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# PARTIAL HYPERBOLICITY AND SPECIFICATION

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Abstract. We study the specification property for partially hyperbolic dynamical systems. In particular, we show that if a partially hyperbolic diffeomorphism has two saddles with different indices, and the stable manifold of one of these saddles coincides with the strongly stable leaf, then it does not satisfy the specification property.

## 1. Introduction and statement of the main results

The purpose of this paper is to give a global perspective for the space of all $C^1$-diffeomorphisms satisfying the specification property. Let $(X, d)$ be a compact metric space, and let $f\colon X \to X$ be a homeomorphism. We say that $f$ satisfies the *specification property* if for each $\varepsilon > 0$, there is an integer $N(\varepsilon)$ for which the following is true: if $I_1, I_2, \cdots, I_k$ are pairwise disjoint intervals of integers with

$$\min\{|m-n| : m \in I_i, n \in I_j\} \geq N(\varepsilon)$$

for $i \neq j$ and $x_1, \cdots, x_k \in X$, then there is a point $x \in X$ such that $d(f^j(x), f^j(x_i)) \leq \varepsilon$ for $j \in I_i$ and $1 \leq i \leq k$. It is well known that every hyperbolic elementary set of a diffeomorphism satisfies the specification property (see [8]). This property was introduced by Bowen in [9] and roughly means that an arbitrary number of pieces of orbits can be "glued" to obtain a real orbit that shadows the previous ones. This is crucial in the study of the uniqueness of equilibrium states ([10]), large deviations theory ([30]), and multifractal analysis ([26, 27]), which justifies the interest of many researchers in several forms of specification ([18, 20, 21, 28, 29]).

In the nineties, Palis proposed a conjecture for a global view of dynamics which has been a routing guide for many works in the last years, which we describe here in the space $C^1$-diffeomorphisms: either a diffeomorphism is uniformly hyperbolic or it can be $C^1$-approximated by a diffeomorphism that exhibits a homoclinic tangency or a heteroclinic cycle. In rough terms, in the complement of uniform hyperbolicity (open condition) the mechanisms that generate non-hyperbolicity in a dense way are tangencies and cycles. We refer the reader to the surveys [4, 19] for reports on the advances towards the conjecture and the current state of the conjecture for $C^1$-diffeomorphisms.

Palis's conjecture and the $C^1$-stability conjecture (cf. [13, 16]) inspired the works of many authors to approach such dichotomy in the space of $C^1$-diffeomorphisms concerning other important dynamical properties that are not necessarily $C^1$-open, namely, expansiveness, shadowing, or specification properties. In [2, 15, 22, 23] it

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was proved that the $C^1$-interior of the set of all diffeomorphisms satisfying any of these properties is contained in the uniformly hyperbolic ones.

In the case of specification, Sakai, the first, and the third authors proved in [23] that the $C^1$-interior of the set of all diffeomorphisms satisfying the specification property coincides with the set of all transitive Anosov diffeomorphisms. Moriyasu, Sakai, and the third author extended the above results to regular maps, and proved that $C^1$-generically, regular maps satisfy the specification property if and only if they are transitive Anosov ([17]). A counterpart of these results for the time-continuous setting was obtained more recently by Arbieto, Senos, and Todero [3]. Owing to these results, the relation to hyperbolicity turns out to be clear.

The current interest in a global description of dynamical systems led us to wonder if these properties can hold generically or at least densely in the complement of the set of uniformly hyperbolic diffeomorphisms. An affirmative answer could permit the use of specification property for diffeomorphisms $C^1$-close to tangencies or heteroclinic cycles. In this paper we will show that the answer is negative even in the stronger context of partial hyperbolicity. This paper is largely motivated by the results of Bonatti-Díaz-Turcat ([6]) and Abdenur-Díaz ([1]) on the shadowing properties. So, before stating our main theorems, we explain their main results.

Throughout, let $M$ be a closed manifold with $\dim M \geq 3$, where $\dim E$ denotes the dimension of $E$, and let $\mathrm{Diff}(M)$ be the space of $C^1$-diffeomorphisms of a closed $C^\infty$ manifold $M$ endowed with the $C^1$-topology. Given $f \in \mathrm{Diff}(M)$, a $Df$-invariant splitting $TM = E \oplus F$ is dominated if there is a constant $k \in \mathbb{N}$ such that

$$\frac{\|D_x f^k(u)\|}{\|D_x f^k(w)\|} < \frac{1}{2},$$

for every $x \in M$ and every pair of unitary vectors $u \in E(x)$ and $w \in F(x)$. Generally, a $Df$-invariant splitting $TM = E_1 \oplus \cdots \oplus E_k$ is dominated if for any $1 \leq l \leq k-1$, $(E_1 \oplus \cdots \oplus E_l) \oplus (E_{l+1} \oplus \cdots \oplus E_k)$ is dominated.

A $Df$-invariant bundle $E$ is uniformly contracting (resp. expanding) if there are $C > 0$ and $0 < \lambda < 1$ such that for every $n > 0$ one has $\|D_x f^n(v)\| \leq C\lambda^n \|v\|$ (resp. $\|D_x f^{-n}(v)\| \leq C\lambda^n \|v\|$) for all $x \in M$ and $v \in E(x)$.

We say that a diffeomorphism $f$ is partially hyperbolic (resp. strongly partially hyperbolic) if there is a $Df$-invariant splitting $TM = E^s \oplus E^c \oplus E^u$ such that $E^s$ and $E^u$ are uniformly contracting and uniformly expanding respectively, and at least one of them is (resp. both of them are) not trivial. A diffeomorphism is hyperbolic if it is strongly partially hyperbolic and $E^c$ is trivial. We say that $E^c$ is the central direction of the splitting.

We say that $f \in \mathrm{Diff}(M)$ is *transitive* if there is $x \in M$ whose orbit is dense in $M$. A diffeomorphism $f$ is *robustly transitive* if there is a $C^1$-neighborhood $\mathcal{U}(f)$ of $f$ in $\mathrm{Diff}(M)$ such that any $g \in \mathcal{U}(f)$ is transitive. Denote by $\mathcal{RNT}$ the set of robustly non-hyperbolic transitive diffeomorphisms in $\mathrm{Diff}(M)$, that is, the set of diffeomorphisms $f$ having a $C^1$-neighborhood $\mathcal{U}(f)$ of $f$ such that every $g \in \mathcal{U}(f)$ is non-hyperbolic and transitive.

A diffeomorphism $f \in \mathrm{Diff}(M)$ satisfies the *shadowing property* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for every sequence $(x_n)_{n\in\mathbb{Z}}$ of points in $M$ satisfying $d(f(x_n), x_{n+1}) < \delta$ $(n \in \mathbb{Z})$, there exists $x \in M$ so that $d(f^n(x), x_n) < \varepsilon$ $(n \in \mathbb{Z})$. In other words, the orbit of $x$ $\varepsilon$-shadows the $\delta$-pseudo-orbit $(x_n)_{n\in\mathbb{Z}}$. In [6], Bonatti, Díaz, and Turcat proved the following theorem.

**Theorem 1.1** ([6, Theorem]). *Let $f\colon M \to M$ be a transitive diffeomorphism with a strongly partially hyperbolic splitting on $M$ with $\dim M = 3$. Assume that $f$ has two hyperbolic periodic points $p$ and $q$ such that $\dim(W^s(p)) = 2$ and $\dim(W^s(q)) = 1$. Then $f$ does not satisfy the shadowing property.*

Moreover, Abdenur and Díaz showed the following.

**Theorem 1.2** ([1, Theorem 2]). *There is a $C^1$-open and dense subset $\mathcal{P}$ in $\mathcal{RNT}$ such that every $f \in \mathcal{P}$ does not satisfy the shadowing property.*

As mentioned before, both the specification and shadowing properties reflect the approachability of pseudo-orbits or finite pieces of orbits of the dynamical system by true orbits, although these two notions, even their $C^1$-interior, do not coincide in general (see [22, 23], for instance). So, it is natural to consider the counterparts of Theorem 1.1 and Theorem 1.2 for the setting of the specification property. Now, we state our main result of this paper.

**Theorem A.** *Let $f\colon M \to M$ be a diffeomorphism admitting a partially hyperbolic splitting $E^s \oplus E^c \oplus E^u$. Assume that there are two hyperbolic periodic points $p$ and $q$ such that either $\dim E^u = \dim W^u(p) < \dim W^u(q)$ or $\dim E^s = \dim W^s(q) < \dim W^s(p)$. Then $f$ does not satisfy the specification property.*

Here $W^u(p)$ (resp. $W^s(p)$) denotes the unstable (resp. stable) manifold of $p$ defined as usual. We should emphasize that the holonomy map along the strong unstable foliation plays a key role in the proof of Theorem A, which is the main difference from that of Theorem 1.1.

Denote by $\mathcal{SPH}_1(M)$ the set of all strongly partially hyperbolic diffeomorphisms with one-dimensional central direction. We note that $\mathcal{SPH}_1(M)$ is open in $\mathrm{Diff}(M)$. In the case that the central direction $E^c$ is one-dimensional, any two hyperbolic periodic points with different indices verify the previous assumptions. Hence, we obtain from the previous result the following consequence.

**Corollary 1.** *Let $f \in \mathcal{SPH}_1(M)$ and suppose that there exist two hyperbolic periodic points $p, q$ with different indices. Then $f$ does not satisfy the specification property.*

Applying this corollary, we obtain the following consequence.

**Corollary 2.** *There is a $C^1$-open and dense subset $\mathcal{P}$ in $\mathcal{RNT} \cap \mathcal{SPH}_1(M)$ such that every $f \in \mathcal{P}$ does not satisfy the specification property.*

In case of $\dim M = 3$, we obtain the counterpart of Theorem 1.2 as follows.

**Corollary 3.** *Suppose that $\dim M = 3$. Then there is a $C^1$-dense open subset $\mathcal{P}$ in $\mathcal{RNT}$ so that every $f \in \mathcal{P}$ does not satisfy the specification property.*

In conclusion, it can be said that the non-hyperbolic transitive diffeomorphisms seldom have the specification property. This answer gives not only a global description of the space of diffeomorphisms as it enhances the need to consider, beyond uniform hyperbolicity, weak forms of specification, such as non-uniform measure theoretical versions of the specification or almost specification properties. One remaining interesting question is to understand which non-hyperbolic systems admit weaker specification properties.

As an application of Theorem A, we also investigate some relation between hyperbolicity, specification property, and presence of homoclinic tangencies. We say

that a diffeomorphism $f$ exhibits a *homoclinic tangency* if $f$ has a hyperbolic periodic point whose stable and unstable manifolds have a non-transverse intersection. Very recently, it was proved [11] that any diffeomorphism $f$ can be approximated in $\mathrm{Diff}(M)$ by diffeomorphisms which exhibit a homoclinic tangency or by partially hyperbolic diffeomorphisms. Inspired by this result, we show the following.

**Corollary 4.** *Let $f$ be a robustly transitive diffeomorphism which has a hyperbolic periodic point with stable index one. Then one of the following properties holds:*

*(1) $f$ is an Anosov diffeomorphism.*
*(2) $f$ can be $C^1$-approximated by a partially hyperbolic diffeomorphism which does not satisfy the specification property.*
*(3) $f$ can be $C^1$-approximated by a diffeomorphism which exhibits a homoclinic tangency.*

We should emphasize that the third case in Corollary 4 cannot be omitted. Indeed, in [7, §6], Bonatti and Viana constructed an open set $\mathcal{O}$ of $\mathrm{Diff}(\mathbb{T}^3)$ such that for any $f \in \mathcal{O}$:

(i) there is a $Df$-invariant dominated splitting $TM = E^u \oplus E^c$ into a 1-dimensional strong unstable subbundle $E^u$ and a 2-dimensional subbundle $E^c$. Moreover, $E^c$ is not uniformly hyperbolic and does not admit a decomposition in invariant subbundles;
(ii) every strong unstable leaf of $f$ is dense in $\mathbb{T}^3$, which implies that $f$ is transitive (and so $f$ is also robustly transitive) and the whole space $\mathbb{T}^3$ coincides with the homoclinic class $H(p)$ of some periodic point $p$.

Assume for a contradiction that no $f \in \mathcal{O}$ admits a homoclinic tangency. Then by [11, Theorem 1.1], there exists a residual subset $\mathcal{R}$ in $\mathcal{O}$ so that for any $g \in \mathcal{R}$ and $p \in \mathbb{T}^3$, $H(p)$ admits a dominated splitting whose central direction $E^c$ has a finest dominated splitting $E^c = E^c_1 \oplus E^c_2$ with $\dim E^c_i = 1$ for $i = 1, 2$, which contradicts the item (i). So, there is a diffeomorphism $f \in \mathcal{O}$ having the homoclinic tangency. This in fact proves that there exists a dense subset $\mathcal{D}$ of $\mathcal{O}$ so that all $f \in \mathcal{D}$ admit a homoclinic tangency.

## 2. Proof of Theorem A

In this section, we prove Theorem A. First, we rewrite the definition of the specification property using the very useful notion of (closed) dynamical balls and prove the preliminary lemma. Given $x \in M$, $\varepsilon > 0$, $m, n \in \mathbb{Z}$ with $m \le n$, and $I = [m, n]$, set

$$B_I(x, \varepsilon) = B_{[m,n]}(x, \varepsilon) = \{y \in M \colon d(f^j(y), f^j(x)) \le \varepsilon, m \le j \le n\}.$$

If no confusion is possible, set $B_n(x, \varepsilon) = B_{[0,n]}(x, \varepsilon)$ and $B_{-n}(x, \varepsilon) = B_{[-n,0]}(x, \varepsilon)$. Then, the specification property can be written as follows: given $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon) \ge 1$ so that for any $x_1, \dots, x_k \in M$ and intervals of integers $I_j = [m_j, n_j]$ $(1 \le j \le k)$ with $m_{j+1} - n_j \ge N$ $(1 \le j \le k-1)$, it holds that

$$\bigcap_{j=1}^{k} f^{-m_j}(B_{I_j}(x_j, \varepsilon)) \neq \emptyset.$$

In the case that $p$ is a hyperbolic periodic point for $f$, there exists $\varepsilon > 0$ small so that the local unstable set

$$W^u_\varepsilon(p) = \{x \in M : d(f^{-n}(x), f^{-n}(p)) \leq \varepsilon \text{ for all } n \geq 0\} = \bigcap_{n\geq 0} B_{-n}(p, \varepsilon)$$

is the local unstable manifold at $p$ with size $\varepsilon$. Analogously, the local stable set $W^s_\varepsilon(p) = \bigcap_{n\geq 0} B_n(p, \varepsilon)$ is the local stable manifold for some $\epsilon > 0$. We refer the reader to [14] and [25] for more details.

**Lemma 2.1.** *Suppose that $f\colon M \to M$ satisfies the specification property. Then for every hyperbolic periodic point $p$, both the stable and unstable manifolds $W^s(p)$ and $W^u(p)$ are dense in $M$.*

*Proof.* Given a hyperbolic periodic point $p$ for $f$, we prove that the unstable manifold $W^u(p)$ is dense in $M$, since the proof for the density of $W^s(p)$ is completely analogous. Let us assume for simplicity that $p$ is a fixed point, since otherwise we just consider $f^k$ where $k$ is the period of $p$.

Let $W^u_{\varepsilon_1}(p)$ denote the local unstable manifold for some $\varepsilon_1 > 0$. Take any point $x \in M$ and $\varepsilon_2 > 0$. It is sufficient to show that there exists a point $w \in M$ such that $d(x, w) \leq \varepsilon_2$ and $w \in W^u(p)$. We set $\varepsilon := \frac{1}{2}\min\{\varepsilon_1, \varepsilon_2\}$ and take an integer $L \geq N(\varepsilon)$. Since $f$ satisfies the specification property, for any $n \geq 1$,

$$f^L(B_{-n}(p, \varepsilon)) \cap B(x, \varepsilon) \neq \emptyset,$$

where $B(x, \varepsilon)$ stands for the closed ball of radius $\varepsilon$ around $x$. Since the previous intersection is a strictly decreasing family of closed sets, by compactness of $M$, there exists a point $w \in M$ such that

$$w \in \bigcap_{n=1}^{\infty} f^L(B_{-n}(p, \varepsilon)) \cap B(x, \varepsilon).$$

Then we have $d(w, x) \leq \varepsilon_2$ and $d(f^{-n}(f^{-L}(w)), f^{-n}(p)) \leq \varepsilon_1$ for any $n \geq 1$. The latter implies that $w \in f^L(W^u_{\varepsilon_1}(p)) = f^L(W^u_{\varepsilon_1}(f^{-L}(p)))$. Thus, we have $w \in W^u(p)$ and $d(x, w) \leq \varepsilon_2$, which proves the lemma. □

It follows from [17, Corollary 2] that $C^1$-generically, non-hyperbolic diffeomorphisms do not have the specification property. On the other hand, maps with the specification property could be dense in the complement of the uniformly hyperbolic diffeomorphisms. Our purpose in Theorem A is to prove that this is not the case even for some partially hyperbolic dynamical systems.

*Proof of Theorem* A. Let $f\colon M \to M$ be a diffeomorphism admitting a partially hyperbolic splitting $E^s \oplus E^c \oplus E^u$ and assume $p$ and $q$ are hyperbolic periodic points for $f$ satisfying $\dim E^u = \dim W^u(p) < \dim W^u(q)$ (the case that $\dim E^s = \dim W^s(q) < \dim W^s(p)$ is analogous).

Assume, by contradiction, that $f$ satisfies the specification property. Then it follows from [24, Proposition 2 (b)] that $f$ is topologically mixing. Thus, $f$ has neither sinks nor sources. In particular, $\dim E^u = \dim W^u(p) > 0$, which implies that $E^u$ is not trivial.

Let us recall a necessary result on the location of the the shadowing point in unstable disks. For $x \in W^u(p)$ and $\eta > 0$ we will consider the local unstable disk around $x$ in $W^u(p)$ given by

$$\gamma^u_\eta(x) := \{z \in W^u(p) : d^u(x, z) \leq \eta\}.$$

Here $d^u$ is the distance in $W^u(p)$ induced by the Riemannian metric.

**Proposition 2.2** ([6, Proposition 3]). *There exists a small positive constant $\varepsilon_1$ such that for any $\varepsilon \in (0, \varepsilon_1)$ the following holds: if $x \in W^u(p)$ and $d(f^{-n}(z), f^{-n}(x)) \leq \varepsilon$ for any $n \geq 1$, then $z \in \gamma^u_{4\varepsilon}(x)$.*

Then we are in a position to prove the next proposition, which is a key ingredient in the proof of our main Theorem A.

**Proposition 2.3.** *Let $\varepsilon_1$ be as in Proposition 2.2. Then there exist $\eta > 0$, $\varepsilon \in (0, \varepsilon_1)$ with $4\varepsilon < \eta$, and a point $x \in W^u(p)$ such that*

$$f^N(\gamma^u_\eta(x)) \cap W^s_\eta(q) = \emptyset,$$

*where $N = N(\varepsilon)$ is as in the definition of the specification property.*

*Proof.* Since $E^u$ is not trivial, it is well known that the subbundle $E^u$ is uniquely integrable and hence we have a foliation $\mathcal{F}^u$ which is tangent to $E^u$, called the strong unstable foliation (see [14]). As usual, let us denote by $\mathcal{F}^u(x)$ the leaf of the foliation $\mathcal{F}^u$ that contains the point $x$. Then Lemma 2.1 guarantees that $W^u(p)$ is dense in $M$. Given $r > 0$, let us consider the family

$$\mathcal{L}(p) = \{V(w) : w \in B(p, r)\},$$

where $V(w)$ is the connected component of $\mathcal{F}^u(w) \cap B(p, r)$ containing $w$. Choose a local disk $D'_0$ and $\eta > 0$ so small that $D'_0$ is transverse to the family $\mathcal{L}(p)$, $p \in D'_0$, and for any open disk $U$ contained in $D'_0$, $\mathcal{A}(U) := \bigcup_{z \in U} \mathcal{F}^u_\eta(z)$ is homeomorphic to $U \times [-\eta, \eta]^{\dim E^u}$. Here we set

$$\mathcal{F}^u_\eta(z) := \{w \in \mathcal{F}^u(z) : d^u(z, w) \leq \eta\},$$

where $d^u$ is the distance in $\mathcal{F}^u(z)$ induced in the Riemannian metric. We set $\varepsilon := \min\{\eta/5, \varepsilon_1/2\}$.

Next, we choose a compact disk $K$ such that $W^s_\eta(q) \subset K$ and $K$ is transverse to $E^u$. Since $K$ is transverse to $E^u$, then $K \cap f^N(\gamma^u_\eta(p))$ consists of finitely many points $\{x_1, x_2, \cdots, x_k\}$. Choose an open subdisk $D_0 \subset D'_0$ containing $p$ such that $K_i \cap K_j = \emptyset$ if $i \neq j$. Here $K_i$ is a connected component of $K \cap f^N(\mathcal{A}(D_0))$ containing $x_i$, for $1 \leq i \leq k$ (see Figure 1).

For each $1 \leq i \leq k$, we set $D_i = f^{-N}(K_i)$ and consider a holonomy map $\pi_i \colon D_i \to D_0$ which is defined by

$$\pi_i(w) := v \text{ if } \{w\} = D_i \cap \mathcal{F}^u(v), \ (v \in D_0).$$

By our choice of $D_0, \cdots, D_k$, for each $1 \leq i \leq k$, $\pi_i$ is a homeomorphism. Since $W^s_\eta(q)$ is a closed submanifold with $\dim W^s_\eta(q) < \dim K_i$, $K_i \setminus W^s_\eta(q)$ is open and dense in $K_i$. Thus, if we set $\gamma^s_i := \pi_i \circ f^{-N}(W^s_\eta(q))$, then $D_0 \setminus (\bigcup_{i=1}^k \gamma^s_i)$ is dense and open in $D_0$. So we can find an open subdisk $U \subset D_0 \setminus (\bigcup_{i=1}^k \gamma^s_i)$ (see Figure 2).

Since $\mathcal{A}(U)$ is homeomorphic to $U \times [-\eta, \eta]^{\dim E^u}$ and $W^u(p)$ is dense in $M$, we can find a point $z' \in \mathcal{A}(U) \cap W^u(p)$. This implies that there exists a point $x \in U$ such that $z' \in \mathcal{F}^u_\eta(x)$. So $x \in W^u(p)$ and $\mathcal{F}^u_\eta(x) = \gamma^u_\eta(x)$. By the choice of $U$, we have $f^N(\gamma^u_\eta(x)) \cap W^s_\eta(q) = \emptyset$, which proves the proposition. □

Now we continue the proof of Theorem A. For each $\varepsilon > 0$, let $N = N(\varepsilon) \geq 1$ be the integer as in the definition of the specification property. Then it follows from

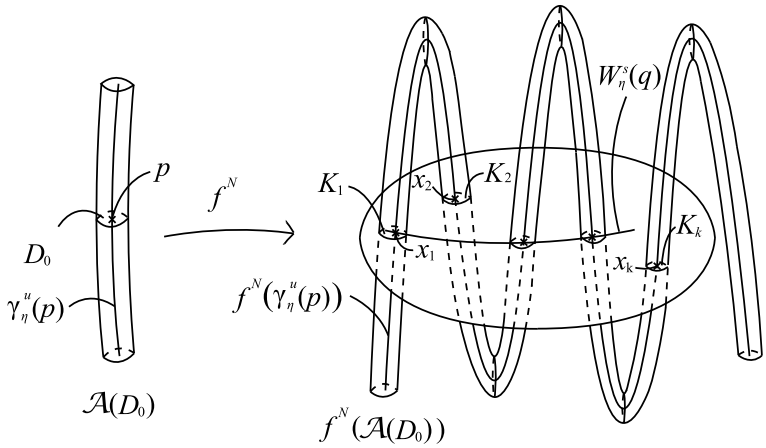


FIGURE 1

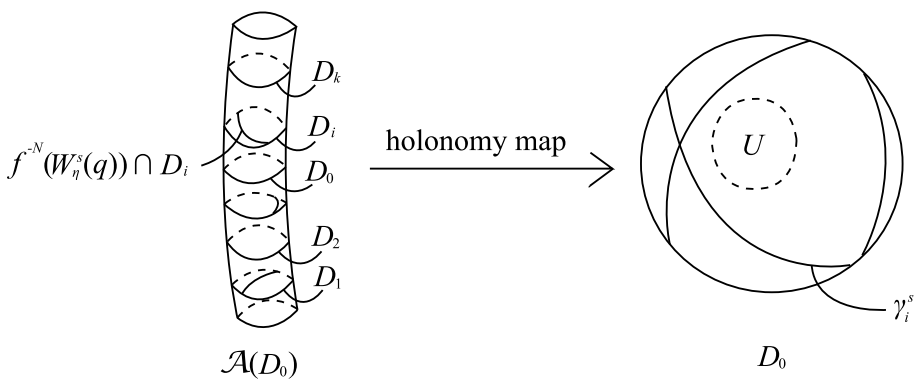


FIGURE 2

Proposition 2.3 that there are $\eta > 0$, $\varepsilon \in (0, \varepsilon_1)$ with $4\varepsilon < \eta$, and a point $x \in W^u(p)$ such that

$$f^N(\gamma_\eta^u(x)) \cap W_\eta^s(q) = \emptyset.$$

On the other hand, it follows from the specification property that for any $n \geq 1$, one has $f^N(B_{-n}(x, \varepsilon)) \cap B_n(q, \varepsilon) \neq \emptyset$ and consequently, using the compactness of $M$, we have

$$\bigcap_{n=1}^{\infty} f^N(B_{-n}(x, \varepsilon)) \cap B_n(q, \varepsilon) \neq \emptyset.$$

Therefore, there exists a point $z \in M$ such that $d(f^{-n}(f^{-N}(z)), f^{-n}(x)) \leq \varepsilon$ for any $n \geq 0$ and $d(f^n(z), f^n(q)) \leq \varepsilon$. Thus, it follows from Proposition 2.2 that $z \in f^N(\gamma_\eta^u(x)) \cap W_\eta^s(q)$, which is a contradiction. This finishes the proof of Theorem A. $\square$

## 3. Proofs of corollaries

*Proof of Corollary* 2. It follows from [1, Theorem 3.1] that there is an open and dense subset $\mathcal{P}'$ in $\mathcal{RNT}$ such that every diffeomorphism in $\mathcal{P}'$ has two saddles with different indices.

We set $\mathcal{P} = \mathcal{P}' \cap \mathcal{SPH}_1(M)$. Then by the openness of $\mathcal{SPH}_1(M)$, $\mathcal{P}$ is open and dense in $\mathcal{RNT} \cap \mathcal{SPH}_1(M)$. Let $f \in \mathcal{P}$. Then there are two saddles $p$ and $q$ so that $\dim W^u(p) < \dim W^u(q)$. Since $\dim E^c = 1$, we see that $\dim W^u(p) = \dim E^u$. So, by Theorem A, we have Corollary 2. □

*Proof of Corollary* 3. Let $M$ be a three-dimensional closed manifold and $\mathcal{RNT}$ be the set of robustly non-hyperbolic transitive diffeomorphisms. Then it follows from [1, Theorem 3.1] that there is an open and dense subset $\mathcal{P}$ in $\mathcal{RNT}$ so that for any $f \in \mathcal{P}$ such that every diffeomorphism in $\mathcal{P}$ has two saddles with different indices.

Since $f \in \mathcal{P}$ is robustly transitive, it follows from [12] that $f$ has a partially hyperbolic splitting $E^u \oplus E^c \oplus E^s$. Thus, the existence of two saddles with different indices, together with Theorem A implies Corollary 3. □

*Proof of Corollary* 4. Let $f$ be a robustly transitive diffeomorphism which has a hyperbolic periodic point $p$ with stable index one. Let $\mathcal{U}_1$ be a $C^1$-neighborhood of $f$ such that for $g \in \mathcal{U}_1$, $g$ is transitive and the continuation $p(g)$ of the periodic point $p$, with stable index one, can be defined. By [1, Lemma 2.5], there exists a residual subset $\mathcal{J}$ of $\mathcal{U}_1$ such that for all $g \in \mathcal{J}$, the homoclinic class of $p(g)$ is the whole ambient manifold $M$.

Let us assume that (1) and (3) do not hold, and we will prove (2). Since (3) does not occur, then $f$ is contained in a $C^1$-open set $\mathcal{U}_2 \subset \mathcal{U}_1$ such that any $g \in \mathcal{U}_2$ does not exhibit a homoclinic tangency. By [11, Theorem 1.1], we can take a residual subset $\mathcal{G}$ of $\mathcal{U}_2$ such that $g \in \mathcal{G}$ has a dominated splitting $E^s \oplus E^c_1 \oplus \cdots E^c_k \oplus E^u$ such that

- $E^s$ is uniformly contracting, and $E^u$ is uniformly expanding. (Here $E^s$ and/or $E^u$ might be trivial.)
- each $E^c_i$ is one-dimensional ($i = 1, \cdots, k$).

We may assume that $\mathcal{G} \subset \mathcal{U}_2 \cap \mathcal{J}$. We claim that $E^s$ is one-dimensional. Indeed, since the stable index of $p(g)$ is 1, the dimension of $E^s$ is smaller than or equal to 1. On the other hand, since $g \in \mathcal{G}$ is robustly transitive, by [5, Theorems 2 and 4], $g$ has its finest dominated splitting $E_1 \oplus E_2 \oplus \cdots \oplus E_k$ which is volume hyperbolic, i.e. there is $n \in \mathbb{N}$ such that the derivative of $g^n$ uniformly contracts the volume in $E_1$ and uniformly expands the volume in $E_k$. Hence, if $E^s$ was trivial, then $E^c_1 = E_1$ would be uniformly contracting because $\dim E^c_1 = 1$, which is a contradiction. This proves our claim.

Since $f$ is not an Anosov diffeomorphism, we can take $h \in \mathcal{U}_2$ arbitrarily close to $f$, which has a hyperbolic periodic point $q$ with stable index different from that of $p$. Since this is an open condition, we may indeed assume without loss of generality that $h \in \mathcal{G}$. Clearly, $h$ satisfies the assumptions of Theorem A and so does not satisfy the specification property. This finishes the proof of the corollary. □

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