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LIPSCHITZ REGULAR COMPLEX ALGEBRAIC SETS ARE SMOOTH

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ABSTRACT. A classical theorem of Mumford implies that a topologically regular complex algebraic surface in $\mathbb{C}^3$ with an isolated singular point is smooth. We prove that any Lipschitz regular complex algebraic set is smooth. No restriction on the dimension and no restriction on the singularity to be isolated is needed.

1. INTRODUCTION

A classical theorem proved by David Mumford in 1961 [4] is stated as follows: *a normal complex algebraic surface with a simply connected link at a point x is smooth in x*. This was a pioneer work in the topology of singular algebraic surfaces. An important sequence of works in this area and some applications to 3-folds topology were deeply stimulated by this theorem.

We say that a closed subset $X \subset \mathbb{R}^n$ is *topologically regular at $x_0 \in X$* if there exists a homeomorphism $\phi : B \to U$ where B is a Euclidean open ball and U is an open neighborhood of x_0 in X. Then, we say that X is *topologically regular* if X is topologically regular at any $x \in X$.

From a modern viewpoint this result can be seen as follows: *a topologically regular normal complex algebraic surface is smooth*. Since a surface in $\mathbb{C}^3$ is normal if and only if it has isolated singularities, the result can be reformulated as follows: *a topologically regular complex surface in $\mathbb{C}^3$ with isolated singularity is smooth*. The condition on the singularity to be isolated is important because there are examples of non-smooth and topologically regular surfaces, with non-isolated singularities.

There are also examples of non-smooth surfaces in $\mathbb{C}^4$ (see Section 5 of [6]) with topologically regular isolated singularity.

Here we study Lipschitz regular complex algebraic sets. We say that a closed subset $X \subset \mathbb{R}^n$ is *Lipschitz regular at $x_0 \in X$* if there exists a subanalytic bi-Lipschitz homeomorphism $\phi : B \to U$ where B is a Euclidean open ball and U is an open neighborhood of x_0 in X. Then, we say that X is *Lipschitz regular* if X is Lipschitz regular at any $x \in X$. The main result of this paper is that Lipschitz regular points of a complex algebraic set are smooth points, i.e. the conclusion

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of the Mumford Theorem holds without any restrictions on the singularities and dimension.

The proof mixes arguments from real algebraic geometry and from complex algebraic geometry. There are two "real" arguments. The first one is the theory of normally embedded sets, introduced by Birbrair and Mostowski [3]. A subanalytic set is called normally embedded if the two natural metrics—the inner metric and the euclidean one—are bi-Lipschitz equivalent. In particular, a subanalytic Lipschitz submanifold must be normally embedded. The second one is the theory of "derivatives" of Lipschitz subanalytic maps, created by Bernig and Lytchak [1] (see also [2]). The derivatives are defined as subanalytic maps between the corresponding metric tangent cones. The theory of Bernig and Lytchak helps us to show that the tangent cone of our set is also a subanalytic Lipschitz manifold and thus a topological manifold. From this point our arguments are "complex". First, we use Prill's Theorem [5]. He proved that algebraic and topologically regular complex cones are smooth. That is why our tangent cone is a complex subspace. The final argument in Proposition 3.4 is also complex; we use the fact that finite holomorphic maps are ramified covers. Combining this with the theory of normal embeddings, we conclude the proof.

2. Normally embedded sets

Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$. A map $f : X \to Y$ is called *Lipschitz* if there exists $k > 0$ such that

$$\|f(x_1) - f(x_2)\| \leq k\|x_1 - x_2\|$$

for all $x_1, x_2 \in X$. A Lipschitz map $f : X \to Y$ is called *bi-Lipschitz* if its inverse map exists and is Lipschitz.

Given a closed and connected subanalytic subset $X \subset \mathbb{R}^m$, the *inner metric* on X is defined as follows: given two points $x_1, x_2 \in X$, $d_X(x_1, x_2)$ is the infimum of the lengths of rectifiable paths on X connecting x_1 to x_2. All the sets considered in this paper are supposed to be equipped with the Euclidean metric. When we consider the inner metric, then we will emphasize it clearly. For instance, considering real or complex cusps $x^2 = y^3$, one can see that the inner metric is not necessarily bi-Lipschitz equivalent to the Euclidean metric on X. A subanalytic subset $X \subset \mathbb{R}^n$ is called *normally embedded* if there exists $\lambda > 0$ such that

$$d_X(x_1, x_2) \leq \lambda\|x_1 - x_2\|$$

for all $x_1, x_2 \in X$.

For instance, we have the following lemma that we shall need in the main proof.

Lemma 2.1. *If a subanalytic subset X is Lipschitz regular at $x_0 \in X$, then there exists a neighborhood U of x_0 in X which is normally embedded.*

Proof of the lemma. let the subanalytic set X be Lipschitz regular at x_0. By definition, there exists an open neighborhood U of x_0 in X and a subanalytic bi-Lipschitz homeomorphism $\psi : B \to U$, where B is an open Euclidean ball. Since ψ induces a bi-Lipschitz homeomorphism between the metric spaces (B, d_B) (B equipped with its inner metric) and (U, d_U) (U equipped with its inner metric) (see [3]), we have positive real constants λ_1 and λ_2 such that

$$\lambda_1\|p-q\| \leq \|\psi(p)-\psi(q)\| \leq \lambda_2\|p-q\| \,\forall p,q \in B$$

and

$$\lambda_1 d_B(p,q) \leq d_U(\psi(p),\psi(q)) \leq \lambda_2 d_B(p,q) \,\forall p,q \in B.$$

On the other hand, Euclidean balls are normally embedded, or more precisely, $d_B(p,q) = |p-q|$ for all $p,q \in B$. By inequalities above, we have that

$$d_U(\psi(p),\psi(q)) \leq \frac{\lambda_2}{\lambda_1}\|\psi(p)-\psi(q)\| \,\forall\, p,q \in B.$$

In other words, U is normally embedded. □

3. Main result

Theorem 3.1. *Let $X \subset \mathbb{C}^n$ be a complex algebraic variety. If X is Lipschitz regular at $x_0 \in X$, then x_0 is a smooth point of X.*

Proof. Let $X \subset \mathbb{C}^n$ be a complex algebraic variety. We first use the result of Bernig and Lytchak [1] which immediately implies

Lemma 3.2. *If a subanalytic subset X is Lipschitz regular at $x_0 \in X$, then the tangent cone of X at x_0 is Lipschitz regular at x_0.*

Now, since we consider complex analytic varieties, the tangent cone at x_0 of an algebraic variety is a complex cone. Recall that an algebraic subset of $\mathbb{C}^n$ is called a *complex cone* if it is a union of one-dimensional complex linear subspaces of $\mathbb{C}^n$. The next result was proved by David Prill in [5].

Lemma 3.3 (Prill's Theorem [5]). *Let $V \subset \mathbb{C}^n$ be a complex cone. If $0 \in V$ has a neighborhood homeomorphic to a Euclidean ball, then V is a linear subspace of $\mathbb{C}^n$.*

Then, our main theorem is a consequence of Prill's Theorem, Lemma 2.1, and the following.

Proposition 3.4. *Let $X \subset \mathbb{C}^n$ be an algebraic subset. Let $x_0 \in X$ be such that the reduced tangent cone $T_{x_0}X := |C_{x_0}X|$ is a linear subspace of $\mathbb{C}^n$. If there exists a neighborhood U of x_o in X such that U is normally embedded, then x_0 is a smooth point of X.*

Proof. Since $T_{x_0}X$ is a linear subspace of $\mathbb{C}^n$, we can consider the orthogonal projection

$$P\colon \mathbb{C}^n \to T_{x_0}X.$$

We may suppose that $x_0 = 0$ and $P(x_0) = 0$.

Notice that the germ of the restriction of the orthogonal projection P to X at x_0 has the following properties:

(1) It is a finite complex analytic map germ.
(2) If $\gamma : [0,\varepsilon) \to X$ is a real analytic arc, such that $\gamma(0) = 0$, then the arcs γ and $P \circ \gamma$ are tangent at 0.

Then the germ at 0 of $P_{|X}\colon X \to T_0X$ is a ramified cover and the ramification locus is the germ of a codimension ≥ 1 complex analytic subset Σ of the linear space T_0X.

The multiplicity of X at 0 can be interpreted as the degree d of this germ of ramified covering map, i.e. there are open neighborhoods U_1 of 0 in X and U_2 of 0 in T_0X, such that d is the degree of the topological covering

$$P_{|X}\colon X\cap U_1\setminus P_{|X}^{-1}(\Sigma)\to T_0X\cap U_2\setminus\Sigma.$$

Let us suppose that the degree d is greater than 1. Since Σ is a complex analytic subset of codimension ≥ 1, the space T_0X, there exists a unit tangent vector $v_0\in T_0X\setminus C_0\Sigma$, where $C_0\Sigma$ is the tangent cone of Σ at 0.

Since v_0 is not tangent to Σ at 0, there exists a positive real number k such that the real cone

$$\{v\in T_0X \mid \|v-tv_0\|<tk\ \forall\ 0<t<1\}$$

does not intersect the set Σ. Since we have assumed that the degree $d\geq 2$, we have at least two different liftings, $\gamma_1(t)$ and $\gamma_2(t)$, of the half-line $r(t)=tv_0$, i.e. $P(\gamma_1(t))=P(\gamma_2(t))=tv_0$ and they satisfy $\|\gamma_i(t)-0\|=t,\ i=1,2$. Since P is the orthogonal projection on the reduced tangent cone T_0X, the vector v_0 is the unit tangent vector to the images $P\circ\gamma_1$ and $P\circ\gamma_2$ of the arcs γ_1 and γ_2 at 0. By construction, we have $\mathrm{dist}(\gamma_\mathrm{i}(\mathrm{t}),\mathrm{P}_{|\mathrm{X}}^{-1}(\Sigma))\geq \mathrm{kt}$ $(i=1,2)$, where by dist we mean the Euclidean distance.

On the other hand, any path in X connecting $\gamma_1(t)$ to $\gamma_2(t)$ is a lifting of a loop, based at the point tv_0, which is not contractible in the germ of $T_0X\setminus\Sigma$ at 0. Thus the length of such a path must be at least $2kt$. It implies that the inner distance, $\mathrm{d_{inner}}(\gamma_1(\mathrm{t}),\gamma_2(\mathrm{t}))$, in X, between $\gamma_1(t)$ and $\gamma_2(t)$, is at least $2kt$. But, since $\gamma_1(t)$ and $\gamma_2(t)$ are tangent at 0, that is $\dfrac{\|\gamma_1(t)-\gamma_2(t)\|}{t}\to 0$ as $t\to 0^+$, and $k>0$, we obtain that X is not normally embedded near 0. Otherwise there will be $\lambda>0$ such that

$$d_X(x_1,x_2)\leq\lambda\|x_1-x_2\|\quad\text{for all}\quad x_1,x_2\in X\quad\text{near } 0,$$

hence

$$\begin{aligned}2k &\leq \frac{\mathrm{d_{inner}}(\gamma_1(\mathrm{t}),\gamma_2(\mathrm{t}))}{t}\\ &\leq \lambda\frac{\|\gamma_1(t)-\gamma_2(t)\|}{t}\to 0,\end{aligned}$$

which is a contradiction. □

Summary of the proof of Theorem 3.1. Let us suppose that the complex algebraic variety X is Lipschitz regular at x_0. Let $C_{x_0}X$ be the tangent cone of X at x_0. Let $h\colon U\to B$ be a subanalytic bi-Lipschitz homeomorphism between an open neighborhood U of x_0 in X and $B\subset\mathbb{R}^N$ be an open Euclidean ball centered at the origin $0\in\mathbb{R}^N$. Let us suppose that $h(x_0)=0$. Then the derivative of h, $dh\colon|C_{x_0}X|\to T_0B$, defined by Bernig and Lytchak [1] (see also [2]), is a bi-Lipschitz homeomorphism between the reduced tangent cones $|C_{x_0}X|$ and $T_0B=\mathbb{R}^N$. In particular, it proves that $|C_{x_0}X|$ is also Lipschitz regular at x_0. By Prill's Theorem (Lemma 3.3), we have that the reduced cone $|C_{x_0}X|$ is a linear subspace of $\mathbb{C}^n$. In order to finish the proof we just need to recall Lemma 2.1 and Proposition 3.4. □

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