{(n)}$ (or the corresponding Young diagrams) are subject to some constraints such as belonging to a specific region of the plane ([1], [8], [10], [12], [13]).

For instance, if each $\lambda^{(n)} = (\lambda_1^{(n)}, \lambda_2^{(n)}, \ldots)$ lies in a strip of fixed height $k \geq 1$ (i.e., $\lambda_{k+1}^{(n)} = 0$), then $f^{\lambda^{(n)}} \leq k^n$, for all $n \geq 1$ ([12]). In case the sequence has the further property that $\lambda_1^{(n)} \leq \frac{n}{\alpha}$, for some fixed $\alpha > 1$, then an exponential lower bound can also be found.

More generally, given $k, l \geq 0$, if we consider a sequence of partitions lying inside the $k \times l$ hook (i.e., $\lambda_{k+1}^{(n)} \leq l$), then $f^{\lambda^{(n)}} \leq (k+l)^n$ ([13]). A lower bound in this case is found (see Proposition 1 below) when the sequence has the further property that $\lambda_1^{(n)}, \lambda_1^{(n)'} \leq \frac{n}{\alpha}$, for some $\alpha > 1$, where $\lambda^{(n)'}$ is the conjugate partition of $\lambda^{(n)}$.

This kind of result, though of interest on its own, can be used as an effective tool in the theory of polynomial identities. In fact, the multilinear polynomials of the free algebra in n fixed variables have a natural structure of module for the symmetric group S_n. In this setting one attaches to every T-ideal of the free algebra (or to a PI-algebra) a sequence of degrees of S_n-characters and the growth of the T-ideal is the growth of such a sequence ([5]).

The above-mentioned results are one of the key ingredients in the proof of the existence of the PI-exponent of an associative PI-algebra ([4]). Nevertheless, in the study of the identities of Lie algebras or more generally of non-associative algebras one needs more general results in the following sense.

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A basic remark is that in the above examples all partitions $\lambda^{(n)}$ have bounded diagonal $\delta(\lambda^{(n)})$, but this is not the case in general. But what can we say if $\delta(\lambda^{(n)})$ goes to infinity with n? Here we shall consider such a question.

In case $\lim_{n\to\infty} \frac{\delta(\lambda^{(n)})^2}{n} = \varepsilon > 0$, then it is not hard to prove that the sequence $\{f^{\lambda^{(n)}}\}_{n\geq 1}$ has overexponential growth (see Proposition 2 below).

More generally, here we shall prove the following result: given any $\alpha \in \mathbb{R}$, $\alpha > 1$, and $\{\lambda^{(n)}\}_{n\geq 1}$ a sequence of partitions such that $\lambda_1^{(n)}, {\lambda_1^{(n)}}' \leq \frac{n}{\alpha}$, then $f^{\lambda^{(n)}} \geq \beta^n$, for any $1 < \beta < \alpha$, for n large enough.

This result was proved by the second author in [9], in case α is an integer and as an application it was proved that no variety of Lie algebras can have exponential growth between one and two. Also in [6], in order to compute the exponential growth of a variety, the authors introduced a real valued function $\Phi(\alpha_1, \dots, \alpha_k)$ with the property that $\Phi(\alpha_1, \dots, \alpha_k)^n$ asymptotically equals f^λ, up to a polynomial factor, where $\lambda = ([\alpha_1 n], \dots, [\alpha_k n])$ (see also [2], [3]). This function is defined only for partitions lying in a strip of height k, while in this paper no restriction on the height is considered.

We would like to point out that in [11] it was proved with different methods that if $\{\lambda^{(n)}\}_{n\geq 1}$ is a sequence of "primary" partitions such that $\lambda_1^{(n)} \leq \frac{n}{2}$, then $f^{\lambda^{(n)}} \geq 2^n$.

2. Preliminaries

Throughout the paper we shall consider the ordinary representation theory of the symmetric group. We refer to [7] for the notation and the basic notions.

If $\lambda = (\lambda_1, \dots, \lambda_k) \vdash n$, we tacitly identify λ with its Young diagram $D_\lambda = D_\lambda\{(i,j)\}$, where (i,j) is the cell of D_λ of coordinates (i,j). We also write $H_{ij} = \{(k,l) \in \lambda \mid k \geq i, l = j \text{ or } k = i, l \geq j\}$ for the hook of the cell (i,j). Recall that the hook number of (i,j) is $h_{ij} = (\lambda_i - j) + (\lambda'_j - i) + 1$ where $\lambda_i - j$ is the length of the arm of the hook and $\lambda'_j - i$ is the length of the leg.

We denote by $\lambda' = (\lambda'_1, \dots, \lambda'_{\lambda_1})$ the conjugate partition of λ, where λ'_i is the length of the i-th column of λ.

In what follows we shall use the hook formula (see [7]) and Stirling formula (see [14]).

Remark 1. Let $\lambda \vdash n$ and let $T_\lambda = (t_{ij})$ be a standard tableau of shape λ. If $N_{ij} = n + 1 - t_{ij}$, then $N_{ij} \geq h_{ij}$, for all i, j.

Proof. Let $B_{ij} = \{t_{kl} | t_{kl} \geq t_{ij}\}$. Clearly $N_{ij} = |B_{ij}|$. Since for each cell (k,l) of the hook H_{ij} of (i,j) we have that $t_{kl} \geq t_{ij}$, it follows that $\{t_{kl} | (k,l) \in H_{ij}\} \subseteq B_{ij}$. Thus $N_{ij} \geq h_{ij}$. □

Recall that a corner cell of a diagram is a cell whose hook number is 1.

Definition 1. If $\lambda \vdash n$, we define the diagonal of λ as

$$\delta(\lambda) = \max_i\{\lambda_i \geq i \text{ and } \lambda'_i \geq i\}.$$

We remark that $\delta = \delta(\lambda)$ is the size of the largest square diagram inside λ.

Definition 2. If $k, l \geq 0$, we define the $k \times l$ hook as the set

$$H(k,l) = \{\lambda \vdash n | \lambda_{k+1} \leq l\}.$$

3. Two special cases

The following result is probably well known.

Proposition 1. *Let $\alpha \in \mathbb{R}$, $\alpha > 1$, and $k \geq l \geq 0$. If $\lambda \vdash n$ is such that $\lambda \in H(k,l)$ and $\lambda_1, \lambda_1' \leq \frac{n}{\alpha}$, then*

$$f^{\lambda^{(n)}} \geq \frac{\alpha^n}{n^m},$$

where $m = \frac{(2l+k-1)k}{2}$.

Proof. Let $t_1, \ldots, t_{k+l}$ be defined as follows:

$$1) \text{ if } 1 \leq s \leq l, \quad \text{set} \quad t_s = \begin{cases} \lambda_s' & \text{if } \lambda_s' > 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$2) \quad \text{if } 1 \leq s \leq k, \quad \text{set} \quad t_{l+s} = \begin{cases} \lambda_s - l & \text{if } \lambda_s - l \geq 0, \\ 0 & \text{if } \lambda_s - l < 0. \end{cases}$$

We remark that $\sum_{i=1}^{k+l} t_i = n$.

Let $\mu = (l+k-1, \ldots, l+1, l) \vdash m = \frac{(2l+k-1)k}{2}$ and define

$$A = \{(i,j) \mid (i,j) \in \lambda \cap \mu\},$$
$$B = \{(i,j) \in \lambda \setminus \mu \mid i \geq k+1\},$$
$$C = \{(i,j) \in \lambda \setminus \mu \mid 1 \leq i \leq k\}.$$

Clearly since $|A| \leq m$,

$$\prod_{(i,j)\in A} h_{ij} \leq n^m.$$

The hook number of a cell in the j-th column of λ, $j \leq l$, is $h_{k+i,j} < t_j - i + 1$, $1 \leq i \leq \lambda_j' - k$. Moreover, consider the hook H_{i,μ_i+j} of a cell of C. Its arm is $\lambda_i - \mu_i - j$ and its leg is $\leq k - i$. Hence

$$h_{i,\mu_i+j} \leq (\lambda_i - \mu_i - j) + (k-i) + 1 = t_{l+i} + l + k - \mu_i - j - i + 1 = t_{l+i} - j + 1,$$

since $\mu_i = l + k - i$.

Hence

$$\prod_{(i,j)\in B\cup C} h_{ij} < \prod_{i=1}^{k+l} t_i!.$$

It follows that

$$f^\lambda \geq \frac{1}{n^m} \frac{n!}{\prod_{i=1}^{k+l} t_i!}.$$

By Stirling's formula, recalling that $t_i \leq \frac{n}{\alpha}$, we get

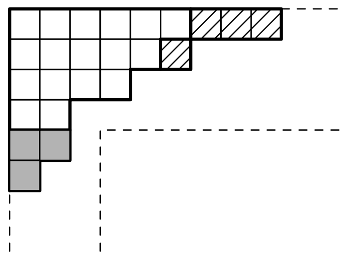
$$\frac{n!}{\prod_{i=1}^{k+l} t_i!} \geq \frac{n^n \prod_{i=1}^{k+l} e^{t_i}}{e^n \prod_{i=1}^{k+l} t_i^{t_i}} \geq \frac{n^n}{\prod_{i=1}^{k+l} \frac{n^{t_i}}{\alpha^{t_i}}} = \prod_{i=1}^{k+l} \alpha^{t_i} = \alpha^n.$$

Hence we get that $f^\lambda \geq \frac{\alpha^n}{n^m}$ and we are done. □

Remark 2. If $k < l$ and $\lambda \in H(k,l)$, then $\lambda' \in H(l,k)$ and $f^\lambda = f^{\lambda'}$. Hence the conclusion of the previous proposition still holds when $k < l$ and $f^\lambda \geq \frac{\alpha^n}{n^m}$ where $m = \frac{(2k+l-1)l}{2}$ in this case.

The following example illustrates the sets A, B, C of the previous proposition.

Example 1. We consider the hook $H(4,3)$ and a partition $\lambda \in H(4,3)$. For $\lambda = (9,6,4,2,2,1) \vdash 24$, we have



where A consists of the white cells, B of the grey cells, and C of the marked cells. In this case we also have that $t_1 = 6, t_2 = 5, t_3 = 3, t_4 = 6, t_5 = 3, t_6 = 1, t_7 = 0$.

Lemma 1. *Let $\{a_s\}_{s\geq 1}$, $\{b_s\}_{s\geq 1}$ be two sequences of natural numbers such that*

$$\lim_{s\to\infty} a_s = \lim_{s\to\infty} b_s = \infty,$$

and define the partition $\mu(s) = (b_s^{a_s})$, $s \geq 1$.

Then for every $\beta > 0$, there exists s_0 such that for every $s \geq s_0$, $f^{\mu(s)} > \beta^{a_s b_s}$.

Proof. Let $n = ab$ and $\mu = \mu(s) = (b^a) \vdash n$. Since $f^{\mu} = f^{\mu'}$, where μ' is the conjugate partition of μ, we assume, as we may, that $b \geq a$.

By the hook formula, we get

$$f^{\mu} = \frac{n!}{(b!)^a} \cdot \frac{1!}{(b+1)} \cdot \frac{2!}{(b+1)(b+2)} \cdot \dots \cdot \frac{(a-1)!}{(b+1)\dots(b+a-1)}$$

$$= \frac{n!}{(b!)^a} \cdot \prod_{t=1}^{a-1} \binom{b+t}{t}^{-1} > \frac{n!}{(b!)^a} \cdot \prod_{t=1}^{a-1} \left(2^{b+t}\right)^{-1} = \frac{n!}{(b!)^a} \cdot 2^{-n-\frac{a^2}{2}+\frac{2b+a}{2}} \geq \frac{n!}{(b!)^a} 2^{-2n},$$

since $a^2 \leq n$. By Stirling's formula, since $n = ab$, we get that $f^{\mu} > \left(\frac{a}{4}\right)^n$.

Now, since $\lim_{s\to\infty} a_s = \infty$, for every β there exists s_0 such that $\frac{a_s}{4} \geq \beta$, for any $s \geq s_0$, and the result follows. □

The bound obtained in the following result, though not very sharp, is all we need for our purpose.

Proposition 2. *Let $\alpha > 1$, $\alpha \in \mathbb{R}$, and let $\{\lambda^{(n)}\}_{n\geq 1}$ be a sequence of partitions, $\lambda^{(n)} \vdash n$. Suppose that for any $\varepsilon > 0$, $\frac{\delta(\lambda^{(n)})^2}{n} \geq \varepsilon > 0$ holds for n large enough. Then for any γ there exists n_0 such that $f^{\lambda^{(n)}} \geq \gamma^n$ for every $n \geq n_0$.*

Proof. For every $n \geq 1$, set $\mu(n) = \left(\delta(\lambda^{(n)}\right)^{\delta(\lambda^{(n)})} \vdash k_n$ where $k_n = \delta(\lambda^{(n)})^2$. Take $\beta = \gamma^{1/\varepsilon}$. Then since $\frac{k_n}{n} \geq \varepsilon$, by Lemma 1 there exists n_0 such that

$$f^{\mu(n)} \geq \beta^{k_n} > \beta^{\varepsilon n} = \gamma^n,$$

for any $n \geq n_0$. Since $f^{\lambda^{(n)}} \geq f^{\mu(n)}$, the proof is complete. □

4. The main results

The following lemma, which is of interest by itself, is the main tool for proving our main result. If $a \in \mathbb{R}$, we write $[a]$ for the integer part of a.

In this lemma we consider the special case when the lengths of the rows and columns outside the $\delta \times \delta$ square are different.

Lemma 2. *Let $\alpha > 1$, $\alpha \in \mathbb{R}$. Let $\lambda \vdash n$ be such that $\delta = \delta(\lambda) \geq 9\alpha$ and $\lambda_1, \lambda_1' \leq \frac{n}{\alpha}$. If $\lambda_1 > \lambda_2 > \cdots > \lambda_\delta > \delta$ and $\lambda_1' > \lambda_2' > \cdots > \lambda_{\tau+1}' \geq \delta$, for some $0 \leq \tau \leq \delta$, then there exists n_0 such that for all $n \geq n_0$*

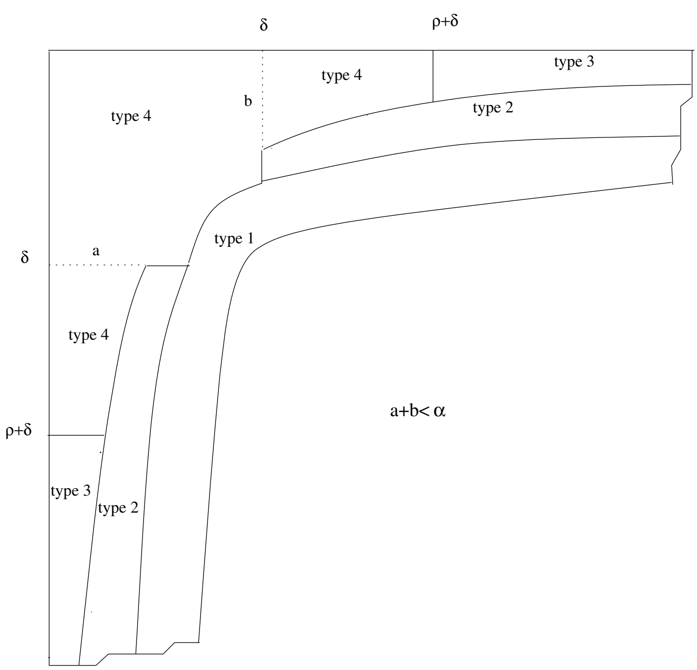
$$f^\lambda \geq \alpha^{n-(\delta^2+\alpha\rho)},$$

where $\rho = \delta^2$ if $\alpha \in \mathbb{N}$ and $\rho = \left[\frac{\delta^2}{\alpha-[\alpha]}\right] + 1$ if $\alpha \notin \mathbb{N}$.

Proof. We shall assign a number N to each cell of λ, $1 \leq N \leq n$, and we shall denote by h_N the corresponding hook number.

We shall split the cells of λ into four eventually empty disjoint sets $T_1, \ldots, T_4$ and we shall call them cells of type $1, \ldots,$ type 4, respectively.

The reader should refer to the following figure for a better understanding of the distribution of the cells in the diagram λ.



We start by defining cells of type 1.

Let s_1 be the number of corner cells of $\lambda = \lambda^{\{1\}}$. By assumption, $s_1 \geq \delta \geq 9\alpha > 2\alpha$. We enumerate the corner cells from top to bottom with the numbers $1, 2, \ldots, s_1$ and we assign to each cell color 1.

Next we consider the new diagram $\lambda^{\{2\}}$, obtained by deleting the s_1 corner cells from λ. Let s_2 be the number of corner cells of $\lambda^{\{2\}}$. Since $s_2 \geq \delta - 1 > 2\alpha$, we repeat the above procedure by enumerating the corner cells of $\lambda^{\{2\}}$ from top to bottom with the numbers $s_1 + 1, \ldots, s_1 + s_2$ and we assign color 2 to each cell.

We repeat this procedure until the obtained uncolored subdiagram of λ has $\geq 2\alpha$ corner cells. Let $1, 2, \ldots, r$ be the assigned colors. The cells obtained in this procedure will be called of type 1. Hence $T_1 = \lambda \setminus \lambda^{\{r+1\}}$ and $|T_1| = \sum_{i=1}^{r} s_i$.

We remark that if we consider a cell of type 1, then the cells in the corresponding hook (whose corner is the given cell) are all colored and each color appears at most twice. Hence if we consider a cell in the i-th step of the above procedure, the corresponding hook number is $h_N \leq 2(i-1)+1$ where N is the number assigned to this given cell. Hence, since $s_1 \geq \delta$ and by hypothesis $s_j \geq 2\alpha$, for all j, $2 \leq j \leq r$, by counting the number of cells up to the $(i-1)$-th step, we get $N > 2\alpha(i-2) + \delta$. Since $\delta > \alpha$, it follows that

$$\alpha h_N \leq N. \tag{1}$$

Next we claim that

$$|T_1| \geq 2\alpha r + \alpha\delta. \tag{2}$$

In fact, since the first δ rows of λ have different lengths, we have

$$s_1 \geq \delta,\ s_2 \geq \delta - 1, \ldots, s_i \geq \delta - i + 1, \ldots,$$

$$s_{\delta - 2[\alpha] - 1} \geq 2[\alpha] + 2 > 2\alpha,\ s_{\delta - 2[\alpha]} \geq 2\alpha, \ldots, s_r \geq 2\alpha.$$

Since

$$s_1 \geq (2[\alpha] + 2) + (\delta - 2[\alpha] - 2),\ s_2 \geq (2[\alpha] + 2) + (\delta - 2[\alpha] - 3), \ldots,$$

$$s_{\delta - 2[\alpha] - 2} \geq (2[\alpha] + 2) + 1,\ s_{\delta - 2[\alpha] - 1} \geq 2[\alpha] + 2,$$

we have

$$|T_1| \geq 2\alpha r + \sum_{j=1}^{\delta - 2[\alpha] - 2} j = 2\alpha r + \frac{(\delta - 2[\alpha] - 2)(\delta - 2[\alpha] - 1)}{2}.$$

Recalling that $\delta \geq 9\alpha$ and $\alpha \geq [\alpha] \geq 1$, we get

$$\delta^2 - 4[\alpha]\delta + 4[\alpha]^2 + 6[\alpha] - 3\delta + 2 - 2\alpha\delta > 9\alpha\delta - 4[\alpha]\delta - 3\delta - 2\alpha\delta \geq 0.$$

Hence

$$\frac{(\delta - 2[\alpha] - 2)(\delta - 2[\alpha] - 1)}{2} > \alpha\delta,$$

and the claim follows.

We continue the process of deleting cells from our diagram λ and we define the set T_2 of cells of type 2.

Consider the diagram $\lambda^{\{r+1\}}$ and its corner cells (i, j) where either $i > \delta$ or $j > \delta$, i.e., we consider corner cells outside the square diagram $\delta \times \delta$.

Let t_1 be the number of corner cells of $\lambda^{\{r+1\}}$ outside the partition $\left(\delta^\delta\right)$. If $t_1 \geq \alpha$, we enumerate these cells with the numbers $|T_1| + 1, \ldots, |T_1| + t_1$ and we color them with color $r + 1$.

Next we consider the diagram $\lambda^{\{r+2\}}$. If the number t_2 of corner cells outside $\left(\delta^\delta\right)$ is $\geq \alpha$, we enumerate these cells with the numbers $|T_1|+t_1+1, \ldots, |T_1|+t_1+t_2$ and we color them with color $r+2$.

We repeat this procedure as long as the number of corner cells outside $\left(\delta^\delta\right)$ is $\geq \alpha$. Let $r+1, \ldots, r+q$ be the colors given to the cells of type 2. The number of cells of type 2 is $|T_2| = \sum_{i=1}^{q} t_i$.

Next we claim that the inequality (1) holds for cells of type 2, i.e., for any N with

$$|T_1| + 1 \leq N \leq |T_1| + |T_2|.$$

In fact, let $0 \leq x \leq q-1$ and consider the corner cells of the diagram $\lambda^{\{r+x+1\}}$ outside $\left(\delta^\delta\right)$. For any such cell (i,j) whose number is N, we have that

$$|T_1| < N \leq |T_1| + t_1, \quad \text{if } \ x = 0,$$

and

$$|T_1| + \sum_{k=1}^{x} t_k < N \leq |T_1| + \sum_{k=1}^{x+1} t_k, \quad \text{if } \ x > 0.$$

Hence

$$N \geq |T_1| + \alpha x, \tag{3}$$

since $t_k \geq \alpha$, $1 \leq k \leq q$.

Next we compute h_N. If $i \leq \delta$ and $j > \delta$, each cell of the arm has different colors. Hence its length is $\lambda_i - j \leq r + x$. Since $j > \delta$, the length of the leg is $\lambda'_j - i \leq \delta - 1$. Therefore

$$h_N \leq r + x + \delta. \tag{4}$$

By (2) putting together (3) and (4), we get (1).

In case $i > \delta$ and $j \leq \delta$, the length of the leg is $\leq r + x$ and the length of the arm is $\leq \delta - 1$ and the result still holds.

Next we define the set T_3 of cells of type 3. We consider the partition $\mu = \lambda^{\{r+q+1\}}$.

Notice that $\mu_\delta + \mu'_\delta \leq \alpha - 1$, if $\alpha \in \mathbb{N}$ and $\mu_\delta + \mu'_\delta \leq [\alpha]$ if $\alpha \notin \mathbb{N}$. In fact, otherwise μ will have $\geq \alpha$ corner cells since the original partition has distinct lengths of rows and columns, and, so, we would be in type 2.

Cells of type 3 are defined as cells of $\mu \setminus \left((\delta+\rho)^{\delta+\rho}\right)$. We enumerate cells of type 3 as follows. For $\delta + \rho < m \leq \max\{\mu_1, \mu'_1\} = k$, define

$$Q_m = \{(i,j) \in \mu \mid i = m \text{ or } j = m\}.$$

First we enumerate the cells of Q_k starting with $|T_1| + |T_2| + 1$ in some order. Then we enumerate the cells of Q_{k-1} in some order starting with $|T_1| + |T_2| + |Q_k| + 1$. We continue this process until $Q_{\delta+\rho+1}$.

Consider a cell (i,j) of type 3 whose number is N. Then either $(i,j) = (i, x+\delta)$, $1 \leq i \leq \delta$, or $(i,j) = (x+\delta, j)$, $1 \leq j \leq \delta$, where $x > \rho$. From the enumeration of the cells of type 3, it follows that there are $\leq \delta^2 + cx$ cells whose number is larger than N, where $c = \alpha - 1$ if $\alpha \in \mathbb{N}$ and $c = [\alpha]$ if $\alpha \notin \mathbb{N}$. Hence $N \geq n - (\delta^2 + cx)$. Also

$$h_N \leq \left(\frac{n}{\alpha} - x - \delta\right) + \delta = \frac{n}{\alpha} - x.$$

If $\alpha \in \mathbb{N}$, then $c = \alpha - 1$ and we have

$$N \geq n - \delta^2 - (\alpha - 1)x = n - \alpha x + x - \delta^2 > \alpha h_N,$$

since $x > \rho = \delta^2$.

If $\alpha \notin \mathbb{N}$, then $c = [\alpha]$ and we have

$$N \geq n - \delta^2 - [\alpha]x = n - \alpha x + (\alpha - [\alpha])x - \delta^2 > \alpha h_N,$$

since $x > \rho = \left[\frac{\delta^2}{\alpha - [\alpha]}\right] + 1 > \frac{\delta^2}{\alpha - [\alpha]}$.

We have proved that the inequality (1) holds for cells of type 3.

Finally, we say that a cell of λ is of type 4 if it is not of type 1, 2, or 3.

Let T_4 be the set of cells of type 4. So $|T_4| = n - |T_1| - |T_2| - |T_3|$. Note that

$$|T_4| \leq \delta^2 + \alpha\rho. \tag{5}$$

We consider a standard tableau $T_\lambda = (t_{ij})$ of shape λ such that $n - |T_4| + 1 \leq t_{ij} \leq n$, for any cell (i,j) of type 4. By Remark 1, if $N = N_{ij}$, then

$$h_{ij} = h_N \leq N, \tag{6}$$

for any cell (i,j) of type 4.

We are now ready to compute a lower bound of f^λ.

We have

$$f^\lambda = \frac{n!}{\prod h_{ij}} = \frac{(n - |T_4|)!}{\prod_{(i,j)\in T_1\cup T_2\cup T_3} h_{ij}} \cdot \frac{n(n-1)\dots(n-|T_4|+1)}{\prod_{(i,j)\in T_4} h_{ij}}$$

$$= \prod_{N=1}^{|T_1|+|T_2|+|T_3|} \frac{N}{h_N} \cdot \prod_{N=|T_1|+|T_2|+|T_3|+1}^{n} \frac{N}{h_N} \geq \alpha^{n-|T_4|} \geq \alpha^{n-(\delta^2+\alpha\rho)},$$

where we have applied the inequalities (1) for cells of type 1, 2, and 3 and (6) for cells of type 4. The last inequality follows from (5).

This completes the proof of Lemma 2. □

Proposition 3. *Let $\lambda \vdash n$ and $\alpha \in \mathbb{R}$, $\alpha > 1$. Suppose that $\lambda_1, \lambda'_1 \leq \frac{n}{\alpha}$ and $\delta = \delta(\lambda) \geq 18\alpha$. Then*

$$f^\lambda \geq \alpha^{n-(\frac{5}{2}\delta^2+\alpha\rho)},$$

where $\rho = \delta^2$ if $\alpha \in \mathbb{N}$ and $\rho = \left[\frac{\delta^2}{\alpha-[\alpha]}\right] + 1$ if $\alpha \notin \mathbb{N}$.

Proof. We may clearly assume that $n > \delta^2$. We shall modify λ so that we can apply Lemma 2.

Define $\widetilde{\lambda}$ as follows:

1) for $1 \leq i \leq \delta$ such that $\lambda_i \geq \delta + i - 1$, define $\widetilde{\lambda}_i = \lambda_i - (i-1)$; otherwise set $\widetilde{\lambda}_i = \delta$;
2) for $1 \leq j \leq \delta$, if $\lambda'_j \geq \delta + j - 1$, set $\widetilde{\lambda}'_j = \lambda'_j - (j-1)$; otherwise set $\widetilde{\lambda}'_j = \delta$.

Note that we erase at most $\delta(\delta-1)$ cells. So $\widetilde{\lambda} \vdash n_1 \geq n - \delta^2 + \delta$.

Let s be the largest integer such that $\widetilde{\lambda}_s > \delta$; otherwise set $s = 0$ if $\widetilde{\lambda}_i \leq \delta$, for all i, $1 \leq i \leq \delta$. Also let t be the largest integer such that $\widetilde{\lambda}'_t > \delta$; otherwise set $t = 0$ if $\widetilde{\lambda}'_i \leq \delta$, for all i, $1 \leq i \leq \delta$.

By eventually considering the conjugate partition, we may assume that $s \geq t$. Since $n > \delta^2$, then $s \geq 1$.

If $s = \delta$, then set $\mu = \widetilde{\lambda}$. Otherwise we define a new partition $\mu \vdash n_2 = n_1 - \frac{(\delta-s-1)(\delta-s)}{2}$ as follows:

1) $\mu_i = \widetilde{\lambda}_i$, if $1 \leq i \leq s+1$, or $\delta + 1 \leq i$,
2) $\mu_{s+2} = \delta - 1, \ldots, \mu_\delta = \delta - (\delta - s - 1)$.

Notice that the largest square inside μ is $\left(\delta(\mu)^{\delta(\mu)}\right)$ where $\delta(\mu) \geq \left[\frac{\delta(\lambda)}{2}\right] + 1 \geq 9\alpha$. Next we shall apply Lemma 2 for the partition $\mu \vdash n_2$. Let $\rho(\mu) = \delta(\mu)^2$ if $\alpha \in \mathbb{N}$ and $\rho(\mu) = \left[\frac{\delta(\mu)^2}{\alpha - [\alpha]}\right] + 1$ if $\alpha \notin \mathbb{N}$.

Since $\rho(\mu) \leq \rho = \rho(\lambda)$ and $s \leq \delta$, by Lemma 2, we have

$$\begin{aligned} f^\mu &\geq \alpha^{n_2 - (\delta(\mu)^2 + \alpha\rho(\mu))} \\ &\geq \alpha^{n_2 - (\delta(\lambda)^2 + \alpha\rho(\lambda))} = \alpha^{n - \delta^2 + \delta - \frac{(\delta-s-1)(\delta-s)}{2} - \delta^2 - \alpha\rho} \\ &\geq \alpha^{n - \frac{5\delta^2 - (2\delta s - s^2) - (3\delta - s)}{2} - \alpha\rho} \geq \alpha^{n - (\frac{5}{2}\delta^2 + \alpha\rho)}. \end{aligned}$$

Since $f^\lambda \geq f^\mu$ the proof is complete. □

Theorem 1. *Let $\alpha \in \mathbb{R}$, $\alpha > 1$, and let $\{\lambda^{(n)}\}_{n \geq 1}$ be a sequence of partitions, $\lambda^{(n)} \vdash n$, such that $\lambda_1^{(n)}, \lambda_1^{(n)\prime} \leq \frac{n}{\alpha}$. Then for any $1 < \beta < \alpha$, there exists n_0 such that for all $n \geq n_0$, $f^{\lambda^{(n)}} \geq \beta^n$.*

Proof. In order to simplify the notation we write $\delta(\lambda^{(n)}) = \delta(n)$. Let $\beta \in \mathbb{R}$ be such that $1 < \beta < \alpha$, and let

$$0 < \gamma \leq \frac{\ln\alpha - \ln\beta}{\ln\alpha}. \tag{7}$$

We partition $\mathbb{N}$ into three disjoint sets $\mathbb{N} = M_1 \cup M_2 \cup M_3$, where the M_i's are defined as follows:

$$\begin{aligned} M_1 &= \{n \in \mathbb{N} \mid \delta(n) < 18\alpha\}, \\ M_2 &= \{n \in \mathbb{N} \mid \delta(n) \geq 18\alpha,\ \gamma n \leq \frac{5}{2}\delta(n)^2 + \alpha\rho(n)\}, \\ M_3 &= \{n \in \mathbb{N} \mid \delta(n) \geq 18\alpha,\ \gamma n > \frac{5}{2}\delta(n)^2 + \alpha\rho(n)\}. \end{aligned}$$

By Proposition 1, there exists n_1 such that for all $n \geq n_1, n \in M_1$, we have that $f^{\lambda^{(n)}} \geq \beta^n$. In fact, $f^{\lambda^{(n)}} \geq \frac{\alpha^n}{n^m}$ where $m = \frac{(3\delta(n)-1)\delta(n)}{2} < \frac{3\delta(n)^2}{2} < 486\alpha^2$.

Suppose now that $n \in M_2$. Then $\gamma n \leq \frac{5}{2}\delta(n)^2 + \alpha\rho(n)$ and suppose first that $\alpha \in \mathbb{N}$. Then in this case $\rho(n) = \delta(n)^2$ and $\gamma n \leq \frac{5}{2}\delta(n)^2 + \alpha\rho(n)$ implies that $\gamma n \leq (\frac{5}{2} + \alpha)\delta(n)^2$. Thus $\frac{\delta(n)^2}{n} \geq \gamma(\frac{5}{2} + \alpha)^{-1} = \varepsilon > 0$. By Proposition 2, there exists n_2 such that for all $n \geq n_2$, $n \in M_2$ we have that $f^{\lambda^{(n)}} \geq \beta^n$.

Suppose now that $\alpha \notin \mathbb{N}$. Then

$$\begin{aligned} \gamma n &\leq \frac{5}{2}\delta(n)^2 + \alpha\left(\left[\frac{\delta(n)^2}{\alpha - [\alpha]}\right] + 1\right) \leq \frac{5}{2}\delta(n)^2 + \alpha \cdot \frac{\delta(n)^2}{\alpha - [\alpha]} + \alpha \\ &\leq \delta(n)^2\left(3 + \frac{\alpha}{\alpha - [\alpha]}\right), \end{aligned}$$

since $\frac{1}{2}\delta(n)^2 > \alpha$. Thus $\frac{\delta(n)^2}{n} \geq \gamma(3 + \frac{\alpha}{\alpha - [\alpha]})^{-1} = \varepsilon > 0$. By Proposition 2 as above, we get $f^{\lambda^{(n)}} \geq \beta^n$, for any $n \geq n_2, n \in M_2$.

By Proposition 3, there exists n_3 such that for all $n \geq n_3, n \in M_3$,

$$f^{\lambda^{(n)}} \geq \alpha^{n-\gamma n} \geq \beta^n,$$

since from (7) we have that $(n - \gamma n) \ln \alpha \geq \ln \beta$.

If we now take $n_0 = max\{n_1, n_2, n_3\}$, we get that $f^{\lambda^{(n)}} \geq \beta^n$ for all $n \geq n_0$. This completes the proof of the theorem. □

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