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ISOVARIANT HOMOTOPY EQUIVALENCES OF MANIFOLDS WITH GROUP ACTIONS

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ABSTRACT. Let f be an equivariant homotopy equivalence f of connected closed manifolds with smooth semifree actions of a finite group G, and assume also that f is isovariant. The main result states that f is a homotopy equivalence in the category of isovariant mappings if the manifolds satisfy a Codimension ≥ 3 Gap Hypothesis; this is done by showing directly that f satisfies the criteria in the Isovariant Whitehead Theorem of G. Dula and the author. Examples are given to show the need for the hypotheses in the main result.

If G is a topological group, then there are two standard notions of morphisms for topological spaces with continuous actions of G. A continuous mapping $f : X \to Y$ of such objects is said to be *equivariant* if $f(g \cdot x) = g \cdot f(x)$ for all $(g, x) \in G \times X$, and it is said to be *isovariant* if one has the identity $G_x = G_{f(x)}$ for isotropy subgroups (compare [2], p. 35, and [13], p. 6). For many purposes equivariant mappings provide an effective framework for studying group actions, but for various classification questions the stronger concept of isovariance is useful and sometimes indispensable (compare [13], [5], [16], [4], [19], [1], or [15]).

In algebraic and geometric topology, it is important to have relatively weak criteria for recognizing when a mapping of topological spaces is a homotopy equivalence, and fundamental results of J. H. C. Whitehead show that, for a large class of well-behaved spaces, a continuous mapping of arcwise connected spaces is a homotopy equivalence if and only if it induces isomorphisms of homotopy groups (see [20] or nearly any other standard reference for homotopy theory), and if the spaces are simply connected one can weaken the hypothesis further, replacing homotopy groups by homology groups. Similar results hold for equivariant homotopy equivalences of well-behaved spaces with well-behaved group actions (for example, see [10] or [12]), and in certain special cases one also has a version of the Whitehead Theorem in the category whose morphisms are isovariant mappings (see [8], pp. 34–37). The purpose of this paper is to give a weaker criterion for recognizing isovariant homotopy equivalences when the spaces in question are compact, connected, unbounded manifolds with smooth semifree actions of finite groups; recall that a compact Lie group action is said to be *semifree* if the only isotropy subgroups are the trivial subgroup and G itself. In order to state the main result we need the following hypothesis on the dimension of the fixed point set.

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Codimension ≥ 3 Gap Hypothesis (Specialized to semifree actions)**.** *If C is a component of the fixed point set M^G, then* $\dim(C) \leq \dim M - 3$.

Clearly one has an analogous hypothesis where 3 is replaced by some fixed positive integer k; in particular, the case $k = 2$ arises repeatedly in the subject.

Theorem 1. *Let n be a positive integer, and let $f : M \to N$ be an equivariant homotopy equivalence of connected, compact, unbounded* (= closed)*, and oriented smooth n-dimensional G-manifolds such that G acts semifreely, the Codimension $\geq$ 3 Gap Hypothesis is satisfied, and f is isovariant. Then f is a homotopy equivalence in the category of isovariant mappings.*

Notational convention. If a mapping is a homotopy equivalence in the category of isovariant mappings, we shall often say that it is an isovariant homotopy equivalence (which is stronger than saying that the mapping is an equivariant homotopy equivalence which happens to be isovariant).

The proof of the the main result relies on some basic properties of smooth manifolds, including Poincaré-Lefschetz Duality and standard implications of the Codimension ≥ 3 Gap Hypothesis for the fundamental group of the complement of the fixed point set. The following examples show that the conclusion of the main result fails if the hypotheses in the latter are not satisfied.

Example 2. The conclusions fail if the spaces in question are not manifolds but can be made into simplicial complexes with simplicial actions of a finite group.

Let $k \geq 2$ be an integer, let $n \geq 2$, and consider the linear action of $\mathbb{Z}_k$ on $\mathbb{C}^{2n}$ by scalar multiplication. This action maps the unit disk D^{2n} to itself, and one can triangulate the latter to make the action simplicial (the orbit space of the unit sphere is a smooth manifold, so we can construct the triangulation of D^{2n} from a smooth triangulation of S^{2n-1}/G). Let $Y = D^{2n} \vee D^{2n}$ where the two subspaces are joined at the origin (hence Y can be triangulated so that the associated group action is simplicial), and let $f : D^{2n} \to Y$ be inclusion into either the first or second disk. Then both D^{2n} and Y are equivariantly contractible, so f is automatically an equivariant homotopy equivalence, and f is isovariant because it is 1–1. However, f is not an isovariant homotopy equivalence. This follows because an isovariant homotopy equivalence induces a homotopy equivalence on the complements of the fixed point sets, and these complements are homotopically inequivalent for D^{2n} and Y because the complement for the first space is connected but the complement for the second has two connected components.

Example 3. The conclusions fail if the spaces in question are noncompact manifolds.

Let $k \geq 2$ be an integer, let $G = \mathbb{Z}_k$, and let $M = \mathbb{R}^p \oplus \mathbb{C}^q$, where $p \geq 0$, $q \geq 2$ and the action of G is trivial on the real coordinates and scalar multiplication on the complex coordinates. Consider the mapping $f : M \to M$ defined by

$$f(t_1, \cdots, t_p; z_1, \cdots, z_q) = \left(t_1, \cdots, t_p; z_1^{k+1}, \cdots, z_q^{k+1}\right) .$$

Once again, this mapping is an equivariant homotopy equivalence because M is equivariantly contractible, and one can check directly that it is also isovariant. However, f is not an isovariant homotopy equivalence; if it were, then f would induce a homotopy equivalence on $M - M^G$, which is homotopy equivalent to S^{2q-1}. Since one can use the construction to check that the degree of the induced mapping $S^{2q-1} \simeq M - M^G \to M - M^G \simeq S^{2q-1}$ is equal to $(k+1)^q$, it is clear that f does not induce a homotopy equivalence on $M - M^G$, and therefore f is not an isovariant homotopy equivalence.

Further remarks about the preceding example. One can check directly that the mapping f is proper, but it is not a homotopy equivalence in the category of proper mappings; this follows because the induced map of one point compactifications $f^\bullet : S^{p+2q} \to S^{p+2q}$ has degree $(k+1)^q$. It is natural to ask if one has the following analog of main result for proper mappings: *Suppose that $f : M \to N$ is a homotopy equivalence in the category of proper equivariant mappings, where M and N are noncompact, connected, unbounded semifree smooth G-manifolds (where G is finite). If f is isovariant and the manifolds satisfy the Codimension ≥ 3 Gap Hypothesis, is f a homotopy equivalence in the category of proper isovariant mappings?*

At least some of the methods in this paper should extend to the setting of the question, so there is some reason to expect the answer might be positive.

Example 4. The conclusion of the main result does not necessarily hold for actions on G-manifolds which satisfy a Codimension 2 Gap Hypothesis.

For each $n \geq 4$ consider the infinite family of smooth $\mathbb{Z}_m$-actions on S^n with knotted $(n-2)$-spheres as fixed point sets in [17]. Let V be the tangent space at a fixed point with the associated linear $\mathbb{Z}_m$-action. By Alexander Duality the complement of the fixed point set has the homology groups of S^1, and one can use the methods of Section 2 in [14] to construct a degree 1 isovariant map from one of these actions to the linear sphere $S(V \oplus \mathbb{R})$ such that the map of fixed point sets is a homeomorphism. This map is an equivariant homotopy equivalence, but it cannot be an isovariant homotopy equivalence because the complement of the fixed point set in the linear action is homotopy equivalent to S^1 but the complements of the fixed point sets in the exotic actions are not (specifically, by Section II of [17] the Alexander polynomials of the examples are nontrivial).

Generalizations to nonsemifree actions. As in [8], two reasons for restricting attention to semifree actions are (i) if $G = \mathbb{Z}_p$ where p is prime, then every action of G is semifree, (ii) in this case it is fairly simple to reconstruct the group action from the submanifolds of points with a common orbit type. One can combine the methods of this paper and the inductive arguments as in [8] to prove an extension of the main theorem to group actions with treelike isotropy structure in the sense of [8] (see pp. 21–22 and 29–31). This class includes all actions of cyclic groups whose orders are a power of a prime. It seems likely that one can extend the approach of [8] to even more general actions by a more detailed analysis of the stratification data associated to a smooth group action (the approach in [7] seems particularly well-suited for this purpose), and if this is worked out, an extension of the main result to such group actions should follow fairly directly using the approach developed here. Even more generally, one can ask if similar results hold for continuous group actions with well-behaved homotopy stratifications as in [19].

Degree one maps of disk bundles. One fundamentally important step in the proof of the main result is the following observation.

Proposition 5. *Let M and N be closed connected smooth n-manifolds, let $f : M \to N$ be a homotopy equivalence, let $k \geq 3$ be an integer, and let α_M and α_N be vector bundles over M and N whose first Stiefel-Whitney classes satisfy $f^*(w_1(\alpha_N)) = w_1(\alpha_M)$. Suppose further that we have a map of the associated disk and sphere bundles*

$$F : (D(\alpha_M), S(\alpha_M)) \longrightarrow (D(\alpha_N), S(\alpha_N)),$$

covering f (for each $x \in M$, the map F sends the fiber over x to the fiber over $f(x)$). If F has degree ± 1, then the map of boundaries $\partial F : S(\alpha_M) \to S(\alpha_N)$ is a fiber homotopy equivalence.

Proof. For the sake of clarity, we shall first consider the case where all the manifolds and vector bundles are orientable, and then we shall describe the modifications needed to treat the general case.

Let $p_0 \in M$, and let $q_0 = f(p_0)$. Since F is fiber preserving, it maps the $(disk, sphere)$ fiber pair in $(D(\alpha_M), S(\alpha_M))$ over p_0 to the corresponding fiber pair in $(D(\alpha_N), S(\alpha_N))$ over q_0. The map of fiber pairs corresponds to a self-map of (D^k, S^{k-1}); let d be its degree. Next, let $U_M \in H^k(D(\alpha_M), S(\alpha_M); \mathbb{Z}) \cong \mathbb{Z}$ and $U_N \in H^k(D(\alpha_N), S(\alpha_N); \mathbb{Z}) \cong \mathbb{Z}$ denote the Thom classes for orientations of α_M and α_N respectively. Since these classes restrict to generators in the fiber pair cohomology $H^k(D^k, S^{k-1}; \mathbb{Z}) \cong \mathbb{Z}$ and are the unique classes with this property up to sign, it follows that $F^*(U_N) = \pm d \cdot U_M$. Let Ω_M and Ω_N denote generating cohomology classes in $H^n(M; \mathbb{Z}) \cong H^n(D(\alpha_M); \mathbb{Z})$ and $H^n(N; \mathbb{Z}) \cong H^n(D(\alpha_N); \mathbb{Z})$ respectively. Since f is a homotopy equivalence we also have $f^*(\Omega_N) = \pm \Omega_N$ (but note that the sign need not be the same as in the preceding sentence!). The cup products $U_M \cdot \Omega_M$ and $U_N \cdot \Omega_N$ generate the groups $H^{n+k}(D(\alpha_M), S(\alpha_M); \mathbb{Z}) \cong \mathbb{Z}$ and $H^{n+k}(D(\alpha_M), S(\alpha_M); \mathbb{Z}) \cong \mathbb{Z}$ respectively, and therefore in cohomology we have

$$F^*(U_n \cdot \Omega_N) \ = \ F^*(U_N) \cdot F^*(\Omega_N) \ = \ \pm d \cdot U_M \cdot \Omega_M,$$

where d is given as above and the sign is again left undetermined. On the other hand, we also know that the degree of f is ± 1, and this means that d must be equal to ± 1. Since the induced map of boundaries ∂F is fiber preserving, it follows that ∂F maps the fiber over p_0 to the fiber over q_0 by a homotopy equivalence, and therefore by a topological Five Lemma for fibrations (*cf.* the proposition at the bottom of p. 80 in [9]), it follows that ∂F maps $S(\alpha_M)$ to $S(\alpha_N)$ by a fiber homotopy equivalence. This completes the proof when everything is orientable.

In the general case, we must work with cohomology groups with twisted coefficients instead of the ordinary cohomology groups $H^*(\cdots; \mathbb{Z})$; more precisely, we need to use local coefficients $\mathbb{Z}^t$ for suitable choices of twisting homomorphisms

$$t : \pi_1(M) \ \cong \ \pi_1(N) \ \longrightarrow \ \mathbb{Z}_2 \ \cong \ \{\pm 1\} \ \cong \ \mathrm{Aut}(\mathbb{Z}).$$

More precisely, we need to consider the twisting homomorphism u given by the first Stiefel-Whitney classes of M and N (which correspond under the isomorphism $\pi_1(M) \cong \pi_1(N)$ induced by the homotopy equivalence f) and the twisting homomorphism v given by the first Stiefel-Whitney classes of α_M and α_N (which also correspond by the hypotheses). It follows that the product homomorphism $u \cdot v$ determines the first Stiefel-Whitney classes of $D(\alpha_M)$ and $D(\alpha_N)$.

We shall now use the setting for Thom isomorphisms with twisted coefficients in Subsection IV.7.9 of [3] (see pp. 253–254 in particular). Specifically, we use this and the preceding paragraph to modify the argument in the orientable case as follows: If $X = M$ or N, then the Thom class generates $H^k(D(\alpha_X), S(\alpha_X); \mathbb{Z}^v) \cong \mathbb{Z}$, the top dimensional cohomology class generates $H^n(X; \mathbb{Z}^u) \cong \mathbb{Z}$, and it follows that their product $U_X \cdot \Omega_X$ generates the top dimensional cohomology group $H^{n+k}(D(\alpha_X), S(\alpha_X); \mathbb{Z}^{u\cdot v}) \cong \mathbb{Z}$. Since F and f define isomorphisms of fundamental groups which preserve u and v, it follows as before that $F^*(U_N) = \pm\, d\, U_M$ where d is the degree of the induced map on the fibers, and in this case the degree $\pm\, 1$ hypothesis implies that F maps $H^{n+k}(D(\alpha_N), S(\alpha_N); \mathbb{Z}^{u\cdot v})$ isomorphically to $H^{n+k}(D(\alpha_N), S(\alpha_N); \mathbb{Z}^{u\cdot v})$. As in the argument for the orientable case, this implies that $d = \pm\, 1$ and F maps $S(\alpha_M)$ to $S(\alpha_N)$ by a fiber homotopy equivalence. $\square$

Some notational conventions. We shall use the following in the proof of the main result:

Let P be a closed smooth G-manifold, where G is a finite group. By local linearity of the action we know that the fixed point set P^G is a union of connected smooth submanifolds; denote these connected components by P_α. Suppose now that M and N are smooth semifree G-manifolds and $f : M \to N$ is an equivariant homotopy equivalence. Then the associated map f^G of fixed point sets defines a $1-1$ correspondence between the components of M^G and N^G, and if N_α denotes a component, we set M_α equal to $f^{-1}[N_\alpha] \cap M^G$, and take f_α to be the continuous map from M_α to N_α determined by f.

Proof of the main theorem. As in the preceding paragraph, denote the components of M^G and N^G by M_α and N_α respectively. Let $E_\alpha(M)$ and $E_\alpha(N)$ denote pairwise disjoint invariant closed tubular neighborhoods M_α and N_α, and set E_M and E_N equal to the unions of these closed tubular neighborhoods. Next, let $S_\alpha(M) = \partial E_\alpha(M)$ and $S_\alpha(N) = \partial E_\alpha(N)$ be the boundaries of the respective components, let S_M and S_N be the unions, and finally let C_M and C_N be the closures of the complements of E_M and E_N respectively. By construction S_M and S_N define G-invariant splittings of $M = E_M \cup C_M$ and $N = E_N \cup C_N$ respectively.

We claim that the pairs (M, C_M) and (N, C_N) are 2-connected. This follows because (1) C_M and C_N are deformation retracts of $M - M^G$ and $N - N^G$ respectively, (2) the pairs $(M, M - M^G)$ and $(N, N - N^G)$ are 2-connected because $\dim M - \dim M^G = \dim N - \dim N^G \geq 3$.

Let M' and N' denote the universal coverings of M and N respectively, and let $f' : M' \to N'$ be a lifting of the equivariant homotopy equivalence f. Furthermore, let E'_M, C'_M, and S'_M be the inverse images of E_M, C_M, and S_M with respect to the universal covering map $M' \to M$, and define E'_N, C'_N, and S'_N in terms of the submanifolds E_N, C_N, and S_N and the universal covering map $N' \to N$. Finally, let Λ denote the group ring $\mathbb{Z}[\pi_1(M)] \cong \mathbb{Z}[\pi_1(N)]$.

By Proposition 3.6 on p. 22 of [8], the isovariant map f is isovariantly homotopic to a map of triads

$$(M; E_M, C_M) \to (N; E_N, C_N),$$

and therefore the induced map f' of universal coverings also splits into a map of triads

$$(M'; E'_M, C'_M) \;\longrightarrow\; (N'; E'_N, C'_N)\;.$$

Since the homotopy equivalence f has degree ± 1, the discussion of degree 1 maps in Chapter 2 of [18] applies, and accordingly the homology mappings

$$\begin{aligned} H_j(E'_M;\Lambda) &\longrightarrow H_j(E'_N;\Lambda),\\ H_j(S'_M;\Lambda) &\longrightarrow H_j(S'_N;\Lambda),\\ H_j(C'_M;\Lambda) &\longrightarrow H_j(C'_N;\Lambda) \end{aligned}$$

are split surjections. Let $K_j(E'_M)$, $K_j(S'_M)$, and $K_j(C'_M)$ denote their respective kernels. Since f is a homotopy equivalence, we can use the reasoning at the top of p. 93 in [6] to conclude that

$$K_j(S'_M) \;\cong\; K_j(E'_M) \;\oplus\; K_j(C'_M)\;.$$

By construction we know that the map of pairs $(E_M, S_M) \to (E_N, S_N)$ splits into maps of connected components $h_\alpha : (E_\alpha(M), S_\alpha(M)) \to (E_\alpha(N), S_\alpha(N))$, and since f is an equivariant homotopy equivalence, it follows that the underlying maps of spaces $h'_\alpha : E_\alpha(M) \to E_\alpha(N)$ are homotopy equivalences.

We now claim that the induced mappings $\partial h_\alpha : S_\alpha(M) \to S_\alpha(N)$ are also homotopy equivalences. As noted in [8] (see lines -11 to -8 on p. 31), the map h_α is equivariantly homotopic, as a map of pairs, to a map of pairs Φ_α which preserves fiber pairs (this is called *normally straightened* in [8]). We claim that the condition on Stiefel-Whitney classes in Proposition 5 is satisfied. Since f_α is a homotopy equivalence, we know that the induced map f^*_α on $H^1(\cdots;\mathbb{Z}_2)$ sends the first Stiefel-Whitney class $a_\alpha(N) \in H^1(N_\alpha;\mathbb{Z}_2)$ to the first Stiefel-Whitney class $a_\alpha(M) \in H^1(N_\alpha;\mathbb{Z}_2)$. Furthermore, since f is a homotopy equivalence which maps $E_\alpha(M)$ to $E_\alpha(N)$, it follows that the analogous cohomology map $(h'_\alpha)^*$ sends the first Stiefel-Whitney class $b'_\alpha(N) \in H^1(E_\alpha(N);\mathbb{Z}_2)$ to the first Stiefel-Whitney class $b'_\alpha(M) \in H^1(E_\alpha(M);\mathbb{Z}_2)$. Let $b_\alpha(N) \in H^1(N_\alpha;\mathbb{Z}_2)$ and $b_\alpha(M) \in H^1(M_\alpha;\mathbb{Z}_2)$ correspond to $b'_\alpha(N)$ and $b'_\alpha(M)$ under the homotopy equivalences $E_\alpha(M) \simeq M_\alpha$ and $E_\alpha(N) \simeq N_\alpha$ given by vector bundle projections. If ξ_α and ω_α are the vector bundles whose total spaces are $E_\alpha(M)$ and $E_\alpha(N)$ respectively, then their first Stiefel-Whitney classes are given by $a_\alpha(M) + b_\alpha(M)$ and $a_\alpha(N) + b_\alpha(N)$. We have shown that the induced cohomology map f^*_α sends $a_\alpha(N)$ to $a_\alpha(M)$ and $b_\alpha(N)$ to $b_\alpha(M)$, and therefore it follows that this cohomology map sends $w_1(\omega_\alpha)$ to $w_1(\xi_\alpha)$.

The preceding discussion implies that Φ_α satisfies the conditions in Proposition 5, and therefore the latter implies that the induced map of boundaries $\partial\Phi_\alpha$ is a fiber homotopy equivalence. Since $\partial\Phi_\alpha$ is homotopic to ∂h_α, it follows that the latter is also a homotopy equivalence. This in turn implies that the kernel groups $K_j(S_\alpha(M)')$ vanish for all j, and therefore the direct sum decomposition of the preceding paragraph implies that $K_j(C'_M) = 0$ for all j.

Finally, by Corollary 4.12 on p. 37 of [8] we have reduced the proof of the main result to showing that the induced map $C_M \to C_N$ is a homotopy equivalence. At the beginning of the proof we noted that the pairs (M, C_M) and (N, C_N) are 2-connected, and since M and N are connected, the same is true for C_M and C_N. The next step is to verify that the map $C_M \to C_N$ induces an isomorphism of fundamental groups. To see this, consider the following commutative diagram:

$$\begin{tikzcd} \pi_1(C_M) \arrow[r] \arrow[d] & \pi_1(C_N) \arrow[d] \\ \pi_1(M) \arrow[r] & \pi_1(N) \end{tikzcd}$$

The vertical morphisms are bijective by the 2-connectivity condition, and the bottom morphism is bijective since $M \to N$ is a homotopy equivalence, so the top diagram must also be an isomorphism by a diagram chase. These isomorphisms of fundamental groups imply that C'_M and C'_N are the universal covering spaces of C_M and C_N respectively, so that the map $C'_M \to C'_N$ is a lifting of $C_M \to C_N$ to universal coverings. By the the final sentence in the preceding paragraph we know that the map $C'_M \to C'_N$ induces isomorphisms in homology, and therefore it follows that $C_M \to C_N$ is a homotopy equivalence, which is what we needed to show in order to complete the proof that f is a homotopy equivalence in the category of isovariant mappings. □

References

[1] Aaron Christian Beshears, *G-isovariant structure sets and stratified structure sets*, ProQuest LLC, Ann Arbor, MI, 1997. Thesis (Ph.D.)–Vanderbilt University. MR2695560

[2] Glen E. Bredon, *Introduction to compact transformation groups*, Academic Press, New York-London, 1972. Pure and Applied Mathematics, Vol. 46. MR0413144 (54 #1265)

[3] Glen E. Bredon, *Sheaf theory*, 2nd ed., Graduate Texts in Mathematics, vol. 170, Springer-Verlag, New York, 1997. MR1481706 (98g:55005)

[4] W. Browder, *Isovariant homotopy equivalence*, Abstracts Amer. Math. Soc. **8** (1987), 237–238.

[5] William Browder and Frank Quinn, *A surgery theory for G-manifolds and stratified sets*, Manifolds—Tokyo 1973 (Proc. Internat. Conf., Tokyo, 1973), Univ. Tokyo Press, Tokyo, 1975, pp. 27–36. MR0375348 (51 #11543)

[6] Sylvain E. Cappell, *A splitting theorem for manifolds*, Invent. Math. **33** (1976), no. 2, 69–170. MR0438359 (55 #11274)

[7] Michael Davis, *Smooth G-manifolds as collections of fiber bundles*, Pacific J. Math. **77** (1978), no. 2, 315–363. MR510928 (80b:57034)

[8] Giora Dula and Reinhard Schultz, *Diagram cohomology and isovariant homotopy theory*, Mem. Amer. Math. Soc. **110** (1994), no. 527, viii+82, DOI 10.1090/memo/0527. MR1209409 (95a:55028)

[9] André Haefliger and Valentin Poenaru, *La classification des immersions combinatoires* (French), Inst. Hautes Études Sci. Publ. Math. **23** (1964), 75–91. MR0172296 (30 #2515)

[10] Soren Arnold Illman, *Equivariant algebraic topology*, ProQuest LLC, Ann Arbor, MI, 1972. Thesis (Ph.D.)–Princeton University. MR2622205

[11] Sören Illman, *Smooth equivariant triangulations of G-manifolds for G a finite group*, Math. Ann. **233** (1978), no. 3, 199–220. MR0500993 (58 #18474)

[12] Takao Matumoto, *On G-CW complexes and a theorem of J. H. C. Whitehead*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **18** (1971), 363–374. MR0345103 (49 #9842)

[13] Richard S. Palais, *The classification of G-spaces*, Mem. Amer. Math. Soc. No. 36, 1960. MR0177401 (31 #1664)

[14] Reinhard Schultz, *Differentiable group actions on homotopy spheres. I. Differential structure and the knot invariant*, Invent. Math. **31** (1975), no. 2, 105–128. MR0405471 (53 #9264)

[15] Reinhard Schultz, *Isovariant homotopy theory and differentiable group actions*, Algebra and topology 1992 (Taejŏn), Korea Adv. Inst. Sci. Tech., Taejŏn, 1992, pp. 81–148. MR1212981 (94i:57056)

[16] Sandor Howard Straus, *Equivariant codimension one surgery*, ProQuest LLC, Ann Arbor, MI, 1972. Thesis (Ph.D.)–University of California, Berkeley. MR2940154

[17] D. W. Sumners, *Smooth Z_p-actions on spheres which leave knots pointwise fixed*, Trans. Amer. Math. Soc. **205** (1975), 193–203. MR0372893 (51 #9097)

[18] C. T. C. Wall, *Surgery on compact manifolds*, 2nd ed., Mathematical Surveys and Monographs, vol. 69, American Mathematical Society, Providence, RI, 1999. Edited and with a foreword by A. A. Ranicki. MR1687388 (2000a:57089)

[19] Shmuel Weinberger, *The topological classification of stratified spaces*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1994. MR1308714 (96b:57024)

[20] George W. Whitehead, *Elements of homotopy theory*, Graduate Texts in Mathematics, vol. 61, Springer-Verlag, New York-Berlin, 1978. MR516508 (80b:55001)

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