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A HÖLDER ESTIMATE FOR ENTIRE SOLUTIONS TO THE TWO-VALUED MINIMAL SURFACE EQUATION

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Abstract. We prove a Hölder estimate near infinity for solutions to the two-valued minimal surface equation over $\mathbb{R}^2 \setminus \{0\}$, and give a Bernstein-type theorem in case the solution can be extended continuously across the origin. The main results follow by modifying methods used to study exterior solutions to equations of minimal surface type.

1. Introduction

We consider in this work solutions to the two-valued minimal surface equation (2MSE) over $\mathbb{R}^2 \setminus \{0\}$, with the goal of showing a Bernstein-type theorem. As such, in Theorem 3.1 we conclude that solutions to the 2MSE over $\mathbb{R}^2 \setminus \{0\}$ which can be extended continuously across the origin, or which are either bounded below or above, are trivial of the form $u_0(r,\theta) = ar^2 \cos 2\theta + br^2 \sin 2\theta + c$ for $a, b, c \in \mathbb{R}$, writing in polar coordinates. This is analogous to Bernstein's Theorem, which states that the only solutions to the minimal surface equation (MSE) over $\mathbb{R}^n$ are linear for $1 \leq n \leq 7$, while non-linear solutions exist over dimensions $n \geq 8$. In general, we show that solutions to the 2MSE over the punctured plane have corresponding two-valued graphs (see §2) asymptotic to a multiplicity two plane near infinity. It is open whether there exist solutions to the 2MSE over $\mathbb{R}^2 \setminus \{0\}$ which cannot be extended continuously across the origin.

The 2MSE was originally introduced in [12], as a result of studying the regularity of stable minimal hypersurfaces; see for example [7], [13], [14]. In [5], [6] the author further studied the 2MSE. To this end, many analogies to the theory of the MSE are drawn. Modern methods of studying the MSE, for example those found in [8], are adapted to the 2MSE in [5], [12]. Nonetheless, particularly in [6], more classical methods of studying the MSE are also employed to the 2MSE, such as the comparison arguments found in [3].

Consequentially, Theorem 3.2 is proved by modifying the methods and results of [9], [10] to the present, two-valued setting. In particular, Theorem 6′ of [10] shows a uniform limit near infinity for the gradient of solutions to equations of minimal surface type over *exterior regions* $\{x \in \mathbb{R}^2 : |x| > R\}$. Given that we can treat $\mathbb{R}^2 \setminus \{0\}$ as an exterior region, and the two-valued graph corresponding to

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a solution to the 2MSE can be written as two single-valued minimal graphs (see §2), then we gain an analogous result, given by Theorem 3.2. Indeed, many of the calculations found in [9], [10] pass with no modifications. We include details and make few direct appeals to [9], [10] to highlight the differences the current two-valued setting demands.

Before we do so, in §2 we present the results from [5], [12] necessary to introduce the 2MSE and to prove the current results. In §3 we state the main results, Theorems 3.1, 3.2, along with Corollaries 3.3, 3.4, 3.5. We prove these in §4, except for Corollary 3.5 which we prove in §5. Corollary 3.5 implies that the total Gaussian curvature of the two-valued graph of a solution to the 2MSE over $\mathbb{R}^2 \setminus \{0\}$ is either zero or -2π. This is proved using Theorem 3.2 together with Gauss-Bonnet.

2. Preliminaries

Throughout we write $x = (x_1, x_2) \in \mathbb{R}$, and in polar coordinates $x = re^{i\theta}$. We denote points in $\mathbb{R}^3$ either by X or (x,t), where $t \in \mathbb{R}$. Also let $\mathcal{D} = \{x : |x| < 1\}$, $\mathcal{D}_R = \{x : |x| < R\}$, and $B_\rho((x,t))$ be an open ball in $\mathbb{R}^3$.

The *two-valued minimal surface operator* is the second order operator

$$\mathcal{M}_0(v) = \sum_{j=1}^{2} \frac{\partial}{\partial x_j} \left(\frac{\frac{\partial v}{\partial x_j}}{\sqrt{1 + \frac{|Dv|^2}{4r^2}}} \right)$$

for $r > 0$. We say u_0 is an *entire* solution to the 2MSE if $\mathcal{M}_0(u_0) = 0$ over $\mathbb{R}^2 \setminus \{0\}$. The significance of the $4r^2$ in the 2MSE is that if $u_0(r,\theta)$, written in polar coordinates, is an entire solution to the 2MSE, then the *two-valued function corresponding to* u_0 given by $u(r,\theta) = u_0(r^{1/2}, \theta/2)$ is locally in r,θ a solution to the MSE globally having period 4π with respect to θ. Therefore, the *two-valued graph corresponding to* u_0 given by

$$G = \{(re^{i\theta}, u(r,\theta)) : r \in (0,\infty), \theta \in \mathbb{R}\}$$

is a smooth immersed minimal surface away from $\{0\} \times [\underline{\lim}_{r\to 0} u_0, \overline{\lim}_{r\to 0} u_0]$.

A compactness argument, shown in §1 of [12] using gradient estimates for non-uniformly elliptic PDEs given in [8], shows that for any $\varphi_0 \in C(\partial\mathcal{D}_R)$ there exists a solution to the 2MSE $u_0 \in C(\overline{\mathcal{D}_R} \setminus \{0\}) \cap C^\infty(\mathcal{D}_R \setminus \{0\})$ with $u_0|_{\partial\mathcal{D}_R} = \varphi_0$. Continuity of solutions to the 2MSE at the origin may fail. In fact, Theorem 6 of [5] shows that the boundary data $\varphi_0(\theta) = \cos\theta$ (for any $\partial\mathcal{D}_R$) yields a solution to the 2MSE which cannot be extended continuously across the origin. [5] and [12] give regularity results determined by continuity at the origin, which we explain.

First, Theorem 1 of [12] concludes that if u_0 is a solution to the 2MSE that can be extended continuously across the origin, then the corresponding two-valued graph G has $C^{1,\alpha}$ branch point at $(0, u_0(0))$ (note that if $u_0(r,\theta+\pi) = u_0(r,\theta)$ for all $r \geq 0$ and $\theta = \mathbb{R}$, so that G is the multiplicity two graph of a single-valued function, then $(0, u_0(0))$ is not a true branch point of G but a regular point). We make this statement more precise, to introduce notation. Take any $\theta_0 \in [0, 2\pi)$, and define the *slit domain* $\Omega_{\theta_0} = \mathbb{R}^2 \setminus \{re^{i\theta_0} : r \in [0,\infty)\}$. We then define the *component functions of* u_0 *corresponding to the slit domain* Ω_{θ_0} to be the functions

$$\begin{cases} u_1(r,\theta) = u(r,\theta) = u_0(r^{1/2}, \theta/2), \\ u_2(r,\theta) = u(r,\theta+2\pi) = u_0(r^{1/2}, \theta/2+\pi), \end{cases}$$

writing every point $x \in \Omega_{\theta_0}$ by $x = re^{i\theta}$ with $r \in (0,\infty), \theta \in (\theta_0, \theta_0 + 2\pi)$. We denote the graphs of u_j over Ω_{θ_0} by

$$G_j = \{(re^{i\theta}, u_j(r,\theta)) : r \in (0,\infty), \theta \in (\theta_0, \theta_0 + 2\pi)\}.$$

We thus have $G \cap (\Omega_{\theta_0} \times \mathbb{R}) = G_1 \cup G_2$. Furthermore, G is $C^{1,\alpha}$ at the origin in the sense that for any $\sigma \in (0,\pi)$, the component functions u_j have $C^{1,\alpha}$ norms bounded by a constant $c(\sigma, \sup_{\theta\in\mathbb{R}} |u(\sigma,\theta)|)$ in the region $\{re^{i\theta} : r \in (0,\sigma), \theta \in (\theta_0 + \sigma, \theta_0 + 2\pi - \sigma)\}$. Theorem 1 of [12] concludes that G is stable, in the sense that the stability inequality holds for all ζ differentiable with compact support in $\mathbb{R}^3$

$$\int_G \zeta^2 |A|^2 \, d\mathcal{H}^2 \leq \int_G |\nabla\zeta|^2 \, d\mathcal{H}^2 \tag{2.1}$$

(ζ need not vanish near $(0, u_0(0))$), where A is the second fundamental form of G and $\nabla = \nabla^G$ is the gradient on G.

Second, suppose u_0 is a solution to the 2MSE which cannot be extended continuously across the origin. By (2) of [5], u_0 satisfies the strong maximum principle, and so $[\underline{\lim}_{r\to 0} u_0, \overline{\lim}_{r\to 0} u_0]$ is a finite interval. Theorem 3 of [5] concludes that for every $t \in (\underline{\lim}_{r\to 0} u_0, \overline{\lim}_{r\to 0} u_0)$ there exists a $\rho > 0$ so that $G \cap B_\rho((0,t)) = L_1(t) \cap L_2(t)$, where each $L_j(t)$ is a smooth embedded minimal surface-with-boundary $\partial L_j(t) \cap B_\rho((0,t)) = \{0\} \times (t-\rho, t+\rho)$.

We also have an asymptotic description of G near the endpoints $(0, \underline{\lim}_{r\to 0} u_0)$, $(0, \overline{\lim}_{r\to 0} u_0)$. Theorem 4 of [5] concludes that G, as a two-dimensional varifold, has unique tangent cones at these points consisting each of a multiplicity one plane containing the vertical axis $\{0\} \times \mathbb{R}$. In fact, letting R be the reflection $R(x,t) = (-x,t)$, for $\rho > 0$ sufficiently small $(\overline{G} \cup R(G)) \cap B_\rho((0, \overline{\lim}_{r\to 0} u_0))$ is the two-valued graph corresponding to a solution to the 2MSE defined over the unique tangent cone of G at $(0, \overline{\lim}_{r\to 0} u_0)$ (the same conclusion holds for $\underline{\lim}_{r\to 0} u_0$).

Finally, an important fact is the following mass bound for balls:

$$\mathcal{H}^2(G \cap B_R((0,t))) \leq \mathfrak{c} R^2 \tag{2.2}$$

for any $t \in \mathbb{R}$, where $\mathfrak{c}$ is a constant not even depending on G, regardless of whether u_0 is continuous at the origin or not. The proof of (2.2), which is (8) of [12], is analogous to showing mass bounds for solutions to the MSE. Therefore, we also suppose that $\mathfrak{c}$ is such that $\mathcal{H}^2(G \cap B_R(X)) \leq \mathfrak{c} R^2$ whenever $|X| \geq 2R$.

3. Results

We list here the results to be proved. We begin with a Bernstein-type theorem.

Theorem 3.1. *If u_0 is an entire solution to the 2MSE which can be extended continuously across the origin, or if u_0 is either bounded from below or from above, then $u_0(r,\theta) = ar^2 \cos 2\theta + br^2 \sin 2\theta + c$ for some $a, b, c \in \mathbb{R}$, meaning the corresponding two-valued graph is a (multiplicity two) plane.*

In general, we give the following asymptotic description at infinity.

Theorem 3.2. *Suppose u_0 is a an entire solution to the 2MSE, and let $R_{u_0} = \max\{|\underline{\lim}_{r\to 0} u_0|, |\overline{\lim}_{r\to 0} u_0|\}$. There exists a vector $\vec{a} \in \mathbb{R}^2$ such that the two-valued function u corresponding to u_0 satisfies*

$$\sup_{r\geq R} |Du - \vec{a}| \leq c \left(\frac{R_{u_0}}{R}\right)^{1/64\pi}$$

for all $R \geq 2R_{u_0}$, where c is a constant not depending on u_0.

Theorem 3.2 leads to the three following corollaries. The first two are derived using the theory of varifolds, for which we refer the reader to [11].

Corollary 3.3. *Consider an entire solution to the 2MSE, and let $\vec{a} \in \mathbb{R}^2$ be as in Theorem 3.2 for that solution. As a varifold, the two-valued graph corresponding to the solution has a unique tangent cone at infinity (choosing the origin as a center) given by the plane with normal vector $\frac{(-\vec{a},1)}{\sqrt{1+|\vec{a}|^2}}$ with multiplicity two.*

Corollary 3.4. *If u_0 is an entire solution to the 2MSE which cannot be extended continuously across the origin with $0 \in [\underline{\lim}_{r\to 0} u_0, \overline{\lim}_{r\to 0} u_0]$, then the density ratios $\frac{\mathcal{H}^2(G\cap B_R(0))}{R^2}$ increase from π to 2π as R increases. Thus, $\int_G \frac{|\nabla \mathbf{r}|^2}{\mathbf{r}^2}\, d\mathcal{H}^2 = \pi$ where $\mathbf{r} = |X|$ and $\nabla = \nabla^G$ is the gradient over G.*

The last corollary is an application of Gauss-Bonnet.

Corollary 3.5. *Suppose u_0 is an entire solution to the 2MSE which cannot be extended continuously across the origin, then the corresponding two-valued graph G has finite total curvature. In fact, $\int_G |A|^2\, d\mathcal{H}^2 = 4\pi$.*

4. Proofs

We begin by proving Theorem 3.1, after which we show how to modify the results of [9], [10] to prove Theorem 3.2. The exact decay exponent $-1/64\pi$ for Du in Theorem 3.2 relies on Corollaries 3.3, 3.4, which we prove in this section. That c in Theorem 3.2 does not depend on u_0 follows in part from Corollary 3.5, the proof of which we leave for §5.

Proof of Theorem 3.1. First, suppose u_0 is an entire solution to the 2MSE which can be extended continuously across the origin. We argue as in the case for the single-valued minimal surface equation. As G satisfies the stability inequality (2.1), then we can choose ζ to be the standard logarithmic cut-off functions. Since G has quadratic area growth by (2.2), then we can show $A = 0$, so that G is a multiplicity two plane.

Second, suppose for contradiction, and without loss of generality, that u_0 is an entire solution to the 2MSE with $\inf_{x\in\mathbb{R}^2\setminus\{0\}} u_0(x) = 0$ which is not identically zero. We argue as in [4], using the lower-half catenoid function. By the strong maximum principle (see (2) of [5]), we have $\lambda_{u_0} := \inf_{x\in\mathcal{D}\setminus\{0\}} u_0 > 0$. Define for $\lambda \in (0,1)$ the family of functions $\tilde{u}_0^\lambda : (\mathbb{R}^2 \setminus \mathcal{D}_\lambda) \to \mathbb{R}$ by

$$\tilde{u}_0^\lambda(x) = \lambda_{u_0} - \lambda^2 \cosh^{-1}[(r/\lambda)^2],$$

which solves the 2MSE over $\mathbb{R}^2 \setminus \mathcal{D}_\lambda$.

Fix any $x \in \mathbb{R}^2 \setminus \{0\}$. Consider any $\lambda \in (0, \min\{|x|, 1\})$, and take any $R \in \mathbb{R}$ with

$$R > \max\{|x|, \lambda\sqrt{\cosh[\lambda^{-2}\lambda_{u_0}]}\}.$$

Note that, by the discussion showing (3) in §3 of [5], the strong maximum principle holds for differences of solutions to the 2MSE. Applying the strong maximum principle to $u_0 - \tilde{u}_0^\lambda$ in $\mathcal{D}_R \setminus \mathcal{D}_\lambda$ therefore gives

$$u_0(x) > \lambda_{u_0} - \lambda^2 \cosh^{-1}[(r/\lambda)^2]$$

for all $\lambda \in (0, \min\{|x|, 1\})$. Letting $\lambda \to 0$, we conclude that $u_0(x) \geq \lambda_{u_0}$ for all $x \in \mathbb{R}^2 \setminus \{0\}$, which is a contradiction. □

Proof of Theorem 3.2. The key idea, used in [9], [10], is to consider the volume form on the upper hemisphere S^2_+, given by $d\omega$ where

$$\omega = \frac{-x_2 dx_1 + x_1 dx_2}{1 + x_3}.$$

Take u_0 an entire solution to the 2MSE and G the corresponding two-valued graph. If K is the Gaussian curvature of G, then

$$K \ d\text{vol}_G = d\nu^{\#}\omega = d\left(\frac{-\nu_2 d\nu_1 + \nu_1 d\nu_2}{1 + \nu_3}\right),$$

where ν is the upward pointing unit normal of G. Observe that $|\nu^{\#}\omega| \leq |A|$.

Our goal is to get a differential inequality for $D(R) := \int_{G \setminus B_R(0)} |A|^2 \ d\mathcal{H}^2$. First, we use Stoke's Theorem via the form ω. Choose by Sard's theorem $R > R_{u_0} := \max\{|\underline{\lim}_{r\to 0} u_0|, \overline{\lim}_{r\to 0} u_0\}$ so that $G \cap \partial B_R(0)$ is the union of smooth immersed closed curves. As G is a minimal surface, $K = -|A|^2/2$, which implies

$$\int_{G \setminus B_R(0)} \zeta^2 |A|^2 \ d\mathcal{H}^2 = 4 \int_{G \setminus B_R(0)} \zeta d\zeta \wedge \nu^{\#}\omega + 2 \int_{G \cap \partial B_R(0)} \zeta^2 \nu^{\#}\omega, \tag{4.1}$$

using Stoke's theorem as in (3.24) of [10], for any differentiable function ζ with compact support in $\mathbb{R}^3$.

Second, we use (4.1) to conclude that G has finite total curvature near infinity. With $\nabla = \nabla^G$, we use $|d\zeta \wedge \nu^{\#}\omega| \leq |\nabla\zeta||A|$ and Cauchy-Schwartz to get from (4.1)

$$\int_{G \setminus B_R(0)} \zeta^2 |A|^2 \ d\mathcal{H}^2 \leq 16 \int_{G \setminus B_R(0)} |\nabla\zeta|^2 \ d\mathcal{H}^2 + 4 \int_{G \cap \partial B_R(0)} \zeta^2 |A| \ d\mathcal{H}^1.$$

As G has quadratic area growth, we can let $\zeta \to 1$ via standard logarithmic cut-off functions so that $\int_{G \setminus B_\rho(0)} |\nabla\zeta|^2 \ d\mathcal{H}^2 \to 0$, giving G finite total curvature near infinity $\int_{G \setminus B_R(0)} |A|^2 \ d\mathcal{H}^2 \leq 4 \int_{G \cap \partial B_R(0)} |A| \ d\mathcal{H}^1$.

Third, we use the fundamental theorem of calculus to derive

$$\int_{G \setminus B_R(0)} |A|^2 d\mathcal{H}^2 \leq 8 \left[\int_{G \cap \partial B_R(0)} |A| \ d\mathcal{H}^1\right]^2. \tag{4.2}$$

To see this, let $\zeta \to 1$ via logarithmic cut-off functions in (4.1) to get

$$\int_{G \setminus B_R(0)} |A|^2 d\mathcal{H}^2 = 2 \int_{G \cap \partial B_R(0)} \nu^{\#}\omega. \tag{4.3}$$

We write $G\cap\partial B_R(0)=\bigcup_{k=1}^n\Gamma_k$, where each smooth curve $\Gamma_k(s)$ is parameterized by arc length. Therefore, by the fundamental theorem of calculus

$$\int_{G\cap\partial B_R(0)}\nu^{\#}\omega=\sum_{k=1}^n\int_{\Gamma_k}\frac{1}{1+\nu_3}\left(-\nu_2\frac{d\nu_1}{ds}+\nu_1\frac{d\nu_2}{ds}\right)ds$$

$$=\sum_{k=1}^n\int_{\Gamma_k}-\left(\frac{\nu_2(s)}{1+\nu_3(s)}-\frac{\nu_2(0)}{1+\nu_3(0)}\right)\frac{d\nu_1}{ds}+\left(\frac{\nu_1(s)}{1+\nu_3(s)}-\frac{\nu_1(0)}{1+\nu_3(0)}\right)\frac{d\nu_2}{ds}\ ds,$$

supposing each Γ_k is defined at $s=0$. On the other hand,

$$\left|\frac{\nu_j(s)}{1+\nu_3(s)}-\frac{\nu_j(0)}{1+\nu_3(0)}\right|=\left|\int_0^s\frac{d}{d\tau}\frac{-\nu_j(\tau)}{1+\nu_3(\tau)}d\tau\right|\le 2\int_{\Gamma_i}|A|ds.$$

Combined with (4.3) this gives (4.2).

Fourth, we get the wanted differential inequality for $\int_{G\setminus B_R(0)}|A|^2\ d\mathcal{H}^2$. Apply Cauchy-Schwartz to (4.2) and the co-area formula with $\mathbf{r}=|X|$ to get

$$\int_{G\setminus B_R(0)}|A|^2\ d\mathcal{H}^2\le -8\frac{d}{dR}\int_{G\setminus B_R(0)}|A|^2\ d\mathcal{H}^1\cdot\frac{d}{dR}\int_{G\cap B_R(0)}|\nabla\mathbf{r}|^2\ d\mathcal{H}^1.$$

To estimate $\frac{d}{dR}\int_{G\cap B_R(0)}|\nabla\mathbf{r}|^2\ d\mathcal{H}^1$, the reflection principle of [1] implies the varifold $G+R_{\#}G$ is stationary in $\mathbb{R}^3$, where $R(x,t)=(-x,t)$. Since

$$\int_{R(G)}\mathrm{div}_{R(G)}\left(\phi(\mathbf{r})X\right)\ d\mathcal{H}^2=\int_G\mathrm{div}_G\left(\phi(\mathbf{r})X\right)\ d\mathcal{H}^2,$$

then $\int_G\mathrm{div}_G\left(\phi(\mathbf{r})X\right)\ d\mathcal{H}^2=0$ so long as ϕ has compact support and depends only on $\mathbf{r}=|X|$. Letting ϕ approximate $\mathcal{X}_{B_R(0)}$, then we can show as in the monotonicity formula using the co-area formula (see the proof of Theorem 17.6 of [11]) that

$$\frac{d}{dR}\int_{G\cap B_R(0)}|\nabla\mathbf{r}|^2=2\cdot\frac{\mathcal{H}^2(G\cap B_R(0))}{R^2}\cdot R.$$

This together with (2.2) gives us the differential inequality

$$D(R)\le -16\mathfrak{c}R\cdot D'(R), \tag{4.4}$$

where $D(R):=\int_{G\setminus B_R(0)}|A|^2\ d\mathcal{H}^2$. As $D(R)$ is decreasing, we integrate (4.4) to get

$$D(R)\le\left(\frac{R_{u_0}}{R}\right)^{1/16\mathfrak{c}}D(R_{u_0}), \tag{4.5}$$

which holds for $R\ge R_{u_0}$.

Our goal now is to get a Hölder continuity estimate near infinity for ν, the upward pointing unit normal G. To do this, we proceed as in [10] using (4.5) together with the Morrey-type lemma given by Lemma 2.2 of [9].

First, let $R\ge R_{u_0}$ and take any $\overline{X}\in G\setminus B_{2R}(0)$, then

$$\tilde{D}(\overline{X},R):=\int_{G\cap B_R(\overline{X})}|A|^2\ d\mathcal{H}^2\le D(R).$$

If $X\in B_{R/2}(\overline{X})$ and $\rho\in(0,R/2)$, then G is stationary in $B_{R/2}(X)$, and (2.2) holds for each $B_\rho(X)$. We can thus repeat the argument leading to (4.5) to get

$$\tilde{D}(X,\rho)\le\left(\frac{\rho}{R/2}\right)^{1/16\mathfrak{c}}\tilde{D}(X,R/2),$$

as $\tilde{D}(X,\rho)$ is increasing. Cauchy-Schwartz gives with $c = \mathfrak{c}^{1/2}$,

$$\int_{G\cap B_\rho(X)} |A|\, d\mathcal{H}^2 \le cD(R)^{1/2}\rho\left(\frac{2\rho}{R}\right)^{1/32\mathfrak{c}}. \tag{4.6}$$

Second, we apply Lemma 2.2 of [9] in order to conclude a Hölder estimate for ν, as a two-valued function, in the ball $B_\rho(\overline{X})$ for any $\rho \in (0, R/4)$. To justify this statement, as well as to make it precise, choose any slit domain Ω_{θ_0} using an angle θ_0 so that the slit ray $\{re^{i\theta_0} : r \in [0,\infty)\}$ does not intersect the ball $B_R(\overline{X})$. It follows that $G \cap B_R(\overline{X}) = (G_1 \cup G_2) \cap B_R(\overline{X})$ where G_j is the graph of the component function u_j. By (4.6), each G_j is a single-valued minimal graph satisfying

$$\int_{G_j\cap B_\rho(X)} |A|\, d\mathcal{H}^2 \le cD(R)^{1/2}\rho\left(\frac{2\rho}{R}\right)^{1/32\mathfrak{c}}$$

for each $X \in B_{R/2}(\overline{X})$ and $\rho \in (0, R/2)$. Since $G_j \cap B_R(\overline{X})$ is a smooth graph, then we can directly apply Lemma 2.2 of [9] to conclude

$$\sup_{X\in G_j\cap B_\rho(\overline{X})^*} |\nu_{G_j}(X) - \nu_{G_j}(\overline{X})| \le cD(R)^{1/2}\left(\frac{\rho}{R}\right)^{1/32\mathfrak{c}} \tag{4.7}$$

for $\rho \in (0, R/4)$. Here, $G_j \cap B_\rho(\overline{X})^*$ is the connected component of $G_j \cap B_\rho(\overline{X})$ containing $\overline{X} \in G \setminus B_{2R}(0)$, and ν_{G_j} is the upward pointing unit normal of G_j at $\overline{X}$, whenever $\overline{X} \in G_j$. Note that in applying Lemma 2.2 of [9], the constant c continues to depend only on $\Lambda_3 = \mathfrak{c}$, as $\Lambda_4 = 0$ in this case. This together with (4.5) implies

$$\sup_{X\in G_j\cap B_\rho(\overline{X})^*} |\nu_{G_j}(X) - \nu_{G_j}(\overline{X})| \le \left(cD(R_{u_0})^{1/2}R_{u_0}^{1/32\mathfrak{c}}\right)R^{-1/32\mathfrak{c}}\left(\frac{\rho}{R}\right)^{1/32\mathfrak{c}}. \tag{4.8}$$

Third, we make two points about connectivity. Firstly, using the fact that G is a minimal surface away from $\{0\} \times [\underline{\lim}_{r\to 0} u_0, \overline{\lim}_{r\to 0} u_0]$, we can show $G \setminus B_R(0)$ is connected for $R > R_{u_0}$, assuming G intersects $\partial B_R(0)$ transversely. Secondly, consider any slit domain Ω_{θ_0}, and let $B_\rho(X)$ be any ball which does not intersect the slit $\{re^{i\theta_0} : r \in [0,\infty)\}$. We can thus refer to Lemma 3.2 of [9] to conclude there is a $\gamma \in (0,1)$ (denoted θ in [9]), depending only on $\Lambda_1 = -2$ and $\Lambda_2\rho^2 = 0$, so that $G_j \cap B_\sigma(X)$ is connected for each $\sigma \in (0, \gamma\rho)$.

Fourth, our goal is to sum (4.8), much as in what follows after equation (3.34) of [10], in order to derive a Hölder estimate for ν near infinity. Consider the slit domain with $\theta_0 = \pi$, which is merely $\mathbb{R}^2 \setminus \{(x_1, 0) : x_1 \le 0\}$. Taking $R \ge R_{u_0}$, choose any $\overline{X} \in G \cap \{(x,t) : x_1 > 0\} \setminus B_{2R}(0)$. Applying Lemma 3.2 of [9] together with (4.8), we conclude that for each G_j corresponding to the slit domain Ω_π, and for $\rho \in (0, \gamma R/4)$,

$$\sup_{X\in G_j\cap B_\rho(\overline{X})} |\nu_{G_j}(X) - \nu_{G_j}(\overline{X})| \le \left(cD(R_{u_0})^{1/2}R_{u_0}^{1/32\mathfrak{c}}\right)R^{-1/32\mathfrak{c}}$$

(since $\gamma \in (0,1)$). The same estimate holds for G_j corresponding to the slit domain $\Omega_0 = \mathbb{R}^2 \setminus \{x : x_1 \ge 0\}$. Since we can cover $B_{(2+\gamma)R}(0) \setminus B_{2R}(0)$ by N balls of radius $\gamma R/8$, with $N \in (0,\infty)$ depending only on γ, then for each connected component $\mathcal{G}$ of $G \cap B_{(2+\gamma)R}(0) \setminus B_{2R}(0)$ (as an immersed surface) we have

$$\sup_{X,\overline{X}\in\mathcal{G}} |\nu(X) - \nu(\overline{X})| \le \left(cD(R_{u_0})^{1/2}R_{u_0}^{1/32\mathfrak{c}}\right)R^{-1/32\mathfrak{c}},$$

where c now depends on γ as well. We observe that this estimate also holds with R replaced by $(2+\gamma)^k R$, together with the corresponding connected components of G. By considering $\overline{R}$ sufficiently close to $2R$ with $\overline{R} < 2R$ and so that $G \setminus B_{\overline{R}}(0)$ is connected, then we can conclude

$$\sup_{X,\overline{X}\in G\setminus B_{2R}(0)} |\nu(X) - \nu(\overline{X})| \leq 2\left(cD(R_{u_0})^{\frac{1}{2}} R_{u_0}^{\frac{1}{32c}}\right) \sum_{k=0}^{\infty} (2+\gamma)^{-\frac{k}{32c}} R^{-\frac{1}{32c}}, \tag{4.9}$$

so long as $R \geq R_{u_0}$, for a constant c not depending on u_0. We make clear, if there are two values for ν at X and $\overline{X}$, then (4.9) holds regardless of which pair of values is being compared.

Therefore $\lim_{X\in G, |X|\to\infty} \nu$ exists. We must now show the limit is a vector in the (open) upper hemisphere. In order to do this, observe that by (4.9) we are done if there is a sequence of $X_k \in G$ with $|X_k| \to \infty$ so that $\nu_3(X_k) \geq 1/2$, where $\nu = (\nu_1, \nu_2, \nu_3)$. We henceforth assume $\nu_3(X) < 1/2$ for all $X \in G$ with $|X| > \mathcal{R}$, for some $\mathcal{R} \geq R_{u_0}$. We argue as in [10], by considering the function $w = \log\left(\frac{1}{\nu_3}\right)$. We derive a Hölder continuity estimate near infinity for w, much as we did for ν.

Before we do so, we modify Theorem 4.2. of [9] to the present setting. Take any slit domain Ω_{θ_0}, and suppose $B_R(X)$ does not intersect the slit. Theorem 4.2 of [9] implies that for any $\rho \in (0, R/2)$

$$\sup_{G_j\cap B_\rho(X)} \left(\frac{1}{\nu_{G_j,3}}\right) \leq c \cdot \inf_{G_j\cap B_\rho(X)} \left(\frac{1}{\nu_{G_j,3}}\right),$$

where $\nu_{G_j} = (\nu_{G_j,1}, \nu_{G_j,2}, \nu_{G_j,3})$ and c depends only on $\Lambda_1 = -2$. If $G \cap \partial B_R(0)$ is a union of smooth curves $\Gamma_1 \ldots \Gamma_n$, then taking c larger independent of R (since we can cover $\partial B_R(0)$ by a finite number, independent of R, of balls of radius $R/4$), we conclude the Harnack inequality for each $k = 1, \ldots, n$,

$$\sup_{G\cap\Gamma_k} \left(\frac{1}{\nu_3}\right) \leq c \cdot \inf_{G\cap\Gamma_k} \left(\frac{1}{\nu_3}\right). \tag{4.10}$$

We now show the Hölder estimate for w. First, assuming $G \cap \partial B_R(0)$ is smooth, we can show using Stoke's theorem exactly as in (3.38) of [10], the equality

$$\int_{G\setminus B_R(0)} \zeta^2 \left(\frac{|A|^2}{\nu_3^2}\right) d\mathcal{H}^2 = 2\int_{G\cap\partial B_R(0)} \zeta^2 \left(\frac{-\nu_2 d\nu_1 + \nu_1 d\nu_2}{\nu_3(1-\nu_3^2)}\right) \tag{4.11}$$
$$+4\int_{G\setminus B_R(0)} \zeta d\zeta \wedge \left(\frac{-\nu_2 d\nu_1 + \nu_1 d\nu_2}{\nu_3(1-\nu_3^2)}\right).$$

Note that we use $K = -|A|^2/2$ in (3.38) of [10]. Here, $G \cap \partial B_R(0)$ is the oriented boundary of $G \cap B_R(0)$.

Second, we use (4.11) in order to show $\int_{G\setminus B_R(0)} |\nabla w|^2 \, d\mathcal{H}^2 < \infty$ for $R \geq \mathcal{R}$. For this, we note by definition of w and the left-hand inequality of (4.6) of [9]:

$$|\nabla w|^2 \leq \left(\frac{|A|^2}{\nu_3^2}\right) \leq \frac{8|\nabla w|^2}{(1-\nu_3^2)}. \tag{4.12}$$

Together with $|-\nu_2 d\nu_1 + \nu_1 d\nu_2| \leq |A|$, $|d\zeta \wedge (-\nu_2 d\nu_1 + \nu_1 d\nu_2)| \leq |\nabla\zeta||A|$, the assumption $\nu_3(X) < 1/2$ for $|X| \geq \mathcal{R}$, and $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$, we can derive from (4.11) that there is a constant c such that

$$\int_{G\setminus B_R(0)} \zeta^2|\nabla w|^2\ d\mathcal{H}^2 \le c\left[\int_{G\cap\partial B_R(0)} \zeta^2|\nabla w|\ d\mathcal{H}^1 + \int_{G\setminus B_R(0)} |\nabla\zeta|^2\ d\mathcal{H}^2\right].$$

We therefore conclude $\int_{G\setminus B_R(0)} |\nabla w|^2\ d\mathcal{H}^2 < \infty$ for $R \ge \mathcal{R}$ by letting $\zeta \to 1$ via logarithmic cut-off functions.

Third, we derive an estimate for w analogous to (4.6). Letting $\zeta \to 1$ via logarithmic cut-off functions in (4.11) gives

$$\int_{G\setminus B_R(0)} |\nabla w|^2\ d\mathcal{H}^2 = \int_{G\cap\partial B_R(0)} W_2 d\nu_1 - W_1 d\nu_2,$$

where $W_i = \frac{-2\nu_i}{\nu_3(1-\nu_3^2)}$. With $G\cap\partial B_R(0) = \bigcup_{k=1}^n \Gamma_k$ where each Γ_k is parameterized by arc-length, by the fundamental theorem of calculus,

$$\int_{G\setminus B_R(0)} |\nabla w|^2\ d\mathcal{H}^2 \le \sum_{k=1}^n \int_{\Gamma_k} |A|\ d\mathcal{H}^1 \cdot \int_{\Gamma_k} |\nabla W_1| + |\nabla W_2|\ d\mathcal{H}^1. \tag{4.13}$$

However, as $\nu_3 < 1/2$ on Γ_k, we can compute $|\nabla W_i| \le \frac{\left(\frac{32(\sqrt{2}+1)}{9}\right)}{\nu_3}|\nabla w|$. Using this in (4.13), together with the Harnack inequality (4.10), we have

$$\int_{G\setminus B_R(0)} |\nabla w|^2\ d\mathcal{H}^2 \le c\sum_{k=1}^n \int_{\Gamma_k} \frac{|A|}{\nu_3}\ d\mathcal{H}^1 \cdot \int_{\Gamma_k} |\nabla w|\ d\mathcal{H}^1,$$

where c here is $\frac{32(\sqrt{2}+1)}{9}$ times the constant from (4.10). Using (4.12), we thus get

$$\int_{G\setminus B_R(0)} |\nabla w|^2\ d\mathcal{H}^2 \le c\left[\int_{G\cap\partial B_R(0)} |\nabla w|\ d\mathcal{H}^1\right]^2,$$

analogous to (4.2). From this, we argue using the co-area formula to get a differential inequality for $D_w(R) := \int_{G\setminus B_R(0)} |\nabla w|^2\ d\mathcal{H}^2$ much as in (4.4). We derive then a decay estimate analogous to (4.5),

$$D_w(R) \le D_w(\mathcal{R})\left(\frac{\mathcal{R}}{R}\right)^\alpha$$

for some $\alpha \in (0,1)$, and all $R \ge \mathcal{R}$. We can also show, analogous to (4.6), that for $R \ge \mathcal{R}$, X with $|X| \ge 2R$, $\rho \in (0,R)$, and $c = \mathfrak{c}^{1/2}$,

$$\int_{G\cap B_\rho(X)} |\nabla w|\ d\mathcal{H}^2 \le cD_w(R)^{1/2}\rho\left(\frac{\rho}{R}\right)^{\alpha/2}. \tag{4.14}$$

Fourth, we observe ν_3 satisfies a maximum principle in the following form. Take any slit domain Ω_{θ_0}, and decompose G into the corresponding component graphs G_j. Let $X \in G_j$ with $|X| \ge 2R$ and $R \ge \mathcal{R}$. Suppose furthermore, that $B_R(X)$ does not intersect the slit. Using the argument following (4.13) of [9], we conclude there is a $\gamma \in (0,1)$ small depending on $\Lambda_1 = -2$, ensuring (3.17) and Lemma 3.2 of [9] hold for G_j in $B_R(X)$, so that $\nu_{G_j,3}$ satisfies the maximum principle on $G_j \cap B_{\gamma R}(X)$. Here, γ is still chosen so that $G_j \cap B_\rho(X)$ is topologically a disk for each $\rho \in (0,\gamma R)$ (the notation θ is used for γ in [9]).

Fifth, take a slit domain Ω_{θ_0} and $\overline{X} \in G_j$ with $|\overline{X}| \geq 2R$, where $R \geq \mathcal{R}$. Suppose $B_R(\overline{X})$ does not intersect the slit. Define for $\rho \in (0, \gamma R)$ the quantities

$$\underline{w_j} = \inf_{G_j \cap B_\rho(\overline{X})} w_j \quad \text{and} \quad \overline{w_j} = \sup_{G_j \cap B_\rho(\overline{X})} w_j$$

for $w_j = \log\left(\frac{1}{\nu_{G_j,3}}\right)$. Estimates analogous to (4.15), (4.16) of [9] thus hold for w_j (replacing θR by ρ in (4.15), (4.16) of [9]), so that we may conclude

$$(\overline{w_j} - \underline{w_j}) \cdot \left(\frac{\rho}{4}\right) \leq \int_{G \cap B_\rho(\overline{X})} |\nabla w_j| \, d\mathcal{H}^2.$$

Together with (4.14), we therefore derive

$$\overline{w_j} - \underline{w_j} \leq 4cD_w(R)^{1/2}\left(\frac{\rho}{R}\right)^{\alpha/2}. \tag{4.15}$$

Sixth, we can follow the same argument after (4.7), applying (4.15) iteratively and summing to conclude

$$\sup_{X, \overline{X} \in G \setminus B_{2R}(0)} |w(X) - w(\overline{X})| \leq c_{\mathcal{R}} R^{-\alpha/2}$$

for all $R \geq \mathcal{R}$, where $c_{\mathcal{R}}$ depends on $\mathcal{R}$ and $D_w(\mathcal{R})$. However, $w = \log\left(\frac{1}{\nu_3}\right)$, so

$$\sup_{G \setminus B_{2R}(0)} \nu_3 \leq e^{c_{\mathcal{R}} R^{-\alpha/2}} \inf_{G \setminus B_{2R}(0)} \nu_3.$$

Therefore, $\lim_{|X| \to \infty} \nu_3$ exists and is nonzero, which together with (4.9) implies $Du \to \vec{a}$ uniformly for some $\vec{a} \in \mathbb{R}^2$.

In order to conclude the proof of Theorem 3.2, we must show that (4.9) holds with $\mathfrak{c}$ replaced by 2π. To prove this involves showing Corollary 3.3.

Proof of Corollary 3.3. Take any sequence $R_k \to \infty$, and consider the rescalings $\eta_{0,R_k\#}G$ where $\eta_{0,R_k}(X) = \frac{X}{R_k}$ (we refer the reader to [11] for an introduction to the theory of varifolds). By (2.2), some subsequence of the $\eta_{0,R_k\#}G$, denoted the same, converges to a varifold $\mathbb{C}$ stationary in $\mathbb{R}^3 \setminus (\{0\} \times \mathbb{R})$. $\mathbb{C}$ is a cone as a consequence of the monotonicity formula holding for G and $\mathbb{C}$ at the origin. (By the reflection principle of [1], $\eta_{0,R_k\#}G$ plus its reflection across $\{0\} \times \mathbb{R}$ is a stationary varifold converging to the sum of $\mathbb{C}$ and its reflection across $\{0\} \times \mathbb{R}$. Hence G and $\mathbb{C}$ both satisfy the monotonicity formula at the origin. Thus, $R^{-2}\|\mathbb{C}\|(B_R(0))$ is constant independent of R because G satisfies the monotonicity formula at the origin, and consequently $\mathbb{C}$ is a cone because $\mathbb{C}$ satisfies the monotonicity formula at the origin). Also, by the structure of one-dimensional stationary varifolds of S^2 given in [2], $\mathbb{C} \cap S^2$ is a locally finite union of great circle arcs.

We now use the theory of stable embedded minimal surfaces given by [7], much as in the arguments found in (9) of [12] and Lemma 2 of [5]. Take any slit domain Ω_{θ_0}, and consider the graphs of the component functions G_j. Since each G_j is a single-valued minimal graph also satisfying the mass bound (2.2) at the origin, we conclude by Theorem 2 of [7] that (some subsequence of) the $\eta_{0,R_k\#}G_j$ converge smoothly in compact subsets of $\Omega_{\theta_0} \times \mathbb{R}$ to $\mathbb{C}_j$ a smooth embedded surface in $\Omega_{\theta_0} \times \mathbb{R}$.

Therefore, $\mathbb{C}$ is either a pair of non-vertical planes, a non-vertical plane together with a finite number of vertical half-planes, or a finite collection of vertical half-planes. Consequentially, each $\mathbb{C}_j$ is either a non-vertical plane or a collection of vertical half-planes. Taking $\delta > 0$ sufficiently small as in Theorem 1 of [7], then for

R_k sufficiently large $G_j \cap \{(re^{i\theta}, t) : \theta \in (\theta_0+\delta, \theta_0+2\pi-\delta), \sqrt{r^2+t^2} \in (\delta R_k, R_k)\}$ is a disjoint union of graphs defined off $\mathbb{C}_j$ with scaled C^1 norm less than δ. However, (4.9) implies $\mathbb{C}_j$ is the plane with normal vector $\frac{(-\vec{a},1)}{\sqrt{1+|\vec{a}|^2}}$. Furthermore, $\mathbb{C}_j$ must have multiplicity one, otherwise we would contradict that G_j is a single-valued graph. We conclude $\mathbb{C}$ is the multiplicity two plane with normal $\frac{(-\vec{a},1)}{\sqrt{1+|\vec{a}|^2}}$. □

Returning to the proof of Theorem 3.2, we appeal to the monotonicity formula. Corollary 3.3 gives us that $\frac{\mathcal{H}^2(G \cap B_R(0))}{R^2}$ increases up to 2π as $R \to \infty$. We thus conclude (4.9) with $\mathfrak{c}$ replaced by 2π. Corollary 3.5, which follows only from what we have shown so far, implies $D(R_{u_0}) \leq 4\pi$. Using this information in (4.9), we conclude Theorem 3.2. □

Proof of Corollary 3.4. If $0 \in [\underline{\lim}_{r\to 0} u_0, \overline{\lim}_{r \to 0} u_0]$, then by Theorem 3 of [5] we have that $\frac{\mathcal{H}^2(G \cap B_R(0))}{R^2}$ decreases to π, the density of two vertical half-planes, as R decreases to 0. Corollary 3.4 thus follows by the Monotonicity Formula. □

5. Finite total curvature

We conclude by proving Corollary 3.5. The proof follows by applying Gauss-Bonnet to $G \cap B_R(0)$ for $R \to \infty$, using the geometric structure of discontinuous solutions to the 2MSE given in [5].

Proof of Corollary 3.5. Suppose u_0 is an entire solution to the 2MSE which cannot be extended continuously across the origin. We shall define a curve using the four components described as follows.

First, let $b = \overline{\lim}_{r\to 0} u_0$. By Theorem 4 of [5], for $\sigma > 0$ sufficiently small there is $\tilde{u}_0$ a solution to the 2MSE over $\mathcal{D}_\sigma \times \mathbb{R}$ which can be extended continuously across the origin with $\tilde{u}_0(0) = 0$, and Q an orthogonal rotation such that

$$G \cap B_\sigma((0,b)) = (Q(\{(re^{i\theta}, \tilde{u}(r,\theta) : r \in (0,\sigma), \theta \in (0,2\pi)\}) + (0,b)) \cap B_\sigma((0,b)),$$

where $\tilde{u}(r,\theta) = \tilde{u}_0(r^{1/2}, \theta/2)$. Fix $\rho \in (0, \sigma/2)$ and let

$$\Gamma_b(\rho) = Q(\{(\rho e^{i\theta}, \tilde{u}(\rho,\theta)) : \theta \in [\pi/2, 3\pi/2]\}) + (0,b).$$

Since $\tilde{u}_0$ can be extended continuously across the origin with $\tilde{u}_0(0) = 0$, then by Theorem 1 of [12] the two-valued graph corresponding to $\tilde{u}_0$ is $C^{1,\alpha}$ at the origin for some $\alpha \in (0,1)$. We can assume Q is such that $D\tilde{u}(0) = 0$. Thus, if we fix $\sigma > 0$ sufficiently small, then we can use Schauder estimates to conclude for all $\rho \in (0,\sigma/2)$ and $\theta \in [\pi/2, 3\pi/2]$ that

$$|\tilde{u}(\rho,\theta)| + \rho|D\tilde{u}(\rho,\theta)| + \rho^2|D^2\tilde{u}(\rho,\theta)| \leq C\rho^{1+\alpha} \sup_{r\in(0,\sigma),\theta\in(0,2\pi)} |\tilde{u}|$$

for a constant $C = C(\alpha,\sigma) \in (0,\infty)$. From this we can show $\Gamma_b(\rho)$ is C^2 asymptotic to a semi-circle as $\rho \searrow 0$, so that we have

$$\lim_{\rho\searrow 0} \int_{\Gamma_b(\rho)} \kappa_b(\rho)\, d\mathcal{H}^1 = \pi,$$

where $\kappa_b(\rho)$ is the geodesic curvature of $\Gamma_b(\rho)$ with respect to G (with $\Gamma_b(\rho)$ oriented so that $(0,b)$ is interior to $\Gamma_b(\rho)$).

Second, with $a = \underline{\lim}_{r\to 0} u_0$ we can similarly define $\Gamma_a(\rho)$ (by replacing b with a in the previous paragraph, choosing $\sigma > 0$ sufficiently small for both a, b). We as well conclude

$$\lim_{\rho\searrow 0} \int_{\Gamma_a(\rho)} \kappa_a(\rho)\ d\mathcal{H}^1 = \pi.$$

Third, Theorems 3, 4 of [5] together imply there are angles

$$\theta_1(a) < \theta_1(b) < \theta_1(a) + 3\pi$$

(see as well Lemma 5 of [5]), so that $\{(re^{i\theta_1(a)}, t) : r, t \in \mathbb{R}\}, \{(re^{i\theta_1(b)}, t) : r, t \in \mathbb{R}\}$ are respectively the tangent cones of G at $(0, a), (0, b)$. Furthermore, with $\sigma > 0$ as above and $\rho \in (0, \sigma/2)$ sufficiently small, we have that $u(\rho, \theta)$ increases monotonically from $u(\rho, \theta_1(a)) \in (a - \sigma, a + \sigma)$ to $u(\rho, \theta_1(b)) \in (b - \sigma, b + \sigma)$. Considering the curve

$$\Gamma_{a\to b}(\rho) = \{(\rho e^{i\theta}, u(\rho, \theta)) : \theta \in [\theta_1(a), \theta_1(b)]\},$$

then $\Gamma_{a\to b}(\rho)$ is C^2 asymptotic to the vertical line segment $\{0\} \times [a, b]$ as $\rho \searrow 0$. In particular,

$$\lim_{\rho\searrow 0} \int_{\Gamma_{a\to b}(\rho)} \kappa_{a\to b}(\rho)\ d\mathcal{H}^1 = 0.$$

To see this more precisely, we first consider $\Gamma_{a\to b}(\rho) \cap B_\sigma((0, b))$ and use Theorem 4 of [5] and Schauder estimates as for $\Gamma_b(\rho)$, after which we use Theorem 3 of [5] along $\{0\} \times [a + \sigma, b - \sigma]$.

Fourth, we consider the curve

$$\Gamma_{b\to a}(\rho) = \{(\rho e^{i\theta}, u(\rho, \theta)) : \theta \in [\theta_1(b) + \pi, \theta_1(a) + 3\pi]\}.$$

We similarly have

$$\lim_{\rho\searrow 0} \int_{\Gamma_{b\to a}(\rho)} \kappa_{b\to a}(\rho)\ d\mathcal{H}^1 = 0.$$

After slightly modifying $\Gamma_b(\rho), \Gamma_a(\rho), \Gamma_{a\to b}(\rho)$ and $\Gamma_{b\to a}(\rho)$, we can define a smooth curve $\Gamma(\rho) \subset G \cap (\mathcal{D}_\sigma \times \mathbb{R})$, for each $\rho > 0$ sufficiently small, so that

$$\lim_{\rho\searrow 0} \int_{\Gamma(\rho)} \kappa(\rho)\ d\mathcal{H}^1 = 2\pi$$

for $\kappa(\rho)$ the geodesic curvature of $\Gamma(\rho)$ with respect to G (with $\Gamma(\rho)$ oriented so that $\{0\} \times [a, b]$ is interior to $\Gamma(\rho)$).

Choose $R \in (0, \infty)$ large and let $G_{\rho,R}$ be the region of G with boundary $\Gamma(\rho)$ and $G \cap (\partial\mathcal{D}_R \times \mathbb{R})$. By Gauss-Bonnet we have

$$\int_{G_{\rho,R}} K\ d\mathcal{H}^2 + \int_{\partial G_{\rho,R}} \kappa\ d\mathcal{H}^1 = 2\pi\mathcal{X}(G_{\rho,R}).$$

Topologically $G_{\rho,R}$ is an annulus, and hence $\mathcal{X}(G_{\rho,R}) = 0$. Furthermore, as $R \to \infty$ we get $\int_{G\cap(\partial\mathcal{D}_R\times\mathbb{R})} \kappa\ d\mathcal{H}^1 \to 4\pi$, the total geodesic curvature of two positively oriented circles, by Theorem 3.2. This together with $\lim_{\rho\searrow 0} \int_{\Gamma(\rho)} \kappa(\rho)\ d\mathcal{H}^1 = -2\pi$ (where $\Gamma(\rho)$ is now oriented with respect to $G_{\rho,R}$) gives Corollary 3.5. □

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