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INJECTIVE MODULES UNDER FAITHFULLY FLAT RING EXTENSIONS

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Abstract. Let R be a commutative ring and let S be an R-algebra. It is well-known that if N is an injective R-module, then $\operatorname{Hom}_R(S,N)$ is an injective S-module. The converse is not true, not even if R is a commutative noetherian local ring and S is its completion, but it is close: It is a special case of our main theorem that, in this setting, an R-module N with $\operatorname{Ext}_R^{>0}(S,N)=0$ is injective if $\operatorname{Hom}_R(S,N)$ is an injective S-module.

Introduction

Faithfully flat ring extensions play an important role in commutative algebra: Any polynomial ring extension and any completion of a noetherian local ring is a faithfully flat extension. The topic of this paper is the transfer of homological properties of modules along such extensions.

In this section, R is a commutative ring and S is a commutative R-algebra. It is well-known that if F is a flat R-module, then $S \otimes_R F$ is a flat S-module, and the converse is true if S is faithfully flat over R. If I is an injective R-module, then $S \otimes_R I$ need not be injective over S, but it is standard that $\operatorname{Hom}_R(S,I)$ is an injective S-module. Here the converse is not true, not even if S is faithfully flat over R: Let $(R,\mathfrak{m})$ be a regular local ring with $\mathfrak{m}$-adic completion $S \neq R$. The module $\operatorname{Hom}_R(S,R)$ is then zero (see e.g. Aldrich, Enoch, and Lopez-Ramos [1]) and hence an injective S-module, but R is not an injective R-module, as the assumption $S \neq R$ ensures that R is not artinian. In this paper, we get close to a converse with the following result.

Main Theorem. *Let R be noetherian and let S be faithfully flat as an R-module; assume that every flat R-module has finite projective dimension. Let N be an R-module; if $\operatorname{Hom}_R(S,N)$ is an injective S-module and $\operatorname{Ext}_R^n(S,N)=0$ holds for all $n>0$, then N is injective.*

The result stated above follows from Theorem 1.7. The assumption of finite projective dimension of flat modules is satisfied by a wide selection of rings, including rings of finite Krull dimension and rings of cardinality at most $\aleph_n$ for some natural number n; see Gruson, Jensen et al. [8, Prop. 6], [10, Thm. II.(3.2.6)], and

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[7, Thm. 7.10]. The projective dimension of a direct sum of modules is the supremum of the projective dimensions of the summands. A direct sum of flat modules is flat, so the assumption implies that there is an upper bound d for the projective dimension of a flat module. Notice also that the condition of $\operatorname{Ext}_R^n(S,N)$ vanishing is finite in the sense that vanishing is trivial for n greater than the projective dimension of S.

The project we report on here is part of the second author's dissertation work. While the question that started the project—when does injectivity of $\operatorname{Hom}_R(S,N)$ imply injectivity of N?—is natural, it was a result of the first author and Sather-Wagstaff [5] that suggested that a non-trivial answer might be attainable. The main result in [5] is essentially the equivalent of our Main Theorem for the relative homological dimension known as *Gorenstein injective dimension*. That the result was obtained for the relative dimension before the absolute is already unusual; it is normally the absolute case that serves as a blueprint for the relative. In the end, our proof of the Main Theorem bears little resemblance with the arguments in [5], and we do not readily see how to employ our arguments in the setting of that paper.

1. Injective modules

In the balance of this paper, R is a commutative noetherian ring and S is a flat R-algebra. By an S-module we always mean a left S-module. For convenience, we recall a few basic facts that will be used throughout without further mention.

1.1. A tensor product of flat R-modules is a flat R-module. For every flat R-module F and every injective R-module I, the R-module $\operatorname{Hom}_R(F,I)$ is injective.

For every flat R-module F, the S-module $S\otimes_R F$ is flat, and every flat S-module is flat as an R-module. For every injective R-module I, the S-module $\operatorname{Hom}_R(S,I)$ is injective, and every injective S-module is injective as an R-module.

An R-module C is called *cotorsion* if one has $\operatorname{Ext}_R^1(F,C)=0$ (equivalently, $\operatorname{Ext}_R^{>0}(F,C)=0$) for every flat R-module F. It follows by Hom-tensor adjointness that $\operatorname{Hom}_R(F,C)$ is cotorsion whenever C is cotorsion and F is flat.

1.2. Under the sharpened assumption that S is faithfully flat, the exact sequence

$$0\longrightarrow R\longrightarrow S\longrightarrow S/R\longrightarrow 0 \tag{1.2.1}$$

is pure. Another way to say this is that (1.2.1) is an exact sequence of flat R-modules; see [9, Theorems (4.74) and (4.85)].

We work mostly in the derived category $\mathsf{D}(R)$ whose objects are complexes of R-modules. The next paragraph fixes the necessary terminology and notation.

1.3. Complexes are indexed homologically, so that the ith differential of a complex M is written $\partial_i^M : M_i\to M_{i-1}$. A complex M is called *bounded above* if $M_v=0$ holds for all $v\gg 0$, *bounded below* if $M_v=0$ holds for all $v\ll 0$, and *bounded* if it is bounded above and below. Brutal *truncations* of a complex M are denoted $M_{\leqslant n}$ and $M_{\geqslant n}$, and good truncations are denoted $M_{\subset n}$ and $M_{\supset n}$; cf. Weibel [11, 1.2.7].

A complex M is *acyclic* if one has $\mathrm{H}(M)=0$, equivalently $M\cong 0$ in $\mathsf{D}(R)$. Finally, $\mathbf{R}\operatorname{Hom}_R(-,-)$ denotes the right derived homomorphism functor, and $-\otimes_R^{\mathbf{L}}-$ denotes the left derived tensor product functor.

The proof of Theorem 1.7 passes through a couple of reductions; the first one is facilitated by the next lemma.

1.4. **Lemma.** *Let N be an R-module of finite injective dimension. If S is faithfully flat,* $\operatorname{Hom}_R(S,N)$ *is an injective R-module, and* $\operatorname{Ext}_R^n(S,N)=0$ *holds for all $n>0$, then N is injective.*

Proof. Let i be the injective dimension of N. There exists then an R-module T such that $\operatorname{Ext}_R^i(T,N)\neq 0$. Let E be an injective envelope of T. The exact sequence $0\to T\to E\to X\to 0$ induces an exact sequence of cohomology modules:

$$\cdots\longrightarrow \operatorname{Ext}_R^i(E,N)\longrightarrow \operatorname{Ext}_R^i(T,N)\longrightarrow \operatorname{Ext}_R^{i+1}(X,N)\longrightarrow\cdots.$$

Since $\operatorname{Ext}_R^{i+1}(X,N)=0$ while $\operatorname{Ext}_R^i(T,N)\neq 0$, we conclude that also $\operatorname{Ext}_R^i(E,N)$ is non-zero. Now apply the functor $-\otimes_R E$ to the pure exact sequence (1.2.1) to get the following exact sequence of R-modules:

$$0\longrightarrow E\longrightarrow S\otimes_R E\longrightarrow S/R\otimes_R E\longrightarrow 0\,.$$

As E is injective the sequence splits, whence E is a direct summand of the module $S\otimes_R E$. This implies $\operatorname{Ext}_R^i(S\otimes_R E,N)\neq 0$. On the other hand, for every $n>0$ one has

$$\begin{aligned}\operatorname{Ext}_R^n(S\otimes_R E,N) &\cong \operatorname{H}_{-n}(\mathbf{R}\operatorname{Hom}_R(S\otimes_R^{\mathbf{L}} E,N))\\ &\cong \operatorname{H}_{-n}(\mathbf{R}\operatorname{Hom}_R(E,\mathbf{R}\operatorname{Hom}_R(S,N)))\\ &\cong \operatorname{H}_{-n}(\mathbf{R}\operatorname{Hom}_R(E,\operatorname{Hom}_R(S,N)))\\ &\cong \operatorname{Ext}_R^n(E,\operatorname{Hom}_R(S,N))\,,\end{aligned}$$

where the first isomorphism uses that S is flat, the second is Hom-tensor adjointness in the derived category, and the third follows by the vanishing of $\operatorname{Ext}_R^{>0}(S,N)$. As $\operatorname{Hom}_R(S,N)$ is injective, this forces $i=0$; that is, N is injective. □

1.5. Let $\operatorname{Spec}R$ be the set of prime ideals in R; for $\mathfrak{p}\in\operatorname{Spec}R$ set $\kappa(\mathfrak{p})=R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. To an R-complex X one associates two subsets of $\operatorname{Spec}R$. The (small) *support*, as introduced by Foxby [6], is the set $\operatorname{supp}_R X=\{\mathfrak{p}\in\operatorname{Spec}R\mid \operatorname{H}(\kappa(\mathfrak{p})\otimes_R^{\mathbf{L}} X)\neq 0\}$, and the *cosupport*, as introduced by Benson, Iyengar, and Krause [4], is the set $\operatorname{cosupp}_R X=\{\mathfrak{p}\in\operatorname{Spec}R\mid \operatorname{H}(\mathbf{R}\operatorname{Hom}_R(\kappa(\mathfrak{p}),X))\neq 0\}$. A complex X is acyclic if and only if $\operatorname{supp}_R X$ is empty if and only if $\operatorname{cosupp}_R X$ is empty; see [6, (proof of) Lem. 2.6] and [4, Thm. 4.13]. The derived category $\mathsf{D}(R)$ is stratified by R in the sense of [3] (see 4.4 *ibid.*), so [4, Thm. 9.5] yields for R-complexes X and Y:

$$\operatorname{cosupp}_R\mathbf{R}\operatorname{Hom}_R(Y,X)=\operatorname{supp}_R Y\cap\operatorname{cosupp}_R X\,.$$

If S is faithfully flat over R, then, evidently, one has $\operatorname{supp}_R S=\operatorname{Spec}R$. In this case an R-complex X is acyclic if $\mathbf{R}\operatorname{Hom}_R(S,X)$ is acyclic.

1.6. **Lemma.** *Let I be an acyclic complex of injective R-modules. Assume that S is faithfully flat and of finite projective dimension over R. If* $\operatorname{Hom}_R(S,I)$ *is acyclic and* $\operatorname{Hom}_R(S,\operatorname{Ker}\partial_n^I)$ *is an injective R-module for every $n\in\mathbb{Z}$, then* $\operatorname{Hom}_R(M,I)$ *is acyclic for every R-module M.*

Proof. Let M be an R-module; in view of 1.5 it is sufficient to show that the complex $\mathbf{R}\operatorname{Hom}_R(S,\operatorname{Hom}_R(M,I))$ is acyclic. Set $d=\operatorname{pd}_R S$ and let $\pi\colon P\to S$ be a projective resolution with $P_i=0$ for all $i>d$. To see that the homology $\operatorname{H}(\mathbf{R}\operatorname{Hom}_R(S,\operatorname{Hom}_R(M,I)))\cong\operatorname{H}(\operatorname{Hom}_R(P,\operatorname{Hom}_R(M,I)))$ is zero, note first that there is an isomorphism

$$\operatorname{Hom}_R(P,\operatorname{Hom}_R(M,I))\;\cong\;\operatorname{Hom}_R(M,\operatorname{Hom}_R(P,I))\,.$$

Fix $m \in \mathbb{Z}$; the truncated complex $J = I_{\leqslant m+d+1}$ is a bounded above complex of injective R-modules, and so is $\operatorname{Hom}_R(P,J)$. It follows that the induced morphism $\operatorname{Hom}_R(\pi,J)$ is a homotopy equivalence; see [11, Lem. 10.4.6]. This explains the first isomorphism in the next display. The second isomorphism, like the equality, is immediate from the definition of Hom. The complex $H = \operatorname{Hom}_R(S, I_{\subset m+d+1})$ is acyclic, as $\operatorname{Hom}_R(S,I)$ is acyclic by assumption and $\operatorname{Hom}_R(S,-)$ is left exact. By assumption $\operatorname{Hom}_R(S, \operatorname{Ker}\partial^I_{m+d+1})$ is injective, so H is a complex of injective modules; it is also bounded above, so it splits. It follows that $\operatorname{Hom}_R(M,H)$ is acyclic.

$$\begin{aligned}
\mathrm{H}_m(\operatorname{Hom}_R(M,\operatorname{Hom}_R(P,I))) &= \mathrm{H}_m(\operatorname{Hom}_R(M,\operatorname{Hom}_R(P,I_{\leqslant m+d+1})))\\
&\cong \mathrm{H}_m(\operatorname{Hom}_R(M,\operatorname{Hom}_R(S,I_{\leqslant m+d+1})))\\
&\cong \mathrm{H}_m(\operatorname{Hom}_R(M,\operatorname{Hom}_R(S,I_{\subset m+d+1})))\\
&= 0\,. \qquad \square
\end{aligned}$$

1.7. **Theorem.** *Let R be a commutative noetherian ring over which every flat module has finite projective dimension. Let N be an R-module and let S be a faithfully flat R-algebra; the following conditions are equivalent.*

(*i*) *N is injective.*

(*ii*) *$\operatorname{Hom}_R(S,N)$ is an injective R-module and $\operatorname{Ext}^n_R(S,N) = 0$ holds for all $n > 0$.*

(*iii*) *$\operatorname{Hom}_R(S,N)$ is an injective S-module and $\operatorname{Ext}^n_R(S,N) = 0$ holds for all $n > 0$.*

Proof. It is well-known that (*i*) implies (*iii*) implies (*ii*), so we need to show that (*i*) follows from (*ii*). Let $N \to E$ be an injective resolution; then $\operatorname{Hom}_R(S,E)$ is a complex of injective R-modules. By assumption $\mathrm{H}_n(\operatorname{Hom}_R(S,E))$ is zero for $n < 0$, so $\operatorname{Hom}_R(S,E)$ is an injective resolution of the module $\operatorname{Hom}_R(S,N)$, which is injective by assumption. It follows that the co-syzygies

$$\operatorname{Ker}\partial_n^{\operatorname{Hom}_R(S,E)} = \operatorname{Hom}_R(S,\operatorname{Ker}\partial^E_n)$$

are injective for all $n \leqslant 0$. As remarked in the Introduction, there is an upper bound d for the projective dimension of a flat R-module. Set $K = \operatorname{Ker}\partial^E_{-d}$; by Lemma 1.4 it is sufficient to show that K is injective. The complex $J = \Sigma^d(E_{\leqslant -d})$ is an injective resolution of K, so we need to show that $\operatorname{Ext}^1_R(M,K) = \mathrm{H}_{-1}(\operatorname{Hom}_R(M,J))$ is zero for every R-module M.

For every flat R-module F and all $i > 0$ one has $\operatorname{Ext}^i_R(F,K) \cong \operatorname{Ext}^{i+d}_R(F,N) = 0$ by dimension shifting; that is, K is cotorsion. For every $i > 0$ the i-fold tensor product $(S/R)^{\otimes i}$ is a flat R-module, and we set $(S/R)^{\otimes 0} = R$. Let η denote the pure exact sequence (1.2.1); splicing together the exact sequences of flat modules $\eta \otimes_R (S/R)^{\otimes i}$ for $i \geqslant 0$ one gets an acyclic complex

$$G = 0 \to R \to S \to S \otimes_R S/R \to S \otimes_R (S/R)^{\otimes 2} \to \cdots \to S \otimes_R (S/R)^{\otimes i} \to \cdots$$

concentrated in non-positive degrees. As K is cotorsion, the functor $\operatorname{Hom}_R(-,K)$ leaves each sequence $\eta \otimes_R (S/R)^{\otimes i}$ exact, so the complex $\operatorname{Hom}_R(G,K)$ is acyclic. For every $n > 0$, the R-module

$$\operatorname{Hom}_R(G,K)_n = \operatorname{Hom}_R(S \otimes_R (S/R)^{\otimes n-1}, K) \cong \operatorname{Hom}_R((S/R)^{\otimes n-1}, \operatorname{Hom}_R(S,K))$$

is injective; indeed, $\operatorname{Hom}_R(S,K)$ is injective and $(S/R)^{\otimes n-1}$ is flat. Moreover, one has $\operatorname{Hom}_R(G,K)_0 \cong K$, so the complexes $\operatorname{Hom}_R(G,K)_{\geqslant 1}$ and J splice together to yield an acyclic complex I of injective R-modules.

We argue that Lemma 1.6 applies to I. For $n < 0$ one has $\mathrm{H}_n(\mathrm{Hom}_R(S,I)) = \mathrm{H}_n(\mathrm{Hom}_R(S,J)) = \mathrm{H}_{n-d}(\mathrm{Hom}_R(S,E)) = 0$, and the module $\mathrm{Hom}_R(S,\mathrm{Ker}\,\partial^I_n) = \mathrm{Hom}_R(S,\mathrm{Ker}\,\partial^E_{n-d})$ is injective. For $n \geqslant 0$ one has

$$\mathrm{Ker}\,\partial^I_n = \mathrm{Hom}_R(\mathrm{Im}\,\partial^G_{-n}, K) = \mathrm{Hom}_R((S/R)^{\otimes n}, K)\ .$$

Since K is cotorsion and $(S/R)^{\otimes n}$ is flat, the module $\mathrm{Ker}\,\partial^I_n$ is cotorsion. The truncated complex $I_{\leqslant n+1}$ is an injective resolution of the module $\mathrm{Ker}\,\partial^I_{n+1}$, so for all $n \geqslant 0$ one has $\mathrm{H}_n(\mathrm{Hom}_R(S,I)) = \mathrm{Ext}^1_R(S,\mathrm{Ker}\,\partial^I_{n+1}) = 0$. Furthermore, the R-module $\mathrm{Hom}_R(S,\mathrm{Ker}\,\partial^I_n) \cong \mathrm{Hom}_R((S/R)^{\otimes n},\mathrm{Hom}_R(S,K))$ is injective.

Now it follows from Lemma 1.6 that $\mathrm{Hom}_R(M,I)$ is acyclic for every R-module M; in particular, one has $\mathrm{H}_{-1}(\mathrm{Hom}_R(M,J)) = \mathrm{H}_{-1}(\mathrm{Hom}_R(M,I)) = 0$. □

2. Injective dimension

To draw the immediate consequences of our theorem, we need some terminology.

2.1. An R-complex I is *semi-injective* if it is a complex of injective R-modules and the functor $\mathrm{Hom}_R(-,I)$ preserves acyclicity. A *semi-injective resolution* of an R-complex N is a semi-injective complex I that is isomorphic to N in $\mathsf{D}(R)$. If N is a module, then an injective resolution of N is a semi-injective resolution in this sense. The *injective dimension* of an R-complex N is denoted $\mathrm{id}_R\,N$ and defined as

$$\mathrm{id}_R\,N = \inf\left\{ i \in \mathbb{Z} \;\middle|\; \begin{array}{c} \text{there is a semi-injective resolution} \\ I \text{ of } N \text{ with } I_n = 0 \text{ for all } n < -i \end{array} \right\}\ ;$$

see [2, 2.4.I], where "DG-injective" is the same as "semi-injective".

2.2. **Theorem.** *Let R be a commutative noetherian ring over which every flat module has finite projective dimension, and let S be a flat R-algebra. For every R-complex N there are inequalities*

$$\mathrm{id}_R\,N \geqslant \mathrm{id}_S\,\mathbf{R}\mathrm{Hom}_R(S,N) \geqslant \mathrm{id}_R\,\mathbf{R}\mathrm{Hom}_R(S,N)\ ,$$

and equalities hold if S is faithfully flat.

Proof. Let N be an R-complex and let I be a semi-injective resolution of N. In $\mathsf{D}(S)$ there is an isomorphism $\mathbf{R}\mathrm{Hom}_R(S,N) \cong \mathrm{Hom}_R(S,I)$. It follows by Hom-tensor adjointness that $\mathrm{Hom}_R(S,I)$ is a semi-injective S-complex, whence the left-hand inequality holds. As S is flat over R, Hom-tensor adjointness also shows that every semi-injective S-complex is semi-injective over R. In particular, any semi-injective resolution of $\mathbf{R}\mathrm{Hom}_R(S,N)$ over S is a semi-injective resolution over R, and the second inequality follows.

Assume now that S is faithfully flat and that $\mathrm{id}_R\,\mathbf{R}\mathrm{Hom}_R(S,N) \leqslant i$ holds for some integer i. Let I be a semi-injective resolution of N; our first step is to prove that the R-module $K = \mathrm{Ker}\,\partial^I_{-i}$ is injective. As $\mathrm{Hom}_R(S,-)$ is left exact one has

$$\mathrm{Ker}\,\partial^{\mathrm{Hom}_R(S,I)}_{-i} \;\cong\; \mathrm{Hom}_R(S,K)\ .$$

In $\mathsf{D}(R)$ there is an isomorphism $\mathrm{Hom}_R(S,I) \cong \mathbf{R}\mathrm{Hom}_R(S,N)$, and by previous arguments the R-complex $\mathrm{Hom}_R(S,I)$ is semi-injective. It now follows from [2, 2.4.I] that the R-module $\mathrm{Hom}_R(S,K)$ is injective, and the truncated complex

$\operatorname{Hom}_R(S,I)_{\supset -i} = \operatorname{Hom}_R(S,I_{\supset -i})$ is isomorphic to $\mathbf{R}\operatorname{Hom}_R(S,N)$ in $\mathsf{D}(R)$. In particular, one has

$$\operatorname{Ext}_R^n(S,K) = \mathrm{H}_{-n}(\operatorname{Hom}_R(S,\Sigma^i(I_{\leqslant -i}))) = \mathrm{H}_{-i-n}(\mathbf{R}\operatorname{Hom}_R(S,N)) = 0$$

for all $n > 0$, so K is injective by Theorem 1.7.

To conclude that N has injective dimension at most i, it is now sufficient to show that $\mathrm{H}_n(N) = 0$ holds for all $n < -i$; see [2, 2.4.I]. Let X be the cokernel of the embedding $\iota\colon I_{\supset -i} \to I$; the sequence $0 \to I_{\supset -i} \to I \to X \to 0$ is a degree-wise split exact sequence of complexes of injective modules. In the induced exact sequence

$$0 \longrightarrow \operatorname{Hom}_R(S,I_{\supset -d}) \longrightarrow \operatorname{Hom}_R(S,I) \longrightarrow \operatorname{Hom}_R(S,X) \longrightarrow 0\ ,$$

the embedding is a homology isomorphism, so $\operatorname{Hom}_R(S,X)$ is acyclic. As X is a bounded above complex of injective modules, it is semi-injective. That is, the complex $\mathbf{R}\operatorname{Hom}_R(S,X)$ is acyclic, and then it follows that X is acyclic; see 1.5. Thus ι is a quasi-isomorphism, whence one has $\mathrm{H}_n(N) = \mathrm{H}_n(I) = 0$ for all $n < -i$. $\square$

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