OPERATOR LIPSCHITZ ESTIMATES
IN THE UNITARY SETTING

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Abstract. We develop a Lipschitz estimate for unitary operators. More specifically, we show that for each $p \in (1, \infty)$ there exists a constant $d_p$ such that $\|f(U) - f(V)\|_p \leq d_p \|U - V\|_p$ for all Lipschitz functions $f : T \to \mathbb{C}$ and unitary operators $U$ and $V$.

1. Preliminaries

We consider a problem in perturbation theory arising from a 1964 conjecture of M. G. Krein [1]. Although the conjecture did not hold [2], an entire class of new problems was brought to light. In fact, it was only recently that D. Potapov and F. Sukochev found a solution in the self-adjoint setting, as follows (notation will be explained below).

Theorem 1 (Potapov–Sukochev [3, Theorem 1]). For each $p \in (1, \infty)$, there is a constant $c_p$ such that

\begin{equation}
\|f(A) - f(B)\|_p \leq c_p \|f\|_{Lip} \|A - B\|_p
\end{equation}

for all Lipschitz functions $f : \mathbb{R} \to \mathbb{C}$ and all self-adjoint operators $A$ and $B$.

The intention of this paper is to translate the result above from the self-adjoint to the unitary setting. We now state our main result.

Theorem 2. For each $p \in (1, \infty)$, there is a constant $d_p$ such that

\begin{equation}
\|f(U) - f(V)\|_p \leq d_p \|f\|_{Lip} \|U - V\|_p
\end{equation}

for all Lipschitz functions $f : T \to \mathbb{C}$ and all unitary operators $U$ and $V$. Further, $d_p \leq 32(c_p + 9)$.

Our method is straightforward and improves the constant $d_p$ significantly over the existing estimates [4, Corollary 6.1].

We now explain the notation. Recall that a function $f : \mathbb{R} \to \mathbb{C}$ is said to be Lipschitz continuous if there is a constant $C$ such that

$$|f(x) - f(y)| \leq C|x - y| \quad \forall x, y \in \mathbb{R};$$

it is well known that this is equivalent to the (distributional) derivative $f'$ being in $L^\infty(\mathbb{R})$ and $\|f'\|_\infty \leq C$. The Lipschitz “norm” $\|f\|_{Lip}$ of $f$ is the smallest possible value of $C$ in either of these inequalities. Evidently the Lipschitz “norm” of a
constant function is 0; however, we may add a constant to the function $f$ in the theorems above without changing either side of (1) or (2).

Throughout, we deal with a fixed Hilbert space $H$ and linear operators thereon. For $p \in [1, \infty)$, the Schatten $p$-norm of the operator $T$ on $H$ is given by

$$\| T \|_p = \left( \sum_{n \in \mathbb{N}} s_n(T)^p \right)^{1/p},$$

where the numbers $s_n(T)$ are the eigenvalues of $(T^*T)^{1/2}$.

Of course, we also deal with functions $f: T \to \mathbb{C}$, where $T$ denotes the unit circle in the complex plane. For such a function $f$, we write $f_r$ for the corresponding $2\pi$-periodic function on $\mathbb{R}$, namely,

$$f_r(\theta) = f(\exp(i\theta)) \quad \forall \theta \in \mathbb{R},$$

and we define $\| f \|_{\text{Lip}}$ to be $\| f_r \|_{\text{Lip}}$.

By Fourier analysis, we may write any (sufficiently smooth) function $f: T \to \mathbb{C}$ in the form

$$f(z) = \sum_{n \in \mathbb{Z}} \hat{f}(n) z^n \quad \forall z \in T,$$

where

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\exp(i\theta)) e^{-in\theta} \, d\theta.$$

We write $\exp(i[a, b])$ for the arc $\{ e^{i\theta} : a \leq \theta \leq b \}$.

2. Proof of the main result

We begin with a simpler version of the result.

**Lemma 3.** Suppose that $f: T \to \mathbb{C}$ is such that $\sum_{n \in \mathbb{Z}} |n \hat{f}(n)| < \infty$. Then

$$\| f(U) - f(V) \|_q \leq \sum_{n \in \mathbb{Z}} |n \hat{f}(n)| \| U - V \|_q$$

for all unitary operators $U$ and $V$ and all $q \in [1, \infty]$.

**Proof.** Clearly, if $n > 0$, then $U^n - V^n = \sum_{k=0}^{n-1} U^k (U - V) V^{n-1-k}$, and so

$$\| U^n - V^n \|_q \leq \sum_{k=0}^{n-1} \| U^k \| \| U - V \|_q \| V^{n-1-k} \| = n \| U - V \|_q,$$

the case where $n < 0$ is similar, whence

$$\| f(U) - f(V) \|_q \leq \sum_{n \in \mathbb{Z}} \left| \hat{f}(n) \right| \| U^n - V^n \|_q \leq \sum_{n \in \mathbb{Z}} |n \hat{f}(n)| \| U - V \|_q,$$

as required. \hfill $\Box$

**Lemma 4.** Suppose that $f: T \to \mathbb{C}$ is a Lipschitz function and that

$$f(z) = \sum_{n \in \mathbb{Z}} \hat{f}(n) z^n \quad \forall z \in T,$$

where $\hat{f}(0) = 0$. Then $\| f \|_{\infty} \leq 2 \| f \|_{\text{Lip}}$. 
Proof. Clearly,

\[ f'(z) = \sum_{n \in \mathbb{Z} \setminus \{0\}} n\hat{f}(n) z^n \quad \forall z \in \mathbb{T}, \]

and so

\[ \|f\|_\infty \leq \sum_{n \in \mathbb{Z} \setminus \{0\}} |\hat{f}(n)| \leq \left( \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{n^2} \right)^{1/2} \left( \sum_{n \in \mathbb{Z} \setminus \{0\}} |n\hat{f}(n)|^2 \right)^{1/2} \leq 2 \|f'\|_2, \]

as claimed. \qed

Proof of Theorem 2. First, we use a partition of unity argument to reduce to the case where \( \text{supp}(f) \subseteq \exp(i[\frac{\pi}{6}, \frac{5\pi}{6}]) \), at the cost of increasing the constant \( d_p \) by the factor 16.

We take the function \( \phi : \mathbb{T} \to [0, 1] \) such that \( \phi_r \) is linear on both \([\frac{\pi}{3}, \frac{2\pi}{3}]\) and \([\frac{2\pi}{3}, \frac{5\pi}{6}]\), and

\[ \phi(e^{i\theta}) = \begin{cases} 1 & \text{if } \theta \in [\frac{\pi}{3}, \frac{2\pi}{3}], \\ 0 & \text{if } \theta \in [\frac{2\pi}{3}, \frac{13\pi}{18}]; \end{cases} \]

then \( \|\phi'\|_\infty = 6/\pi \) and \( \phi_1 + \phi_2 + \phi_3 + \phi_4 = 1 \), where \( \phi_k(e^{i\theta}) = \phi(ie^{i\theta}) \) for all \( \theta \in \mathbb{R} \). Without loss of generality, we suppose that \( f \) has mean 0, and so Lemma 4 implies that

\[ \|f\phi_k\|_{\text{Lip}} \leq \|f'\phi_k\|_\infty + \|f(\phi_k)^{\prime}\|_\infty \leq 4 \|f\|_{\text{Lip}}. \]

Since \( f = f\phi_1 + f\phi_2 + f\phi_3 + f\phi_4 \), it suffices to show Theorem 2 for each \( f\phi_k \). By a simple argument using rotations, it suffices to treat the case where \( k = 0 \).

Suppose that \( \text{supp}(f) \subseteq \exp(i[\frac{\pi}{6}, \frac{5\pi}{6}]) \). We define the symmetrized function \( \tilde{f} : \mathbb{T} \to \mathbb{C} \) by

\[ \tilde{f}(e^{i\theta}) = f(e^{i\theta}) + f(e^{-i\theta}) \quad \forall \theta \in \mathbb{R}. \]

The corresponding function \( \tilde{f} \) on \( \mathbb{R} \) is given by \( \tilde{f}(\theta) = f_r(\theta) + f_r(-\theta) \) for all \( \theta \in \mathbb{R} \). The functions \( f_r(\cdot) \) and \( f_r(-\cdot) \) have disjoint supports, so \( \|\tilde{f}\|_{\text{Lip}} = \|f\|_{\text{Lip}} \). We now show that

\[ \|\tilde{f}(U) - \tilde{f}(V)\|_p \leq 2 c_p \|f\|_{\text{Lip}} \|U - V\|_p \]

for all unitary operators \( U \) and \( V \). Define \( g : \mathbb{R} \to \mathbb{C} \) by

\[ g(\lambda) = \begin{cases} \tilde{f}_r(\cos^{-1}(\lambda)) & \text{if } \lambda \in [-1, 1], \\ 0 & \text{otherwise}. \end{cases} \]

If \( \lambda \in [\frac{\sqrt{3}}{2}, 1] \), then \( \cos^{-1}(\lambda) \in [0, \frac{\pi}{4}] \), and so \( g(\lambda) = 0 \). Similarly, \( g(\lambda) = 0 \) if \( \lambda \in [-1, -\frac{\sqrt{3}}{2}] \). Now \( g'(\lambda) = 0 \) unless \( \lambda \in [-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}] \), and for almost all these \( \lambda \),

\[ |g'(\lambda)| = \left| \tilde{f}_r'(\cos^{-1}(\lambda)) \frac{1}{(1 - \lambda^2)^{1/2}} \right| \leq 2 \|f\|_{\text{Lip}}; \]

that is, \( \|g\|_{\text{Lip}} \leq 2 \|f\|_{\text{Lip}} \). Further, by definition, for all \( \theta \in \mathbb{R} \),

\[ g\left(\frac{1}{2}(e^{i\theta} + e^{-i\theta})\right) = g(\cos(\theta)) = \tilde{f}_r(\theta) = \tilde{f}(e^{i\theta}). \]
For any unitary operator $W$, the operator $\frac{1}{2}(W + W^*)$ is self-adjoint, and by the spectral theorem,

$$g(\frac{1}{2}(W + W^*)) = \hat{f}(W).$$

Theorem [4] therefore implies that

$$\left\|\hat{f}(U) - \hat{f}(V)\right\|_p \leq \left\|g(\frac{1}{2}(U + U^*)) - g(\frac{1}{2}(V + V^*))\right\|_p \leq c_p \|g\|_{\text{Lip}} \left\|\frac{1}{2}(U + U^*) - \frac{1}{2}(V + V^*)\right\|_p \leq 2c_p \|\hat{f}\|_{\text{Lip}} \|U - V\|_p.$$

We continue to suppose that $\text{supp}(f) \subseteq \exp(i[\frac{\pi}{6}, \frac{5\pi}{6}])$. By Lemma [3] $\|f\|_{\infty} \leq 2\|f\|_{\text{Lip}}$. To conclude, we show that

$$\|f(U) - f(V)\|_p \leq (2c_p + 18) \|f\|_{\text{Lip}} \|U - V\|_p.$$

Define the function $\psi : \mathbb{T} \to [0,1]$ by requiring that $\psi_\lambda$ is linear on both $[-\frac{\pi}{18}, \frac{\pi}{18}]$ and $[\frac{17\pi}{18}, \frac{19\pi}{18}]$, and

$$\psi(e^{i\theta}) = \begin{cases} 1 & \text{if } \theta \in \left[\frac{\pi}{18}, \frac{17\pi}{18}\right], \\ 0 & \text{if } \theta \in \left[\frac{19\pi}{18}, \frac{35\pi}{18}\right]. \end{cases}$$

Further, let $\chi : \mathbb{T} \to \mathbb{R}$ be the step function such that $\chi(e^{i\theta}) = 9$ if $\theta \in [-\frac{\pi}{9}, \frac{\pi}{9}]$ and $\chi(e^{i\theta}) = 0$ otherwise, and define $\phi$ to be the convolution $\chi \ast \psi$; that is,

$$\phi(e^{i\theta}) = \frac{9}{2\pi} \int_{-\pi/9}^{\pi/9} \psi(e^{i(\theta-\eta)}) \, d\eta \quad \forall \theta \in \mathbb{R}.$$

We see easily that $\phi(e^{i\theta}) = 1$ if $\theta \in \left[\frac{\pi}{9}, \frac{5\pi}{9}\right]$ and $\phi(e^{i\theta}) = 0$ if $\theta \in \left[\frac{7\pi}{9}, \frac{11\pi}{9}\right]$; further, $\hat{\phi} = \hat{\psi} \hat{\chi}$, and a routine computation shows that

$$\sum_{n \in \mathbb{Z}} \left| n \hat{\phi}(n) \right| = \sum_{n \in \mathbb{Z} \setminus \{0\}} \left| n \sin\left(\frac{n\pi}{9}\right) \left| \frac{9e^{-\frac{i\pi}{2}n/2}}{n^2\pi^2} \right| \left( \cos\left(\frac{17n\pi}{18}\right) - \cos\left(\frac{19n\pi}{18}\right) \right) \right| \leq \frac{162}{n^2\pi^3} = \frac{54}{\pi} < 18.$$ 

Then $f = \phi \hat{f}$. Further, $\phi(U)$ is well defined and $\|\phi(U)\|_{\infty} = 1$ by the spectral theorem. Thus, by [3] and Lemma [3]

$$\|f(U) - f(V)\|_p = \|\phi(U)\hat{f}(U) - \phi(V)\hat{f}(V)\|_p \leq \|\phi(U)\hat{f}(U) - \hat{f}(V)\|_p + \|\phi(U) - \phi(V)\|_p \|\hat{f}\|_p \leq 2c_p \|f\|_{\text{Lip}} \|U - V\|_p + 18 \|U - V\|_p \|f\|_{\text{Lip}},$$

as desired. This step introduces a factor of $2(c_p + 9)$ into $d_p$. \hfill \Box

Our final remark concerns the dependence of the constant $d_p$ on $p$. It was shown in [5] that $c_p = O(p^2/(p - 1))$. It is clear that the behaviour of the constant $d_p$ is similar.
REFERENCES


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