EXISTENCE AND UNIQUENESS OF GLOBAL CLASSICAL SOLUTIONS OF A GRADIENT FLOW OF THE LANDAU-DE GENNES ENERGY

XINFU CHEN AND XIANG XU

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Abstract. In this paper we establish the existence and uniqueness of global classical solutions to a gradient flow in $\mathbb{R}^d$, $d \geq 2$. This gradient flow is generated by the Landau-de Gennes energy functional that involves four elastic-constant terms describing nematic liquid crystal configurations in the space of $Q$-tensors. We work in Hölder spaces, and deal with $d = 2$ and $d \geq 3$ separately.

1. Introduction

In this paper we study the existence and uniqueness of global classical solutions to an evolution problem in $\mathbb{R}^d$ ($d \geq 2$) that arises from the Landau-de Gennes theory for nematic liquid crystals. The specific feature of this evolution problem is that it is generated by an energy functional that is unbounded from below.

The Landau-de Gennes energy functional is

$$E[Q] = \int_{\mathbb{R}^d} F(\nabla Q(x), Q(x)) \, dx,$$

where $Q$ is a matrix valued function that takes values in the $Q$-tensor space (cf. [2,3,12]) defined by

$$S^{(d)} \triangleq \left\{ (M^{ij})_{d \times d} \mid \sum_{i=1}^{d} M^{ii} = 0, M^{ij} = M^{ji} \in \mathbb{R}, \forall i, j = 1, \ldots, d \right\}.

The energy density $F$ is composed of an elastic part and a bulk part (see [2,12]):

$$F(\nabla Q, Q) \triangleq F_{\text{el}}(\nabla Q, Q) + F_{\text{bulk}}(Q).$$

We consider one of the simplest isotropic forms,

$$F_{\text{el}}(\nabla Q, Q) \triangleq L_1 Q^{ij}_{x_k} Q^{ij}_{x_k} + L_2 Q^{ik}_{x_j} Q^{jk}_{x_k} + L_3 Q^{ij}_{x_j} Q^{ik}_{x_k} + L_4 Q^{ik} Q^{ij}_{x_k} Q^{ij}_{x_k},

F_{\text{bulk}}(Q) \triangleq \frac{a}{2} \text{tr}(Q^2) + \frac{b}{3} \text{tr}(Q^3) + \frac{c}{4} \text{tr}^2(Q^2),$$

where $L_1, L_2, L_3, L_4, a, b, c$ are material dependent constants; see [10,13]. Here and after, $\text{tr}(A)$ denotes the trace of a matrix $A$ and $Q^{ij}_{x_k}$ the partial derivative $\partial Q^{ij}/\partial x_k$; also the Einstein summation convention is used, i.e., repeated indices are automatically summed from 1 to $d$. 

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The special feature of the above energy functional is that it contains an unusual cubic $L_4$ term, whose retention in our settings is due to the fact that it allows the reduction of the elastic energy $F_{el}(\nabla Q, Q)$ to the classical Oseen-Frank energy of nematic liquid crystals (see [3,8]). As mentioned in [4,8], the “unpleasant” feature of $\mathcal{E}[Q]$ is that it is unbounded from below due to the $L_4$ term and non-positivity of $Q \in S^{(d)}$.

In this paper, we study the global (in time) well-posedness of the $L^2$ gradient flow
\[
\frac{\partial Q^{ij}}{\partial t} = -\left(\frac{\delta \mathcal{E}}{\delta Q}\right)^{ij} + \lambda \delta^{ij} + \mu^{ij} - \mu^{ji}, \quad \forall i, j = 1, \ldots, d; \quad Q \in S^{(d)}.
\]

Here $\delta^{ij}$ stands for the Kronecker delta and $\delta \mathcal{E}/\delta Q$ represents the Frechet derivative of $\mathcal{E}$ with respect to $Q$. The restriction $Q(x, t) \in S^{(d)}$ for $x \in \mathbb{R}^d$ and $t \geq 0$ is obtained by the Lagrange multipliers, $\lambda$, for the constraint $\text{tr}(Q) = 0$, and $\mu = (\mu^{ij})_{d \times d}$, for the constraint $Q^{ij} = Q^{ji}$; that is,
\[
\lambda = \frac{1}{d} \text{tr} \left( \frac{\delta \mathcal{E}}{\delta Q} \right), \quad \mu = \frac{1}{2} \frac{\delta \mathcal{E}}{\delta Q}.
\]

In expanded form (see Appendix A in [8] for details), the gradient flow reads as
\[
Q^{ij}_t = 2L_1 \Delta Q^{ij} + L_4 \left\{ 2(Q^{ik}Q^{ij}_{x_k})_{x_k} - Q^{kl}Q^{lji} + d^{-1}|\nabla Q|^2 \delta^{ij} \right\} - [a + c \text{tr}(Q^2)] Q^{ij} + (L_2 + L_3) \left( Q^{ik}_{x_kx_j} + Q^{lk}_{x_kx_i} - 2d^{-1}Q^{lk}_{x_kx_i} \delta^{ij} \right) - b \{ Q^{ik}Q^{kj} - d^{-1} \text{tr}(Q^2) \delta^{ij} \}.
\]

We shall study the Cauchy problem to this system of equations in $\mathbb{R}^d \times (0, \infty)$ with initial condition
\[
Q(\cdot, 0) = Q_0(\cdot) \in S^{(d)} \quad \text{on} \quad \mathbb{R}^d \times \{0\}.
\]

This is a subsequent work of [8] where for $d = 2$ and the Dirichlet problem in a smooth bounded domain $\Omega$, the existence and uniqueness of global $(L^2_{loc}([0, \infty); H^2(\Omega)))$ weak solutions for small $L^\infty(\Omega)$ initial data with bounded $H^1(\Omega)$ norm is established; the analysis relies on a maximum principle (valid only for 2D) and energy estimates inherited from the gradient flow nature of the problem and Sobolev imbedding (which also seems to work only in 2D). In this paper, we first improve the result of [8] by establishing the existence of a unique global and classical ($C^\infty(\mathbb{R}^d \times (0, \infty))$) solution, in the 2D case, for any initial data with suitably small $L^\infty$ norm and bounded $C^{1+\alpha}(\mathbb{R}^2)$ norm; see Theorem 2.1 in Section 2. The analysis relies on the maximum principle for $\text{tr}(Q^2)$ discovered in [8], but does not use Sobolev space and imbedding and therefore does not depend on the nature of the gradient flow and does not require the integrability of $|\nabla Q|^2$. Then we handle the corresponding higher dimensional case. Compared to 2D, the essential difficulty in higher space dimension is that one cannot find an analogous maximum principle for $\text{tr}(Q^2)$, due to those terms related to $L_2$ and $L_3$, and Sobolev imbedding does not seem to work. To overcome this difficulty, we shall use H"older space and assume that $L_1$ is large, in comparison to the other parameters $L_2 + L_3, L_4, a, b, c$, and to the $C^{2+\alpha}(\mathbb{R}^d)$ initial data. We shall establish the existence of unique global and classical solutions; see Theorem 3.1 in Section 3.

It is worth mentioning that, with proper assumptions of boundary data on any smooth bounded domain, the results in Theorems 2.1 and 3.1 for the Cauchy problem can be easily adapted to the corresponding Dirichlet problem as discussed in [8].
Notational convention. Throughout this paper we denote the Frobenius norm of the matrix $Q$ by
\begin{equation}
\rho \overset{\text{def}}{=} \sqrt{\text{tr}(Q^2)},
\end{equation}
and the $d \times d$ identity matrix by $I$. For any smooth scalar function $u : \mathbb{R}^d \to \mathbb{R}$, we define the following norms and semi-norms:
\begin{align*}
[u]_{\alpha} &= \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^{\alpha}}, & 0 \leq \alpha \leq 1, \\
[u]_{1+\alpha} &= \max_{i} [u_{x_i}]_{\alpha}, & 0 < \alpha \leq 1, \\
[u]_{2+\alpha} &= \max_{i,j} [u_{x_i, x_j}]_{\alpha}, & 0 < \alpha < 1, \\
[u]_{0,T} &= \sup_{x \in \mathbb{R}^d} |u(x)|, \\
[u]_{0,T}^1 &= \sup_{0 \leq t \leq T} \|u(\cdot, t)\|_0, \\
[u]_{2+\alpha,T} &= \sup_{0 \leq t \leq T} \|u(\cdot, t)\|_{1+\alpha}, \\
[u]_{0,T}^2 &= \sup_{0 \leq t \leq T} \max_{i,j} \|u_{x_i, x_j}(\cdot, t)\|_0.
\end{align*}

For a tensor valued function $Q : \mathbb{R}^d \to \mathbb{R}^{d \times d}$, the corresponding norms are defined to be the maximum of each component, for instance, $[Q]_{\alpha} = \max_{i,j} [Q^{ij}]_{\alpha}$.

In the sequel, $C(A)$ denotes a positive constant that depends only on $A$; they may differ from one line to the next.

2. The 2D case

This section is devoted to the proof of the well-posedness result in the two space dimension case. In contrast to the work of [8] where the global existence is established in the framework of Sobolev spaces and therefore for weak solutions, here we work in the Hölder space and therefore for classical solutions.

We begin with the following a priori estimate for the heat equation, valid in any space dimension.

Lemma 2.1. Let $u$ be a classical solution to the Cauchy problem
\begin{equation*}
\begin{cases}
  u_t - \Delta u = f & \text{in } \mathbb{R}^d \times [0, \infty), \\
  u(\cdot, 0) = g(\cdot) & \text{on } \mathbb{R}^d \times \{0\}.
\end{cases}
\end{equation*}

For each $\alpha \in (0, 1)$, there exists a constant $C(\alpha, d)$ depending only on $\alpha$ and $d$ such that for any $T > 0$,
\begin{equation*}
[u]_{2+\alpha,T} \leq [g]_{2+\alpha} + C(\alpha, d)[f]_{\alpha,T}.
\end{equation*}

This is a standard result in potential theory, with the notable feature that $C(\alpha, d)$ does not depend on $T$. For the reader’s convenience, we provide the proof.

Proof. Observe that the solution $u$ can be represented by
\begin{equation*}
u = u_1 + u_2, \quad u_1 \overset{\text{def}}{=} \Gamma(t) * g, \quad u_2 \overset{\text{def}}{=} \int_0^t \Gamma(\tau) * f(t - \tau) \, d\tau, \quad \Gamma(z, t) \overset{\text{def}}{=} (2\pi t)^{-\frac{d}{2}} e^{-\frac{|z|^2}{2t}}.
\end{equation*}

Applying the maximum principle for $u_{1x, x_j}(\cdot, t) - u_{1x, x_j}(\cdot + h, t)$ with arbitrary $h \in \mathbb{R}^d$, one finds that
\begin{equation*}
[u_1]_{2+\alpha} \leq [g]_{2+\alpha} \quad \forall t \in [0, T].
\end{equation*}
Next, we write \( u_2 \) as \( u_2 = \int_0^t w(\tau) \, d\tau \) where \( w(\tau) = \Gamma(\tau) * f(t - \tau) \). Then

\[
w_{x_i x_j}(x, \tau) = \int_{\mathbb{R}^d} \Gamma_{x_i x_j} (x - y, \tau) [f(y, t - \tau) - f(x, t - \tau)] \, dy,
\]

which implies

\[
\begin{aligned}
(w_{x_i x_j}(x, \tau)) & \leq |f|_{\alpha,T} \int_{\mathbb{R}^d} |z|^\alpha |\Gamma_{z_i z_j}(z, \tau)| \, dz = C(\alpha, d) |f|_{\alpha,T} \frac{\alpha^2}{2}, \\
(w_{x_i x_j x_k}(x, \tau)) & \leq |f|_{\alpha,T} \int_{\mathbb{R}^d} |z|^\alpha |\Gamma_{z_i z_j z_k}(z, \tau)| \, dz = C(\alpha, d) |f|_{\alpha,T} \frac{\alpha^3}{2}.
\end{aligned}
\]

Consequently, for any \( x_1, x_2 \in \mathbb{R}^d \) and \( h > 0 \), denoting \( (t - h)^+ = \max\{t - h, 0\} \), we have

\[
\begin{aligned}
&|D^2 u_2(x_1, t) - D^2 u_2(x_2, t)| \\
&\quad \leq \int_0^{(t-h)^+} |x_1 - x_2| \|D^3 w(\cdot, t - \tau)\|_{L^\infty} \, d\tau + 2 \int_{(t-h)^+}^t \|D^2 w(\cdot, t - \tau)\|_{L^\infty} \, d\tau \\
&\quad \leq C(\alpha, d) |x_1 - x_2| |f|_{\alpha,T} \int_{-\infty}^{t-h} (t - \tau)^\frac{\alpha^2}{2} \, d\tau + C(\alpha, d) |f|_{\alpha,T} \int_{t-h}^t (t - \tau)^\frac{\alpha^2}{2} \, d\tau \\
&\quad \leq C(\alpha, d) |x_1 - x_2| \left[h^\frac{\alpha-1}{2} + h^\frac{\alpha}{2}\right] |f|_{\alpha,T}.
\end{aligned}
\]

Choosing \( h = |x_1 - x_2|^2 \), we then obtain the assertion of the lemma. \( \square \)

In the two dimensional case, the structure of \( Q \) implies that there are scalar functions \( p \) and \( q \) such that

\[
Q = \begin{pmatrix}
  p & q \\
  q & -p
\end{pmatrix}, \quad Q^2 = (p^2 + q^2) I = \frac{1}{2} \text{tr}(Q^2) I.
\]

Hence \( \text{tr}(Q^3) = 0 \). The 2D problem \((1.2)\) becomes

\[
(2.3)
\]

\[
Q_{ij}^t = L \Delta Q_{ij} + L_4 \left\{ 2(Q_{ik} Q_{kj})_{x_i} - Q_{x_i} Q_{x_j}^2 + \frac{1}{2} \nabla Q^2 \delta_{ij} \right\} - [a + c \text{tr}(Q^2)] Q_{ij},
\]

where

\[
(2.4)
\]

\[
L \overset{\text{def}}{=} 2L_1 + L_2 + L_3.
\]

The special feature of 2D is that the terms on the second line of \((1.2)\) “disappeared” in \((2.3)\). This allows us to apply maximum principle for \( \rho \) when initial data is suitably small.

**Lemma 2.2.** For \( d = 2 \), suppose the coefficients of \((2.3)\) satisfy

\[
L > 0, \quad c > 0, \quad -\frac{a}{c} < \frac{2L^2}{9L_4^2},
\]

where \( L \) is defined in \((2.4)\). Let \( Q \) be a classical solution of \((2.3)\) in \( \mathbb{R}^2 \times (0, T) \) with initial data \( Q(\cdot, 0) = Q_0 \in S^{(2)} \) satisfying

\[
\| \text{tr}(Q_0^2) \|_{L^\infty(\mathbb{R}^2)} < \frac{2L^2}{9L_4^2}.
\]

Then

\[
\| \text{tr}(Q^2(\cdot, t)) \|_{L^\infty(\mathbb{R}^2)} \leq \max \left\{ \| \text{tr}(Q_0^2) \|_{L^\infty(\mathbb{R}^2)}, \quad -\frac{a}{c} \right\}, \quad \forall t \in (0, T).
\]
Proof. Set \( \rho(x,t) = \sqrt{\text{tr}(Q^2(x,t))} \). Taking the inner product of (2.3) with \( 2Q \) gives
\[
(2.5) \quad \langle \rho^2 \rangle_t = \left[ (L\delta^{kl} + 2L_4 Q^{kl})(\rho^2)_{x_k} \right]_{x_l} - 2(L\delta^{kl} + 3L_4 Q^{kl}) Q^{ij}_{x_k x_l} Q^{ij}_{x_k} - 2(a + c\rho^2) \rho^2.
\]
Define
\[
T_* = \sup \left\{ t \in [0,T] \mid \|\rho^2\|_{L^\infty(\mathbb{R}^2 \times [0,t])} \leq \frac{2L^2}{9L_4^2} \right\}.
\]
We know from \( Q \in S^{(2)} \) that the eigenvalues of \( Q \) are \( \pm \rho/\sqrt{2} \). This implies that, in \( \mathbb{R}^d \times [0,T_*] \),
\[
LI + 2L_4 Q > 0 \quad \text{and} \quad LI + 3L_4 Q \leq 0.
\]
Hence the term \( 2(L\delta^{kl} + 3L_4 Q^{kl}) Q^{ij}_{x_k x_l} Q^{ij}_{x_k} \) in (2.5) is non-negative, so by the maximum principle,
\[
\|\rho^2(\cdot,t)\|_{L^\infty(\mathbb{R}^2)} \leq \max \left\{ \|\rho^2(0)\|_{L^\infty(\mathbb{R}^2)}, \frac{a}{c} \right\} < \frac{2L^2}{9L_4^2}, \quad \forall t \in [0,T_*].
\]
Since \( \|\rho(\cdot,t)\|_{L^\infty(\mathbb{R}^2)} \) is a continuous function of \( t \) (for which we omit the proof, simply making it as our assumption for classical solutions), by the definition of \( T_* \), we conclude that \( T_* = T \). The proof is complete. \( \square \)

Using Lemma 2.1 and interpolation, we now show the following:

**Lemma 2.3.** For each \( \alpha \in (0,1) \), there exists a constant \( C_1(\alpha) \) such that for any \( T > 0 \) and any classical solution to the 2D evolution problem (2.3) in \( \mathbb{R}^2 \times (0,T) \) with initial data \( Q(\cdot,0) = Q_0 \in S^{(2)} \),
\[
(2.6) \quad [Q]_{2+\alpha,T} \leq [Q_0]_{2+\alpha} + \frac{C_1(\alpha)}{L} \left[ L_4 \|Q\|_{0,T} [Q]_{2+\alpha,T} \right. \\
+ \left. ([a] + 3c\|Q\|^2_{0,T}) \|Q\|_{0,T} [Q]_{2+\alpha,T} \right].
\]

**Proof.** By time rescaling \( (Lt \to t) \), we can write equation (2.3) as
\[
Q^{ij}_t - \Delta Q^{ij} = L^{-1} F^{ij},
\]
where
\[
F^{ij} = L_4 \left( 2Q^{kl} Q^{ij}_{x_k x_l} + 2Q^{kl} Q^{ij}_{x_k} Q^{ij}_{x_k} - Q^{kl}_x Q^{kl}_x + \frac{1}{2} |\nabla Q|^2 \delta^{ij} \right) - [a + c\text{tr}(Q^2)] Q^{ij}.
\]
The interpolation
\[
[u]_{\alpha+(1-\theta)\beta} \leq C(\alpha,\beta,\theta)[u]_{\alpha}^\theta [u]_{\beta}^{1-\theta} \quad \text{for} \quad 0 \leq \alpha < \beta, \quad 0 \leq \theta \leq 1,
\]
implies
\[
[Q^{kl}_x Q^{ij}_{x_k x_l}]_{\alpha} \leq \|Q\|_{0} [Q]_{2+\alpha} + [Q]_{\alpha} \|D^2 Q\|_{0} \leq C(\alpha)\|Q\|_{0} [Q]_{2+\alpha},
\]
\[
[Q^{ij}_{x_m} Q^{kl}_{x_n}]_{\alpha} \leq 2\|\nabla Q\|_{0} [Q]_{1+\alpha} \leq C(\alpha)\|Q\|_{0} [Q]_{2+\alpha},
\]
\[
[(a + c\rho^2)Q^{ij}]_{\alpha} \leq ([a] + 3c\|Q\|^2_{0}) [Q]_{\alpha} \leq C(\alpha) ([a] + 3c\|Q\|^2_{0}) \|Q\|_{0}^{\theta} [Q]_{2+\alpha}^{1-\theta}.
\]
These estimates, together with Lemma 2.1, then give the assertion of the lemma. \( \square \)
Lemma 2.3 indicates that if the coefficient of $[Q]_{2+\alpha,T}$ on the R.H.S. of (2.6) is less than one, then the terms involving $[Q]_{2+\alpha,T}$ on the R.H.S. can be absorbed by the L.H.S., so that the $C^{2+\alpha}$ semi-norm of $Q$ is controlled by its $C^0$ norm, which can be estimated by Lemma 2.2.

Now we are ready to prove the existence of global classical solutions to the evolution problem (2.3) for initial data suitably small.

**Theorem 2.1.** For $d = 2$ and $\alpha \in (0, 1)$, suppose the coefficients of the problem (2.3) satisfy

$$
L > 0, \quad c > 0, \quad -\frac{a}{c} < \eta_1 = \min \left\{ \frac{1}{4C_1^2(\alpha)} \frac{2}{9} \right\} L^2, \quad L^2
$$

where $C_1(\alpha)$ is as in Lemma 2.3. Then for each $Q_0 : \mathbb{R}^2 \to S^{(2)}$ satisfying

$$
\| \text{tr}(Q_0^2) \|_{L^\infty(\mathbb{R}^2)} < \eta_1, \quad [Q_0]_{1+\alpha} < \infty,
$$

the 2D evolution problem (2.3) in $\mathbb{R}^2 \times (0, \infty)$ with initial data $Q(\cdot, 0) = Q_0$ admits a unique classical solution $Q \in C^{1+\alpha,(1+\alpha)/2}(\mathbb{R}^2 \times [0, \infty)) \cap C^\infty(\mathbb{R}^2 \times (0, \infty))$.

**Proof.** We divide the proof into two steps.

1. **Local existence.** Let $T \in (0, 1]$ be a small positive constant to be determined. For $R \in C^{1,1/2}(\mathbb{R}^2 \times [0, T] \to S^{(2)})$ satisfying

$$
\| R \|_{C^{1,1/2}(\mathbb{R}^2 \times [0, T])} \leq 1 + \| Q_0 \|_{C^{1+\alpha}(\mathbb{R}^2)}, \quad \| \text{tr}(R^2) \|_{L^\infty(\mathbb{R}^2 \times [0, T])} \leq \frac{L^2}{4L_4^2},
$$

consider the linear scalar equation, for $Q^{ij} \in \mathbb{R}^2 \times (0, T]$,

$$
Q^{ij} - \left[ (L\delta^{kl} + 2L_4 R^{kl}) Q^{jk}_{\cdot x_i} \right]_{x_j} = \frac{L_4}{2} |\nabla R|^2 \delta^{ij} - L_4 R^{kl} R^{kj}_{\cdot x_i} - [a + c \text{tr}(R^2)] R^{ij},
$$

subject to the initial condition $Q^{ij}(\cdot, 0) = Q_0^{ij}(\cdot)$. Since $LI + 2L_4 R$ is positive definite and $R \in C^{1,1/2}$, by classical theory of parabolic equation [5,7], this problem admits a unique $C^{1+\alpha,(1+\alpha)/2}(\mathbb{R}^2 \times [0, T])$ solution. Moreover, the Lagrange multipliers $\lambda$ and $\mu$ introduced in 2.4 ensure that the solution satisfies $Q \overset{\text{def}}{=} (Q^{ij})_{d \times d} \in S^{(2)}$.

In all,

$$
\| Q \|_{C^{1+\alpha,(1+\alpha)/2}(\mathbb{R}^2 \times [0, T])} \leq C(\| Q_0 \|_{C^{1+\alpha}(\mathbb{R}^2)}).
$$

Then by interpolation,

$$
\| Q - Q_0 \|_{C^{1,1/2}(\mathbb{R}^2 \times [0, T])} \leq C(\| Q_0 \|_{C^{1+\alpha}(\mathbb{R}^2)})^{T^{\alpha/2}}.
$$

Since $\text{tr}(Q_0^2) < 2L^2/(9L_4^2)$, we can find a small positive $T$ that does not depend on $R$ such that

$$
\text{tr}(Q^2) \leq \frac{L^2}{4L_4^2}, \quad \| Q \|_{C^{1,1/2}(\mathbb{R}^2 \times [0, T])} \leq 1 + \| Q_0 \|_{C^{1+\alpha}(\mathbb{R}^2)}.
$$

In addition, by taking smaller $T$ if necessary, one can show that the map from $R$ to $Q$ is a contraction; here we need to use $C^{1+\alpha}$ estimate for parabolic equation of divergence form. Hence, there exists a unique fixed point which gives us a $C^{1+\alpha,(1+\alpha)/2}$ solution. By a local parabolic estimate and a bootstrap argument, we have

$$
\| Q \|_{C^{n+\alpha,(n+\alpha)/2}(\mathbb{R}^2 \times [\varepsilon, T])} < \infty, \quad \forall \varepsilon \in (0, T], \quad n = 2, 3, \cdots.
$$

Hence, $Q \in C^{1+\alpha,(1+\alpha)/2}(\mathbb{R}^2 \times [0, T]) \cap C^\infty(\mathbb{R}^2 \times (0, T])$ and $\| \rho \|_{L^\infty(\mathbb{R}^2)} \in C^{(1+\alpha)/2}([0, T]).$ Here $\rho$ is defined in (1.4).
2. A priori estimate. To establish the global in time existence, it remains to establish a priori estimate for classical solutions. For this purpose, suppose we have a solution \( Q(x, t) \in C(\mathbb{R}^2 \times [0, T]) \cap C^\infty(\mathbb{R}^2 \times (0, T]) \) where \( T > 0 \) is arbitrary. Let us take \( \tilde{Q}(\cdot, t) = Q(\cdot, \varepsilon + t) \) with \( 0 < \varepsilon \ll 1 \), and \( Q_0(\cdot) = \tilde{Q}(\cdot, 0) \). By Lemma 2.2 and definition of \( \eta_1 \) in (2.7), there holds
\[
\| \tilde{\rho}^2(\cdot, t) \|_{L^\infty(\mathbb{R}^2)} \leq \max \left \{ \| \text{tr}(\tilde{Q}_0^0) \|_{L^\infty(\mathbb{R}^2)}, \frac{a}{c} \right \} < \eta_1, \quad \forall t \in [0, T - \varepsilon].
\]
Next, by Lemma 2.3 and again the definition of \( \eta_1 \) in (2.7), we have
\[
[\tilde{Q}]_{2+\alpha,T} \leq [Q_0]_{2+\alpha} + \frac{C_1(\alpha)|L_4|}{L} \| \tilde{Q} \|_{0,T}[\tilde{Q}]_{2+\alpha,T}
\]
\[
+ \frac{C_1(\alpha)}{L} (|a| + 3c\|\tilde{Q}\|_0^2, t) \| \tilde{Q} \|_{2+\alpha,T}/\| \tilde{Q} \|_{2+\alpha,T}
\]
\[
\leq [Q_0]_{2+\alpha} + \frac{C_1(\alpha)|L_4|\sqrt{\eta_1}}{L} [\tilde{Q}]_{2+\alpha,T}
\]
\[
+ \frac{1}{4} [\tilde{Q}]_{2+\alpha,T} + C(\alpha, a, c, L, \|Q_0\|_{L^\infty(\mathbb{R}^2)})
\]
\[
\leq [Q_0]_{2+\alpha} + \frac{3}{4} [\tilde{Q}]_{2+\alpha,T} + C(\alpha, a, c, L, \|Q_0\|_{L^\infty(\mathbb{R}^2)}),
\]
where we used the Young’s inequality in the second inequality and definition of \( \eta_1 \) in the third inequality. Thus,
\[
[\tilde{Q}(\cdot, t)]_{2+\alpha} \leq 4[Q_0]_{2+\alpha} + C(\alpha, a, c, L, \eta_1) \quad \forall t \in [0, T - \varepsilon].
\]
The a priori estimates (2.9) and (2.12), together with the local existence, give us a unique global and classical solution \( Q \) in \( [\varepsilon, \infty) \). This establishes the global in time a priori estimate and also completes the proof of our theorem. □

Remark 2.1.1. Uniqueness of the solution of the underlying problem, written in quasi-linear parabolic system in the divergence form,
\[
Q_t - \text{div}[(LI + 2L_4 Q)\nabla Q] = F(\nabla Q, Q),
\]
can also be obtained via Gronwall’s inequality using cut-off function and the localized \( L^2 \) norm
\[
\|u\|_2^2 \overset{\text{def}}{=} \sup_{x \in \mathbb{R}^2} \int_{|y| < 1} u^2(x - y) \, dy.
\]
Here, we need \( |L_4| \|Q_0\|_{L^\infty}/L \) suitably small, say \( |L_4| \|Q_0\|_{L^\infty}/L \leq \frac{1}{8} \) (see the conditions (2.7) and (2.8)). To be more precise, let \( \zeta \in C^\infty(\mathbb{R}^2) \) be a fixed cut-off function that satisfies \( \zeta(z) = 1 \) when \( |z| < 1 \) and \( \zeta(z) = 0 \) when \( |z| > 2 \). For any fixed \( x \in \mathbb{R}^2 \), integrating over \( y \in \mathbb{R}^2 \) and \( t \in [0, T] \) the equation (2.13) multiplied by \( \zeta^2(x - y)Q(y, t) \) and using routine calculation, we can derive that
\[
\int_{\mathbb{R}^2} \zeta^2(x - y)|Q|^2(y, t) \, dy \bigg|_{t=T}^{t=0}
\]
\[
\leq C \int_0^T \int_{\mathbb{R}^2} \left[ \zeta^2(x - y) + |\nabla \zeta|^2(x - y) \right] |Q(y, t)|^2 \, dy dt,
\]
where \( C \) is a constant depending on \( Q \). Taking the supreme over \( x \in \mathbb{R}^2 \) and defining
\[
\|w\|_2^2 = \sup_{x \in \mathbb{R}^2} \int_{\mathbb{R}^2} \zeta^2(x - y)|w(y)|^2 \, dy,
\]
we then obtain
\[ \|Q(T)\|^2 \leq \|Q(0)\|^2 + C \int_0^T \|Q(t)\|^2 \, dt. \]

It follows from Gronwall’s inequality that if \( Q(0) \equiv 0 \), then \( Q(t) \equiv 0 \) for all \( t > 0 \). The uniqueness follows directly from the above argument by considering \( Q \) as the difference of two solutions.

2. Applying a classical \( L^\infty \) estimate \([5,7,9]\) for the following parabolic equation in divergence form,
\[ u_t - \text{div}(LI + 2L_4Q)\nabla u = f + f^i, \]
to the function \( u = \zeta(\cdot + y)Q_{ij}^jx_k \), where \( \zeta \) is a cut-off function, one can establish the global existence of classical solution for initial data with small \( L^\infty \) norm and bounded gradient; i.e., the condition \([Q_0]_{1+\alpha} < \infty \) in (2.8) can be weakened by \( \|\nabla Q_0\|_{L^\infty(\mathbb{R}^2)} < \infty \). For simplicity of presentation, we omit this extension.

3. In theory, since \( F \) in (2.13) has only the critical quadratic growth in \( \nabla Q \), applying the a priori estimate for quasi-linear parabolic equation of divergence form \([9]\), the weak solution \( Q \in L^\infty(\Omega \times (0, \infty)) \cap L^2(0, \infty; H^1(\Omega)) \cap L^2_{\text{loc}}((0, \infty), H^2(\Omega)) \) with strictly positive definite matrix \( LI + 2L_4Q \), established in \([8]\), is indeed a classical solution with proper assumptions on the boundary data; here again, we omit the details.

3. THE \( d \geq 3 \) CASE

In this section, we deal with the high space dimensional case. The method in \([8]\) does not seem to apply since (i) the maximum principle for \( \rho^2 \) does not work and (ii) the energy estimates and Sobolev imbedding do not provide enough regularity for \( L^\infty \) estimate and \( L^2(0, T; H^2) \) integrability of \( Q \) that are essential in Sobolev space settings.

To this end, here we shall use a method different from \([8]\). To begin with, we provide two useful algebraic results that are valid for any \( d \).

**Lemma 3.1.** For any \( Q \in \mathcal{S}(d) \), \( |\text{tr}(Q^3)| \leq (\text{tr}(Q^2))^{3/2} \).

**Proof.** For any \( Q \in \mathcal{S}(d) \), by using the Cauchy-Schwarz inequality repeatedly, we have
\[
|\text{tr}(Q^3)| \leq \sum_i \sum_j \sum_k |Q^{ji}| \sum_k |Q^{jk}Q^{ik}|
\leq \sum_i \sum_j \left|Q^{ji}\right| \left(\sum_{k'} \left|Q^{ik'}\right|^2\right)^{1/2} \left(\sum_k \left|Q^{ik}\right|^2\right)^{1/2}
\leq \sum_i \left\{ \left(\sum_j \left|Q^{ji}\right|^2\right)^{1/2} \left(\sum_{j', k'} \left|Q^{j'k'}\right|^2\right)^{1/2} \left(\sum_k \left|Q^{ik}\right|^2\right)^{1/2} \right\}
\leq \left(\sum_{i', j'} \left|Q^{i'j'}\right|^2 \sum_{i, k} \left|Q^{ik}\right|^2\right)^{1/2} \left(\sum_{j', k'} \left|Q^{j'k'}\right|^2\right)^{1/2} = (\text{tr}(Q^2))^{3/2}.
\]

\( \square \)
Remark 3.1. It is proven in [11] that $|\text{tr}(Q^3)| \leq \text{tr}(Q^2)^{3/2}/\sqrt{6}$ for every $Q \in S^{(3)}$ where the biaxiality structure of $Q$ is used. The above estimate is weaker, but the proof is simpler and valid for any dimension.

Next we estimate the minimum eigenvalue of $Q$ in terms of its Frobenius norm $\rho = \sqrt{\text{tr}(Q^2)}$.

Lemma 3.2. Let $\lambda(Q)$ be the minimum eigenvalue of $Q \in S^{(d)}$. Then

$$\lambda(Q) \geq -\sqrt{\frac{d-1}{d} \text{tr}(Q^2)}.$$  

Proof. For any fixed $Q \in S^{(d)}$, suppose $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_d$ are the eigenvalues of $Q$. Since $Q$ is symmetric and traceless, we know that all $\lambda_i, i=1, \cdots, d$, are real valued, $\sum_{i=1}^d \lambda_i = 0$ and $\text{tr}(Q^2) = \sum_{i=1}^d \lambda_i^2$. Therefore,

$$d\lambda_1^2 = (d-1)\lambda_1^2 + \left( \sum_{i=2}^d \lambda_i \right)^2 \leq (d-1) \sum_{i=1}^d \lambda_i^2 = (d-1) \text{tr}(Q^2).$$

Based on Lemmas 3.1 and 3.2, we introduce the following two constants:

(3.1) \[ \rho_L := \frac{2}{3} \sqrt{\frac{d-1}{d}} \frac{L_1}{|L_4|}, \quad \rho_s := \frac{|b| + \sqrt{b^2 + 4|a|c}}{2c}. \]

Note that $c\rho_s^2 = |b|\rho_s + |a|$.

Remark 3.2. Next we shall first provide a proof of the following Lemma 3.3 before the statement of the lemma itself. This is quite atypical. The reason for this arrangement is that there are several complicated constants involved in the proof that are difficult to define in advance.

We now derive a priori estimates for classical solutions. We write (1.2) as

(3.2) \[ Q_t^{ij} - 2L_1 \Delta Q_t^{ij} = G_t^{ij}[Q], \]

where

(3.3) \[ G_t^{ij}[Q] = L_4 \left[ (Q_t^{kl} Q_t^{ij})_{x_k} + Q_t^{kl} Q_t^{ij} - d^{-1} |\nabla Q_t|^2 \delta^{ij} \right] + (L_2 + L_3) \left[ Q_t^{ik} Q_t^{jx_k} + Q_t^{jk} Q_t^{ix_k} - 2d^{-1} Q_t^{kl} \delta_{kx_l} \delta^{ij} \right] - aQ_t^{ij} - b (Q_t^{ik} Q_t^{jx_k} + Q_t^{jk} Q_t^{ix_k}) - c \text{tr}(Q_t^2) Q_t^{ij}. \]

Taking the inner product of equation (3.2) with $Q$ and setting $\rho = \sqrt{\text{tr}(Q^2)} = \sqrt{Q_t^{ij} Q_t^{ij}}$, we obtain

(3.4) \[ \frac{1}{2} (\rho^2)_t = \left[ (L_1 \delta^{kl} + L_4 Q_t^{kl}) (\rho^2)_x_k \right]_{x_l} - (2L_1 \delta^{kl} + 3L_4 Q_t^{kl}) \rho_c Q_t^{ij} + (L_2 + L_3) Q_t^{ij} \left( Q_t^{jk} Q_t^{ix_k} + Q_t^{ik} Q_t^{jx_k} \right) - [a\rho^2 + b \text{tr}(Q^2) + c \rho^4]. \]

Let us temporarily suppose

(3.5) \[ \rho \leq \rho_L \quad \text{in} \quad \mathbb{R}^d \times [0, T_*], \]
where $T_*$ is the maximum time that (3.5) is valid. Note that $\rho_L$ is defined in (3.1). Then $L_1 \mathbf{1} + L_4 Q > 0$ and $2L_1 \mathbf{1} + 3L_4 Q \geq 0$ in $\mathbb{R}^d \times [0, T_*]$.

By approximating the solution with compactly supported initial value, we assume, for simplicity of presentation, that there exists $(x_0, t_0) \in \mathbb{R}^2 \times [0, T_*)$ such that

$$
\rho(x_0, t_0) = \bar{\rho} \overset{\text{def}}{=} \sup_{\mathbb{R}^d \times [0, T_*]} \rho.
$$

If $t_0 = 0$, then $\bar{\rho} = \|\rho(0)\|_{L^\infty(\mathbb{R}^d)}$. If $t_0 > 0$, then evaluating equation (3.4) at $(x_0, t_0)$ and using $\rho_t \geq 0, (D^2 \rho) \leq 0$ and $\nabla \rho = 0$ at $(x_0, t_0)$ and Lemma 3.1 for $\text{tr}(Q^3)$, we obtain

$$
0 \leq |L_2 + L_3| \tilde{\rho} (2d^{3/2}|D^2 Q||0, T_*) - \tilde{\rho} H(\tilde{\rho}),
$$

where

$$
H(\rho) = \rho(c\rho^2 - |b|\rho + a) = c\rho(\rho - \rho_+)(\rho - \rho_-),
$$

with $\rho_+ = (|b| \pm \sqrt{b^2 - 4ac})/(2c)$. Note that $d^2$ on the R.H.S. of (3.6) comes from the Cauchy-Schwarz inequality and the definitions of $\rho$ and $[Q]_{2, T_*}$.

It follows immediately from (3.6) and interpolation that

$$
H(\tilde{\rho}) \leq 2d^{3/2}|L_2 + L_3| \|D^2 Q\|_0, T_*) \leq |L_2 + L_3| C(\alpha, d) \tilde{\rho}^{\frac{2+\alpha}{3+\alpha}} [Q]_{2+\alpha, T_*}^{\frac{2}{3+\alpha}}.
$$

Thus, if $\tilde{\rho} > \max\{|\rho(0)|_{L^\infty(\mathbb{R}^d)}, \rho_*\}$, using

$$
H(\tilde{\rho}) > c\tilde{\rho}(\tilde{\rho} - \rho^*)^2 \geq c\tilde{\rho}^{\frac{2+\alpha}{3+\alpha}} (\tilde{\rho} - \rho_*)^{\frac{6+2\alpha}{3+\alpha}},
$$

we obtain

$$
\tilde{\rho} \leq \rho_* + C_2(\alpha, d) \left(\left(\frac{|L_2 + L_3|}{C} \right)^{\frac{2+\alpha}{\alpha}} [Q]_{2+\alpha, T_*}^{\frac{2}{3+\alpha}} \right)^{\frac{1}{3+\alpha}},
$$

where $C_2(\alpha, d)$ is a constant depending only on $\alpha$ and $d$.

Next, we use Lemma 2.1 and a similar argument as in the proof of Lemma 2.3 to derive

$$
L_1 [Q]_{2+\alpha, T_*} \leq L_1 [Q_0]_{2+\alpha} + C(\alpha, d) [G(Q)]_{2+\alpha, T_*} \leq L_1 [Q_0]_{2+\alpha} + C(\alpha, d) \left(\|L_2 + L_3\|_0 + L_4|\rho|_0 \right) [Q]_{2+\alpha, T_*}^{\frac{2}{3+\alpha}}.
$$

Using $c\tilde{\rho}^2 - |b|\tilde{\rho} - |a| \geq c\rho_*^2 - |b|\rho_* - |a| = 0$ when $\tilde{\rho} \geq \rho_*$ and estimate (3.7), we have

$$
|a| + |b|\tilde{\rho} + c\tilde{\rho}^2 \tilde{\rho}^{\frac{6+2\alpha}{3+\alpha}} \leq 2c \max\{\tilde{\rho}, \rho_*\}^{2+\frac{6+2\alpha}{3+\alpha}} \leq C(\alpha)c\rho_*^{\frac{6+2\alpha}{3+\alpha}} + 3c|\tilde{\rho} - \rho_*|^{\frac{6+2\alpha}{3+\alpha}}
$$

$$
\leq C(\alpha)c\rho_*^{\frac{6+2\alpha}{3+\alpha}} + C(\alpha, d)|L_2 + L_3|[Q]_{2+\alpha, T_*}^{\frac{2}{3+\alpha}}.
$$

It then follows from (3.8) and Young’s inequality that

$$
L_1 [Q]_{2+\alpha, T_*} \leq L_1 [Q_0]_{2+\alpha} + C_3(\alpha, d) \left(\|L_2 + L_3\|_0 + |L_4|\tilde{\rho} \right) [Q]_{2+\alpha, T_*}^{\frac{2}{3+\alpha}} \leq L_1 [Q_0]_{2+\alpha} + \frac{L_4}{4} [Q]_{2+\alpha, T_*}^{\frac{1}{\alpha/2}} + \frac{C_4(\alpha, d)c^{1+\alpha/2}}{L_1^{\alpha/2}} \rho_3^{3+\alpha}.
$$

Finally we close the above looping estimates as follows. Let $\alpha \in (0, 1)$ be fixed and let $M_0$ and $M_1$ be positive constants satisfying

$$
M_0 > \max\left\{\rho_*, \ C_2(\alpha, d) L_2 + L_3 \frac{2+\alpha}{6+2\alpha} (4M_1)^{\frac{1}{3+\alpha}}\right\},
$$
\[(3.11)\quad L_1 > \max \left\{ 4C_3(\alpha, d)(|L_2 + L_3| + 2M_0|L_4|), \quad C_4(\alpha, d) \rho_*^{\frac{6+2\alpha}{2+\alpha}} M_1^{-\frac{2}{2+\alpha}} \right\}.
\]
Assume that
\[(3.12)\quad \left\| \sqrt{\text{tr}(Q_0^2)} \right\|_{L^\infty(\mathbb{R}^d)} \leq M_0, \quad [Q_0]_{2+\alpha} \leq c^{\frac{2+\alpha}{2}} M_1.
\]
Suppose \(Q\) is a classical solution of \((3.2)\) in \(\mathbb{R}^d \times (0, T)\) subject to \(Q(\cdot, 0) = Q_0(\cdot)\).
We define
\[
T_* := \sup \left\{ t \in [0, T] \mid \left\| \sqrt{\text{tr}(Q^2)} \right\|_{L^\infty(\mathbb{R}^d \times [0, t])} \leq 2M_0, \quad \sup_{s \in [0, t]} [Q(\cdot, s)]_{2+\alpha} \leq 4c^{\frac{2+\alpha}{2}} M_1 \right\}.
\]
Choosing \(C_3(\alpha, d)\) properly large in \((3.11)\) such that \(2M_0 < \rho_L\), then our assumption \((3.5)\) is satisfied. Consequently, either \(\bar{\rho} \leq \max\{\rho_*, \|\rho_0\|_{L^\infty(\mathbb{R}^d)}\} \leq M_0\) or \((3.7)\) holds, in which case, using \([Q]_{2+\alpha, T_*} \leq 4c^{\frac{2+\alpha}{2}} M_1\) and \((3.10)\), we obtain
\[
(3.13) \quad \bar{\rho} \leq \rho_* + C_2(\alpha, d) \left( \left\| \frac{L_2 + L_3}{c} \right\|^{\frac{2+\alpha}{\alpha}} \right)_{2+\alpha, T_*} \leq \rho^* + M_0 < 2M_0.
\]
Next, by \((3.11)\), we have \(C_3(\alpha, d)(|L_2 + L_3| + |L_4|) \leq L_1/4\), so from \((3.9)\) and \((3.11)\), we obtain
\[
(3.14) \quad [Q]_{2+\alpha, T_*} \leq 2[Q_0]_{2+\alpha, T_*} + \left\| \frac{C_4(\alpha, d)c^{1+\alpha/2} \rho_*^{3+\alpha}}{L_1^{1+\alpha/2}} \right\| \leq 3c^{1+\alpha/2} M_1.
\]
This, together with \((3.13)\) and the definition of \(T_*\) (and continuity of \(\|\rho(t)\|_{L^\infty}\)) and \([Q(t, \cdot)]_{2+\alpha}\) in \(t\), we find that \(T_* = T\). Hence we have the following a priori estimate:

**Lemma 3.3.** Assume that \((3.10), (3.11), \) and \((3.12)\) hold. If \(Q\) is a classical solution of \((1.2)\) in \(\mathbb{R}^d \times (0, T)\) with initial value \(Q(\cdot, 0) = Q_0\), then
\[\left\| \sqrt{\text{tr}(Q^2)} \right\|_{L^\infty(\mathbb{R}^d \times [0, T])} \leq 2M_0, \quad [Q]_{2+\alpha, T} \leq 3c^{1+\alpha/2} M_1.\]

Now we are ready to prove the following:

**Theorem 3.1.** For \(d \geq 3\), suppose \(L_1 > 0\), \(\alpha \in (0, 1)\), and \(\rho_*, \rho_L\) are defined in \((3.1)\). There exist positive constants \(C_2(\alpha, d), C_3(\alpha, d), \) and \(C_4(\alpha, d)\) such that if the coefficients of the problem \((1.2)\) together with positive constants \(M_0, M_1\) satisfy \((3.10)\) and \((3.11)\), then for any \(Q_0 : \mathbb{R}^d \to S^{(d)}\) satisfying \((3.12)\), the evolution equation \((1.2)\) in \(\mathbb{R}^d \times (0, \infty)\) with initial value \(Q(\cdot, 0) = Q_0\) admits a unique solution \(Q \in C^{2+\alpha, 1+\alpha/2}(\mathbb{R}^d \times [0, \infty)) \cap C(\mathbb{R}^d \times (0, \infty))\) satisfying
\[
(3.15) \quad \left\| \sqrt{\text{tr}(Q^2)} \right\|_{L^\infty(\mathbb{R}^d \times [0, \infty))} \leq 2M_0, \quad \sup_{t \geq 0} [Q(\cdot, t)]_{2+\alpha} \leq 3c^{1+\alpha/2} M_1.
\]

**Proof.** Since we already have a priori estimates, we need only prove local existence. For this we use a fixed point argument. Fix small \(T > 0\). Given \(R\) satisfying
\[\left\| \sqrt{\text{tr}(R^2)} \right\|_{L^\infty(\mathbb{R}^d \times [0, T])} \leq 2M_0, \quad [R]_{2+\alpha, T} \leq 2M_1,
\]
we define \(Q_{ij}^t\) as the unique solution of the scalar linear problem
\[Q_{ij}^t - 2L_1 \Delta Q_{ij}^t = G_{ij}(R) \quad \text{in} \quad \mathbb{R}^d \times (0, T), \quad Q_{ij}^t(\cdot, 0) = Q_0(\cdot).\]
Since \(|L_2 + L_3| + |L_4|/L_1\) is quite small (taking suitably large \(C_3(\alpha, d)\) in (3.11)), one can show that when \(T\) is small, the map from \(R\) to \(Q\) is a contraction in the \(\|\cdot\|_{0,T} + \|\cdot\|_{2+\alpha,T}\) norm. Thus, there exists a unique fixed point, which gives a unique solution. The equation \(Q_t = 2L_1\Delta Q + G(Q)\) implies that \([Q_t]_{\alpha,T} < \infty\), so we have a local in time classical solution.

The local existence and uniqueness and global a priori estimate give a unique global in time classical solution. Finally, regarding (1.2) as a quasilinear-linear parabolic system, we derive from standard theory of parabolic system \(\square\) that \(Q \in C^{2+\alpha,1+\alpha/2}(\mathbb{R}^d \times [0, \infty)) \cap C^\infty(\mathbb{R}^d \times (0, \infty))\). This completes the proof.

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