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A BERNSTEIN TYPE THEOREM OF ANCIENT SOLUTIONS TO THE MEAN CURVATURE FLOW

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Abstract. We derive a curvature estimate for the entire graphic solution to the mean curvature flow. As a consequence we show a Bernstein type theorem for ancient solutions (or eternal solutions) to the mean curvature flow.

1. Introduction

In this paper we study ancient solutions to the mean curvature flow. The mean curvature flow is a natural and important geometric flow because this flow decreases the volume of the submanifold most rapidly. A particularly interesting topic in this theory is the study of the long time existence and the asymptotic behavior of the submanifold. However, during the flow, singularities often arise and the flow can not continue smoothly any more. Hence we have to deal with singularities.

Now we recall ancient solutions which are defined for all negative time $(-\infty, 0]$. They arise as tangent flows near singularities, and they are good asymptotic models of singularities (see, for example, [5], [6]). Therefore, in order to investigate singularities, it is effective to study ancient solutions.

On the other hand, since the mean curvature flow is a parabolic type evolution equation, we can not solve it backward. Hence ancient solutions are highly special solutions of the mean curvature flow. Furthermore, the class of ancient solutions includes all self-shrinkers and translating solitons which are submanifolds having an interesting property, namely, the self-similarity under the flow.

Since the late 1980's many mathematicians have studied the geometry of self-shrinkers and translating solitons. However, the geometry of generic ancient solutions is less known so far. Therefore, we want to consider this problem, especially in the entire graphic case. Under certain curvature and slope conditions, we obtain a Bernstein type theorem of ancient solutions. Roughly speaking, our theorem says that there are no non-trivial ancient solutions with bounded slope. Our result gives some characterization of ancient solutions and also gives information on singularities of the mean curvature flow.

In the proof of our curvature estimate, we mainly use the technique by Souplet-Zhang [9] and Meng Wang [10] for harmonic map heat flow on non-compact complete manifolds. It turns out that their technique is also available for the entire graphic mean curvature flow.

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1.1. **Mean curvature flow.** Let M^n be an oriented Reimannian manifold. The mean curvature flow is a one-parameter family of smooth immersions $F : M^n \times [-T, 0] \to \mathbb{R}^{n+1}$, $T > 0$, which satisfies the following evolution equation:

$$\frac{\partial F}{\partial t}(p,t) = \vec{H}(p,t),\ p \in M^n, t \in [-T, 0], \tag{1.1}$$

where $\vec{H}(p,t) = \mathrm{trace}B$ is the mean curvature vector of $F(M^n, t) \subset \mathbb{R}^{n+1}$ and B is its second fundamental form. We denote by $M_t := F(M^n, t)$ a time slice of the mean curvature flow $\mathcal{M}_T := \{M_t\}_{t\in[-T,0]}$. Note that we may consider $F(p,t)$ as the position vector of $M_t \subset \mathbb{R}^{n+1}$.

1.2. **Main result.** Let M_t be a complete hypersurface in $\mathbb{R}^{n+1}$, ν be a choice of unit normal to M_t and $V \in \mathbb{R}^{n+1}$ be a constant unit vector. Define a function on M_t by

$$\omega = \langle \nu, V \rangle.$$

Note that $-1 \leq \omega \leq 1$. If a hypersurface is written as an entire graph on some hyperplane, then we can find V such that $\omega > 0$ everywhere on M_t. On the other hand, if there exists a constant vector $V \in \mathbb{R}^{n+1}$ and a positive constant $c > 0$ such that $\omega \geq c$ on M_t, then M_t can be written as an entire graph on the hyperplane which is orthogonal to V. This means that the image of the Gauss map of M_t lies in a closed subset in the upper half disk $B_\Lambda^{\mathbb{S}^n}$ of $\mathbb{S}^n$, where the radius $\Lambda < \pi/2$.

Let $u \in \mathbb{R}^{n+1}$ be a unit constant vector. A translating soliton is a hypersurface $M^n \subset \mathbb{R}^{n+1}$ which satisfies the equation $\vec{H} = u^\perp$. Using the property of the Gauss map, Bao-Shi [1] showed the Bernstein type theorem of translating solitons.

Theorem A (Bao-Shi [1], 2013)**.** *Let $M^n \subset \mathbb{R}^{n+1}$ be an n-dimensional complete translating soliton. If there exists a positive constant $c > 0$ such that $\omega \geq c$, then M^n must be a hyperplane.*

Here note that $v = 1/\omega$ is the coefficient of the volume form of a graph and therefore v represents the slope of the graph. The above theorem says that there are no non-trivial entire graphic translating solitons with bounded slope. Moreover, recently we generalized Theorem A to an arbitrary co-dimensional case with flat normal bundle (see [7]).

Our main result for ancient solutions is a time dependent version of this theorem. At first we show a curvature estimate for ancient solutions. Let $B_R(o)$ be a Euclidean closed ball of radius R centered at the origin $o \in \mathbb{R}^{n+1}$ and $B_{R,T}(o) := B_R(o) \times [-T, 0] \subset \mathbb{R}^{n+1} \times (-\infty, \infty)$ be a cylindrical domain in the space-time. Now we may consider $\mathcal{M}_T$ as the space-time domain

$$\{(F(p,t), t) | p \in M^n, t \in [-T, 0]\} \subset \mathbb{R}^{n+1} \times (-\infty, \infty).$$

If M_t can be written as an entire graph for each t, then $\mathcal{M}_T$ also can be written as a graph on $\mathbb{R}^n \times [-T, 0]$. Without loss of generality we may consider that $F(p_0, 0)$ is at the origin o in $\mathbb{R}^{n+1}$. Finally we define the space-time domain $D_{R,T}(o) := B_{R,T}(o) \cap \mathcal{M}_T$.

Theorem 1.1. *Let $F : M^n \times [-T, 0] \to \mathbb{R}^{n+1}$ be a solution to the mean curvature flow. Assume that there exist a positive constant c and a nonnegative constant C_H*

such that $\omega(p,t) \geq c$ and $|\vec{H}(p,t)| \leq C_H$ for any point in $\mathcal{M}_T$. Define the function $\varphi := 1 - \omega$. Then there exists a constant C which is independent of R and T such that

$$\sup_{D_{R/2,T/2}(o)} \frac{|B|}{b-\varphi} \leq C\Big(\frac{1}{R} + \frac{1}{\sqrt{R}} + \frac{1}{\sqrt{T}}\Big),$$

where b is a constant such that $\sup_{\mathcal{M}_T} \varphi \leq 1 - c < b < 1$.

As a consequence of Theorem 1.1 we obtain our main result.

Theorem 1.2. *Let $F : M^n \times (-\infty, 0] \to \mathbb{R}^{n+1}$ be an ancient solution to the mean curvature flow. If there exist a positive constant c and a nonnegative constant C_H such that $\omega(p,t) \geq c$ and $|\vec{H}(p,t)| \leq C_H$ for any point in $\mathcal{M}_\infty$, then M_t must be a hyperplane for any $t \in (-\infty, 0]$.*

Remark 1.3. Our theorem is a generalization of Theorem A. In fact translating solitons do not change their shape under the flow, and hence we only have to consider a time slice (a hypersurface at some time) which satisfies $\vec{H} = V^\perp$. Then we automatically have $|\vec{H}| \leq C_H$. Now our theorem reduces to a time independent version, namely, Theorem A.

Furthermore, since all self-shrinkers are also ancient solutions, our result covers the Bernstein type theorem for self-shrinkers, although the Bernstein type theorem for the self-shrinkers with polynomial volume growth was already known by Ecker-Huisken [2]. Later L. Wang [11] removed the growth condition.

Remark 1.4. Eternal solutions to the mean curvature flow are solutions which are defined for all time. Translating solitons are fundamental examples of eternal solutions. Our theorem is also valid for eternal solutions to the mean curvature flow if we take the time interval as $[-T, T]$ in the proof of the curvature estimate.

In our setting, the entire graphic ancient solutions are automatically eternal solutions by the result of Ecker-Huisken [2].

2. Preliminaries

Our setting follows [4]. Let $\{x^i\}$ be a local coordinate on M^n. We write $\partial_i := \partial/\partial x^i$ and $\partial_t := \partial/\partial t$ for short. Now the metric tensor and the scalar second fundamental form can be defined as

$$g_{ij} = \langle F_i, F_j \rangle,$$
$$h_{ij} = -\langle F_{ij}, \nu \rangle,$$

where $F_i := \partial_i F$ and $F_{ij} := \partial_j \partial_i F$. We define the scalar mean curvature $H := g^{ij} h_{ij}$. Here we use Einstein's summation convention. Note that the relation $B_{ij} := B(\partial_i, \partial_j) = -h_{ij}\nu$ holds. Now the mean curvature vector can be written as

$$\vec{H} = -H\nu.$$

For any smooth hypersurfaces in $\mathbb{R}^{n+1}$, the following Gauss and Weingarten equations hold:

$$F_{ij} = \Gamma_{ij}^k F_k - h_{ij}\nu, \tag{2.1}$$

$$\nu_j = h_j^k F_k, \tag{2.2}$$

where $\nu_j := \partial_j \nu$.

Our computation relies on the following well-known evolution equations of important quantities for the mean curvature flow.

Lemma 2.1. *For the solutions to the mean curvature flow, we have*

$$\partial_t \nu = \nabla H, \tag{2.3}$$

$$\partial_t |B|^2 = \Delta |B|^2 - 2|\nabla B|^2 + 2|B|^4. \tag{2.4}$$

For the proof see [4]. We next compute the derivatives of the function ω which plays an important role in our proof of the main theorem.

Lemma 2.2. *For the solutions to the mean curvature flow, we have*

$$\nabla_j \omega = h_j^k \langle F_k, V\rangle, \tag{2.5}$$

$$\Delta \omega = \partial_t \omega - \omega |B|^2, \tag{2.6}$$

$$|\nabla \omega|^2 \le |B|^2. \tag{2.7}$$

Proof. We may compute by using a normal coordinate system at a point $p \in M^n$ for the first two equations. Using the Weingarten equation (2.2), we obtain

$$\nabla_j \omega = \partial_j \langle \nu, V\rangle = h_j^k \langle F_k, V\rangle.$$

Similarly we can compute the Laplacian of ω as

$$\begin{aligned}\Delta \omega &= g^{ij}\nabla_i \nabla_j \omega = g^{ij}\nabla_i \big(h_j^k \langle F_k, V\rangle\big)\\ &= g^{ij}\nabla_i h_j^k \langle F_k, V\rangle + g^{ij} h_j^k \langle F_{ik}, V\rangle\\ &= g^{ij}\nabla^k h_{ij}\langle F_k, V\rangle - g^{ij} h_j^k h_{ik}\langle \nu, V\rangle\\ &= \langle \nabla H, V\rangle - \omega |B|^2\\ &= \partial_t \omega - \omega |B|^2,\end{aligned}$$

where we use the Codazzi equation for the third equality, the Gauss equation (2.1) for the fourth equality, and (2.3) for the last equality.

In order to show (2.7), we use a local orthonormal frame $\{e_i\}$ around a point $p \in M^n$ such that at p,

$$g_{ij} = \delta_{ij}, \qquad h_{ij} = \kappa_i \delta_{ij},$$

where κ_i are principal curvatures of M^n. Now we obtain the following from (2.5):

$$|\nabla \omega|^2 \le \sum_{i=1}^n \kappa_i^2 = |B|^2.$$

□

3. Proof of the main theorem

In this section we prove Theorem 1.1. Our calculation is really similar to [10] which treats the harmonic map heat flow. However, the point of our proof is using the quantities $|B|^2$ and ω.

Let $F : M^n \times [-T, 0] \to \mathbb{R}^{n+1}$ be a solution to the mean curvature flow. First we assume that there exists a positive constant $c > 0$ such that $\omega \ge c$. Recall $\varphi = 1 - \omega$. Then since $\sup \varphi < 1$, there exists a constant b such that

$$\sup_{\mathcal{M}_T} \varphi \le 1 - c < b < 1.$$

Define the function on $\mathcal{M}_T$ by

$$\phi = \frac{|B|^2}{(b-\varphi)^2}.$$

A direct calculation shows that

$$\nabla\phi = \frac{\nabla|B|^2}{(b-\varphi)^2} + \frac{2|B|^2\nabla\varphi}{(b-\varphi)^3}. \tag{3.1}$$

Similarly we can compute

$$\Delta\phi = \frac{\Delta|B|^2}{(b-\varphi)^2} + \frac{4\langle\nabla\varphi, \nabla|B|^2\rangle}{(b-\varphi)^3} + \frac{2|B|^2\Delta\varphi}{(b-\varphi)^3} + \frac{6|\nabla\varphi|^2|B|^2}{(b-\varphi)^4}. \tag{3.2}$$

Substitute (2.4) and (2.6) into (3.2) to obtain

$$\begin{aligned}\Delta\phi =&\frac{2|\nabla B|^2 + \partial_t|B|^2 - 2|B|^4}{(b-\varphi)^2} + \frac{4\langle\nabla\varphi, \nabla|B|^2\rangle}{(b-\varphi)^3}\\ &+ \frac{2\omega|B|^4 - 2|B|^2\partial_t\omega}{(b-\varphi)^3} + \frac{6|\nabla\varphi|^2|B|^2}{(b-\varphi)^4}.\end{aligned}$$

On the other hand, the time derivative of ϕ is given by

$$\partial_t\phi = \frac{\partial_t|B|^2}{(b-\varphi)^2} - \frac{2|B|^2\partial_t\omega}{(b-\varphi)^3}. \tag{3.3}$$

We continue the calculation by using (3.3) as

$$\begin{aligned}\Delta\phi =&\frac{2|\nabla B|^2 - 2|B|^4}{(b-\varphi)^2} + \frac{4\langle\nabla\varphi, \nabla|B|^2\rangle}{(b-\varphi)^3}\\ &+ \frac{2\omega|B|^4}{(b-\varphi)^3} + \frac{6|\nabla\varphi|^2|B|^2}{(b-\varphi)^4} + \partial_t\phi.\end{aligned}$$

Note that the following relations hold:

$$\frac{2|\nabla B|^2}{(b-\varphi)^2} + \frac{2|\nabla\varphi|^2|B|^2}{(b-\varphi)^4} \geq \frac{4|\nabla B||\nabla\varphi||B|}{(b-\varphi)^3}, \tag{3.4}$$

$$\frac{2\langle\nabla|B|^2, \nabla\varphi\rangle}{(b-\varphi)^3} + \frac{4|B|^2|\nabla\varphi|^2}{(b-\varphi)^4} = \frac{2\langle\nabla\varphi, \nabla\phi\rangle}{(b-\varphi)}. \tag{3.5}$$

By using (3.4) and (3.5) with the Cauchy-Schwarz inequality and Kato's inequality, we finally obtain

$$\Delta\phi - \partial_t\phi \geq \frac{2(1-b)|B|^4}{(b-\varphi)^3} + \frac{2\langle\nabla\varphi, \nabla\phi\rangle}{b-\varphi}. \tag{3.6}$$

Now we consider a smooth function $\eta(r,t) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ supported on $[-R, R] \times [-T, 0]$ which has the following properties:

(1) $\eta(r,t) \equiv 1$ on $[-R/2, R/2] \times [-T/2, 0]$ and $0 \leq \psi \leq 1$.
(2) $\eta(r,t)$ is decreasing if $r \geq 0$, i.e., $\partial_r\eta \leq 0$.
(3) $|\partial_r\eta|/\eta^a \leq C_a/R$, $|\partial_r^2\eta|/\eta^a \leq C_a/R^2$ when $0 < a < 1$.
(4) $|\partial_t\eta|/\eta^a \leq C_a/T$ when $0 < a < 1$.

Such a function can be easily given in a canonical way (for example, see [8]). In fact, we construct η explicitly in the Appendix. Fix a point $(p_0, 0) \in M^n \times [-T, 0]$ and assume that $F(p_0, 0)$ is the origin o of $\mathbb{R}^{n+1}$. We use a cut-off function supported on $D_{R,T}(o)$ given by $\psi(F(p,t)) := \eta(r(F), t)$, where $r(F) := |F|$ is the distance function on $\mathbb{R}^{n+1}$.

Let $L := -2\nabla\varphi/(b-\varphi)$. By using (3.6), we can calculate

$$\begin{aligned}(3.7)\quad &\Delta(\psi\phi) + \langle L, \nabla(\psi\phi)\rangle - 2\Big\langle \frac{\nabla\psi}{\psi}, \nabla(\psi\phi)\Big\rangle - \partial_t(\psi\phi)\\ &= \psi(\Delta\phi - \partial_t\phi) + \phi(\Delta\psi - \partial_t\psi) + \langle \psi L, \nabla\phi\rangle + \langle \phi L, \nabla\psi\rangle - 2\frac{|\nabla\psi|^2}{\psi}\phi\\ &\geq 2\psi\frac{(1-b)|B|^4}{(b-\varphi)^3} + \phi(\Delta\psi - \partial_t\psi) - 2\frac{\langle\nabla\varphi, \nabla\psi\rangle}{b-\varphi}\phi - 2\frac{|\nabla\psi|^2}{\psi}\phi.\end{aligned}$$

Note that $D_{R,T}(o)$ is compact, since any time slice M_t can be written as an entire graph. Hence $\psi\phi$ attains its maximum at some point $F(p_1, t_1)$ in $D_{R,T}(o)$. At this point, we have

$$\nabla(\psi\phi) = 0, \quad \Delta(\psi\phi) \leq 0, \quad \partial_t(\psi\phi) \geq 0.$$

Hence by (3.7), we obtain

$$\begin{aligned}2\psi(1-b)\frac{|B|^4}{(b-\varphi)^3} &\leq 2\phi\frac{\langle\nabla\varphi, \nabla\psi\rangle}{b-\varphi} + 2\phi\frac{|\nabla\psi|^2}{\psi} + \phi(\partial_t\psi - \Delta\psi)\\ &= I + II + III.\end{aligned}$$

Note that the following holds:

$$|\nabla\psi|^2 = |\partial_r\eta|^2|\nabla r|^2 \leq n|\partial_r\eta|^2.$$

By using (2.7), Young's inequality and the property of η, we can estimate I as follows:

$$\begin{aligned}(3.8)\quad I &\leq 2\phi\frac{|\nabla\varphi|}{b-\varphi}|\nabla\psi| \leq 2\phi\frac{|B|}{b-\varphi}|\nabla\psi| = 2\phi^{\frac{3}{2}}|\nabla\psi|\\ &\leq \frac{\varepsilon}{4}\psi\phi^2 + \frac{C(\varepsilon)|\nabla\psi|^4}{\psi^3} \leq \frac{\varepsilon}{4}\psi\phi^2 + \frac{n^2C(\varepsilon)|\partial_r\eta|^4}{\psi^3}\\ &\leq \frac{\varepsilon}{4}\psi\phi^2 + \frac{C(\varepsilon, n)}{R^4},\end{aligned}$$

where $\varepsilon > 0$ is an arbitrary constant, $C(\varepsilon)$ and $C(\varepsilon, n)$ are constants depending only on ε and n. Similarly, we can calculate by using Young's inequality and the property of η,

$$(3.9)\quad II = 2\phi\frac{|\nabla\psi|^2}{\psi} \leq \frac{\varepsilon}{4}\psi\phi^2 + C(\varepsilon)\frac{|\nabla\psi|^4}{\psi^3} \leq \frac{\varepsilon}{4}\psi\phi^2 + \frac{C(\varepsilon, n)}{R^4}.$$

Now we assume $|\vec{H}(p,t)| \leq C_H$. Since $\partial_r\eta \leq 0$, we have

$$\Delta\psi = (\Delta r)(\partial_r\eta) + |\nabla r|^2(\partial_r^2\eta) \geq (C_H + \frac{n}{r})(\partial_r\eta) - n|\partial_r^2\eta|.$$

Hence we obtain for the second term of III in the same way as above,

$$\begin{aligned}(3.10)\quad -\phi\Delta\psi &\leq n\phi|\partial_r^2\eta| + \frac{n|\partial_r\eta|}{r}\phi + C_H|\partial_r\eta|\phi\\ &\leq n\phi|\partial_r^2\eta| + \frac{2n|\partial_r\eta|}{R}\phi + C_H|\partial_r\eta|\phi\\ &\leq \frac{\varepsilon}{4}\psi\phi^2 + C(\varepsilon, n)\Big(\frac{1}{R^4} + \frac{1}{R^2}\Big).\end{aligned}$$

(Note that we may assume $R/2 \leq r$ for the second inequality, since $\partial_r\eta \equiv 0$ for $r \leq R/2$.)

As for the first term of III, we have

$$\begin{aligned}\phi(\partial_t\psi) &\leq \phi C_H|\partial_r\eta| + \phi|\partial_t\eta| \leq \frac{\varepsilon}{4}\psi\phi^2 + C(\varepsilon, C_H)\Big(\frac{|\partial_r\eta|^2}{\psi} + \frac{|\partial_t\eta|^2}{\psi}\Big) \\ &\leq \frac{\varepsilon}{4}\psi\phi^2 + C(\varepsilon, C_H)\Big(\frac{1}{R^2} + \frac{1}{T^2}\Big).\end{aligned} \tag{3.11}$$

From (3.8), (3.9), (3.10), and (3.11), we finally obtain

$$2(1-b)(b-\varphi)\psi\phi^2 \leq \varepsilon\psi\phi^2 + C(\varepsilon, n, C_H)\big(\frac{1}{R^4} + \frac{1}{R^2} + \frac{1}{T^2}\big).$$

Since $\varepsilon > 0$ is arbitrary, we can take a sufficiently small ε such that

$$2(1-b)(b-\varphi) - \varepsilon > 0.$$

Then we have

$$(\psi\phi)^2 \leq \psi\phi^2 \leq C\Big(\frac{1}{R^4} + \frac{1}{R^2} + \frac{1}{T^2}\Big).$$

Since $\psi \equiv 1$ on $D_{R/2,T/2}(o)$,

$$\sup_{D_{R/2,T/2}(o)} \frac{|B|}{b-\varphi} \leq C\Big(\frac{1}{R} + \frac{1}{\sqrt{R}} + \frac{1}{\sqrt{T}}\Big).$$

This completes the proof of Theorem 1.1. Theorem 1.2 immediately follows taking R and T to ∞.

4. Appendix

In this appendix, we construct the cut-off function $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ used in the proof of the main theorem. Let $f : \mathbb{R} \to \mathbb{R}$ be

$$f(s) := \begin{cases} e^{-\frac{1}{s}} & (s > 0), \\ 0 & (s \leq 0). \end{cases}$$

Note that f is a C^∞ function. Set

$$g(s) := \frac{f(s)}{f(s) + f(1-s)}.$$

We can easily check that $g(s), g'(s), g''(s)$ are bounded. For $s \geq 0$, we consider the following function:

$$h(s) := g(2-s)^k, \quad \forall k > 2. \tag{4.1}$$

By calculating the first derivative of $h(s)$, we see that

$$\begin{aligned} h'(s) &= -kg(2-s)^{k-1}g'(2-s) \\ &= -kh(s)^{\frac{k-1}{k}}g'(2-s). \end{aligned}$$

Since $g'(2-s)$ is bounded, we obtain an estimate

$$\frac{|h'(s)|}{h(s)^{\frac{k-1}{k}}} \leq C_k, \tag{4.2}$$

which assures that the division by $h(s)^{\frac{k-1}{k}}$ is possible.

Next we calculate the second derivative of $h(s)$ as follows:

$$h''(s) = kg^{k-1}(2-s)g''(2-s) + \frac{k-1}{k}\frac{h'(s)^2}{h(s)}. \tag{4.3}$$

Divide (4.3) by $h(s)^{\frac{k-2}{k}}$ and obtain

$$\frac{h''(s)}{h(s)^{\frac{k-2}{k}}} = kg(2-s)g''(2-s) + \frac{k-1}{k}\Big(\frac{h'(s)}{h(s)^{\frac{k-1}{k}}}\Big)^2.$$

From (4.2) and the fact that $g(2-s), g''(2-s)$ are bounded, we get a bound

$$\frac{|h''(s)|}{h(s)^{\frac{k-2}{k}}} \leq \widetilde{C_k}.$$

Since $k > 2$ is arbitrary, we can take sufficiently large k in (4.1), and fix it. Then for any $0 < a < (k-2)/k$, we have

$$\frac{|h'(s)|}{h(s)^a} \leq C_a, \quad \frac{|h''(s)|}{h(s)^a} \leq C_a.$$

Finally we define

$$\widetilde{h}(r) := h\Big(\frac{2r}{R}\Big), \quad \overline{h}(t) := h\Big(\frac{2t}{T}\Big),$$

and then the desired cut-off function $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is given by

$$\eta(r,t) := \frac{\widetilde{h}(r) + \overline{h}(t)}{2}.$$

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